William Poore's book
bought of S. B. Balch Geo. Town. –
Cost 2½ Dollars.
Augt. 28th. 1801.

Mathematical speculation, will inure us to attend closely to any subject, to seek and gain clear ideas, to distinguish truth from falsehood, to judge justly, and to argue strongly; and these studies do more directly furnish us with all the various rules of those useful ts of life viz measuring, building, sailing &c.

I. Watts.

NLE
ed.gov

THE
ELEMENTS
OF
EUCLID

VIZ.

THE FIRST SIX BOOKS,

TOGETHER WITH THE

ELEVENTH AND TWELFTH.

The Errors, by which THEON, or others, have long ago
vitiated theſe Books, are corrected;

And ſome of EUCLID'S Demonſtrations are reſtored.

ALSO

THE BOOK OF

EUCLID'S DATA,

In like manner corrected.

BY

ROBERT SIMSON, M.D.

Emeritus Profeſſor of Mathematics in the Univerſity of Glaſgow.

To this NINTH EDITION are alſo annexed
ELEMENTS of PLAIN and SPHERICAL TRIGONOMETRY.

EDINBURGH:
PRINTED FOR E. WINGROVE, LONDON,
AND J. BALFOUR, EDINBURGH.

M,DCC,XCIII.

1793

TO THE

KING,

THIS EDITION

OF THE

PRINCIPAL BOOKS

OF THE

ELEMENTS OF EUCLID,

AND OF THE

BOOK OF HIS DATA,

IS MOST HUMBLY DEDICATED,

BY

HIS MAJESTY'S

MOST DUTIFUL, AND

MOST DEVOTED

SUBJECT AND SERVANT,

ROBERT SIMSON.

PREFACE.

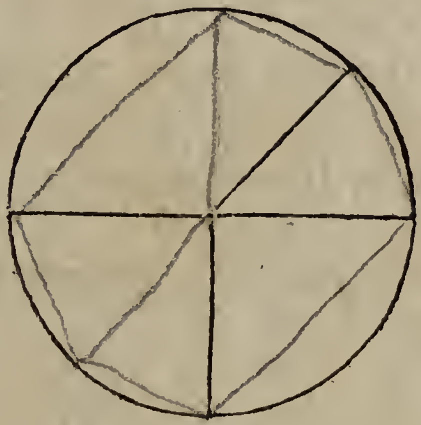
THE Opinions of the Moderns concerning the Author of the Elements of Geometry, which go under Euclid's name, are very different and contrary to one another. Peter Ramus aſcribes the Propoſitions, as well as their Demonſtrations, to Theon; others think the Propoſitions to be Euclid's, but that the Demonſtrations are Theon's; and others maintain that all the Propoſitions and their Demonſtrations are Euclid's own. John Buteo and Sir Henry Savile are the Authors of greateſt Note who aſſert this laſt; and the greater part of Geometers have ever ſince been of this Opinion, as they thought it the moſt probable. Sir Henry Savile after the ſeveral Arguments he brings to prove it, makes this Concluſion, (Page 13. Praelect.) "That, excepting a very few "Interpolations, Explications, and Additions, Theon altered "nothing in Euclid." But, by often conſidering and comparing together the Definitions and Demonſtrations as they are in the Greek Editions we now have, I found that Theon, or whoever was the Editor of the preſent Greek Text, by adding ſome things, ſuppreſſing others, and mixing his own with Euclid's Demonſtrations, had changed more things to the worſe than is commonly ſuppoſed, and thoſe not of ſmall moment, eſpecially in the Fifth and Eleventh Books of the Elements, which this Editor has greatly vitiated; for inſtance, by ſubſtituting a ſhorter, but inſufficient Demonſtration of the 18th Prop. of the 5th Book, in place of the legitimate one which Euclid had given; and by taking out of this Book, beſides other things, the good Definition which Eudoxus or Euclid had given of Compound Ratio, and given an abſurd one in place of it in the 5th Definition of the 6th Book, which neither Euclid, Archimedes, Appolonius, nor any Geometer before Theon's time, ever made uſe of, and of which there is not to be found the leaſt appearance in any of their Writings; and, as this Definition did much embarraſs Beginners, and is quite uſeleſs, it is now thrown out of the Elements, and another, which, without doubt, Euclid had given, is put in its proper place among the Definitions of the

5th Book, by which the doctrine of compound Ratios is rendered plain and eaſy. Beſides, among the Definitions of the 11th Book, there is this, which is the 10th, viz. "Equal and "ſimilar ſolid Figures are thoſe which are contained by ſimilar "Planes of the ſame Number and Magnitude." Now this Propoſition is a Theorem, not a Definition; becauſe the equality of Figures of any kind muſt be demonſtrated, and not aſſumed; and, therefore, though this were a true Propoſition, it ought to have been demonſtrated. But, indeed, this Propoſition, which makes the 10th Definition of the 11th Book, is not true univerſally, except in the caſe in which each of the ſolid angles of the Figures is contained by no more than three plane Angles; for in other caſes, two ſolid Figures may be contained by ſimilar Planes of the ſame Number and Magnitude, and yet be unequal to one another, as ſhall be made evident in the Notes ſubjoined to theſe elements. In like manner, in the Demonſtration of the 26th prop. of the 11th Book, it is taken for granted, that thoſe ſolid Angles are equal to one another which are contained by plain Angles of the ſame Number, and Magnitude, placed in the ſame order; but neither is this univerſally true, except in the caſe in which the ſolid Angles are contained by no more than three plain Angles; nor of this Caſe is there any Demonſtration in the Elements we now have, though it be quite neceſſary there ſhould be one. Now, upon the 10th Definition of this Book depend the 25th and 28th Propoſitions of it; and, upon the 25th and 26th depend other eight, viz. the 27th, 31ſt, 32d, 33d, 34th, 36th, 37th, and 40th of the ſame Book; and the 12th of the 12th Book depends upon the eighth of the ſame; and this eighth, and the Corollary of Propoſition 17. and Prop. 18th of the 12th Book, depend upon the 9th Definition of the 11th Book, which is not a right Definition; becauſe there may be Solids contained by the ſame number of ſimilar plane Figures, which are not ſimilar to one another, in the true Senſe of Similarity received by Geometers; and all theſe Propoſitions have, for theſe Reaſons, been inſufficiently demonſtrated ſince Theon's time hitherto. Beſides, there are ſeveral other things, which have nothing of Euclid's accuracy, and which plainly ſhew, that his Elements have been much corrupted by unſkilful Geometers; and, though theſe are not ſo groſs as the others now mentioned, they ought by no means to remain uncorrected.

Upon theſe Accounts it appeared neceſſary, and I hope will prove acceptable to all Lovers of accurate Reaſoning, and of Mathematical Learning, to remove ſuch Blemiſhes, and reſtore the

XV.

A circle is a plane figure contained by one line, which is called the circumference, and is ſuch that all ſtraight lines drawn from a certain point within the figure to the circumference, are equal to one another.



XVI.

And this point is called the centre of the circle.

XVII.

A diameter of a circle is a ſtraight line drawn through the centre, and terminated both ways by the circumference.

XVIII.

A ſemicircle is the figure contained by a diameter and the part of the circumference cut off by the diameter.

XIX.

" A ſegment of a circle is the figure contained by a ſtraight line, and the circumference it cuts off."

XX.

Rectilineal figures are thoſe which are contained by ſtraight lines.

XXI.

Trilateral figures, or triangles, by three ſtraight lines.

XXII.

Quadrilateral, by four ſtraight lines.

XXIII.

Multilateral figures, or polygons, by more than four ſtraight lines.

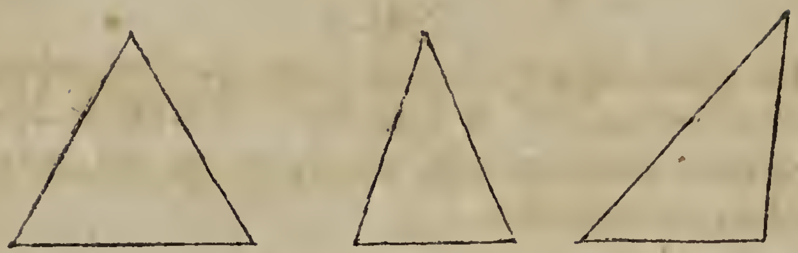
XXIV.

Of three ſided figures, an equilateral triangle is that which has three equal ſides.

XXV.

An iſoſceles triangle is that which has only two ſides equal.

XXVI.



XXVI.

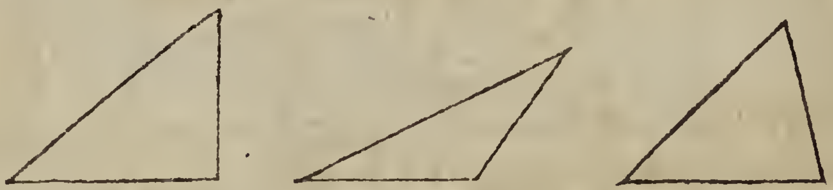
A ſcalene triangle, is that which has three unequal ſides.

XXVII.

A right angled triangle, is that which has a right angle.

XXVIII.

An obtuſe angled triangle, is that which has an obtuſe angle.

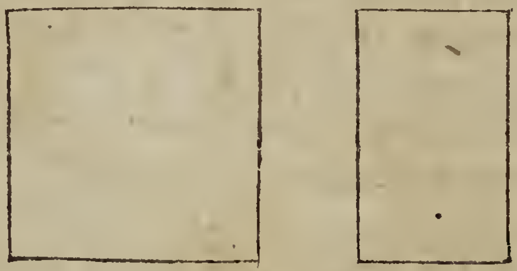


XXIX.

An acute angled triangle, is that which has three acute angles.

XXX.

Of four ſided figures, a ſquare is that which has all its ſides equal, and all its angles right angles.

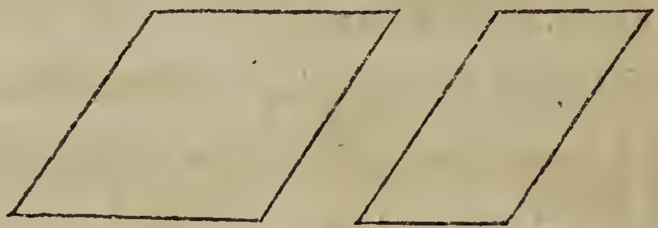


XXXI.

An oblong, is that which has all its angles right angles, but has not all its ſides equal.

XXXII.

A rhombus, is that which has all its ſides equal, but its angles are not right angles.



XXXIII.

A rhomboid, is that which has its oppoſite ſides equal to one another, but all its ſides are not equal, nor its angles right angles.

XXXIV.

All other four ſided figures beſides theſe, are called Trapeziums.

XXXV.

Parallel ſtraight lines, are ſuch as are in the ſame plane, and which, being produced ever ſo far both ways, do not meet.

POSTULATES.

I.

LET it be granted that a ſtraight line may be drawn from any one point to any other point.

II.

That a terminated ſtraight line may be produced to any length in a ſtraight line.

III.

And that a circle may be deſcribed from any centre, at any diſtance from that centre.

AXIOMS.

I.

THINGS which are equal to the ſame are equal to one another.

II.

If equals be added to equals, the wholes are equal.

III.

If equals be taken from equals, the remainders are equal.

IV.

If equals be added to unequals, the wholes are unequal.

V.

If equals be taken from unequals, the remainders are unequal.

VI.

Things which are double of the ſame, are equal to one another.

VII.

Things which are halves of the ſame, are equal to one another.

VIII.

Magnitudes which coincide with one another, that is, which exactly fill the ſame ſpace, are equal to one another.

IX.

The whole is greater than its part.

X.

Two ſtraight lines cannot incloſe a ſpace.

XI.

All right angles are equal to one another.

XII.

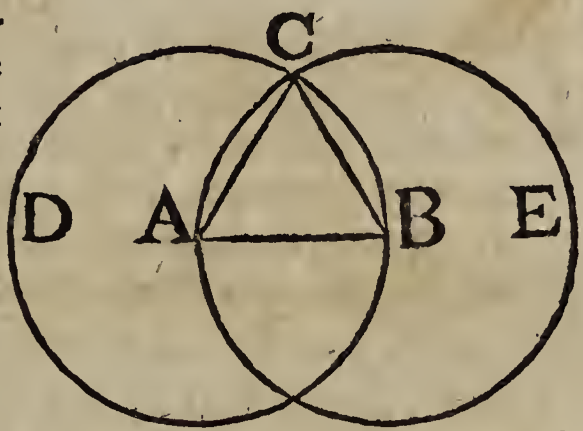
" If a ſtraight line meets two ſtraight lines, ſo as to make the " two interior angles on the ſame ſide of it taken together " leſs than two right angles, theſe ſtraight lines being continually produced, ſhall at length meet upon that ſide on " which are the angles which are leſs than two right angles. " See the notes on Prop. 29. of Book I."

PROPO-

PROPOSITION I. PROBLEM.

TO deſcribe an equilateral triangle upon a given finite ſtraight line.

Let AB be the given ſtraight line; it is required to deſcribe an equilateral triangle upon it.



From the centre A, at the diſtance AB, deſcribe [a] the circle BCD, and from the centre, B, at the diſtance BA, deſcribe the circle ACE; and from the point C, in which the circles cut one another, draw the ſtraight lines[b] CA, CB to the points A, B; ABC ſhall be an equilateral triangle.

a. 3. Poſtulate.

b. 1. Poſt.

Becauſe the point A is the centre of the circle BCD, AC is equal [c] to AB; and becauſe the point B is the centre of the circle ACE, BC is equal to BA: But it has been proved that CA is equal to AB; therefore CA, CB are each of them equal to AB; but things which are equal to the ſame are equal to one another [d]; therefore CA is equal to CB; wherefore CA, AB, BC are equal to one another; and the triangle ABC is therefore equilateral, and it is deſcribed upon the given ſtraight line AB. Which was required to be done.

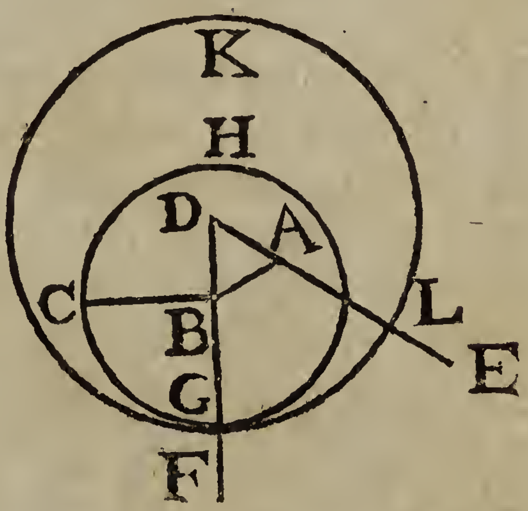
c. 15. Definition.

d. 1ſt Axiom.

PROP. II. PROB.

FROM a given point to draw a ſtraight line equal to a given ſtraight line.

Let A be the given point, and BC the given ſtraight line; it is required to draw from the point A a ſtraight line equal to BC.



From the point A to B draw [a] the ſtraight line AB; and upon it deſcribe [b] the equilateral triangle DAB, and produce [c] the ſtraight lines DA, DB, to E and F; from the centre B, at the diſtance BC, deſcribe [d] the circle CGH, and from the centre D, at the diſtance DG, deſcribe the circle GKL. AL ſhall be equal to BC.

a. 1. Poſt.

b. 1. 1.

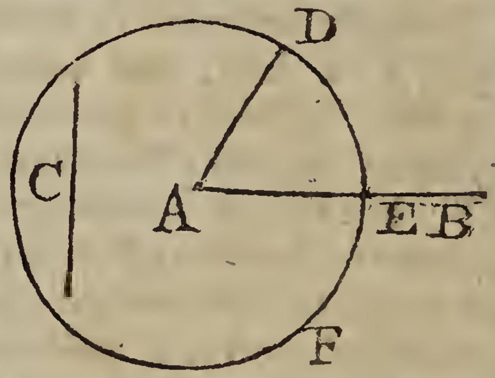
c. 2. Poſt.

d. 3. Poſt.

Becauſe the point B is the centre of the circle CGH, BC is
e. 15. Def. equal [e] to BG; and becauſe D is the centre of the circle GKL, DL is equal to DG, and DA, DB, parts of them, are equal;
f. 3. Ax. therefore the remainder AL is equal to the remainder [f] BG: But it has been ſhown, that BC is equal to BG; wherefore AL and BC are each of them equal to BG; and things that are equal to the ſame are equal to one another; therefore the ſtraight line AL is equal to BC. Wherefore from the given point A a ſtraight line AL has been drawn equal to the given ſtraight line BC. Which was to be done.

PROP. III. PROB.

FROM the greater of two given ſtraight lines to cut off a part equal to the leſs.



Let AB and C be the two given ſtraight lines, whereof AB is the greater. It is required to cut off from AB, the greater, a part equal to C, the leſs.

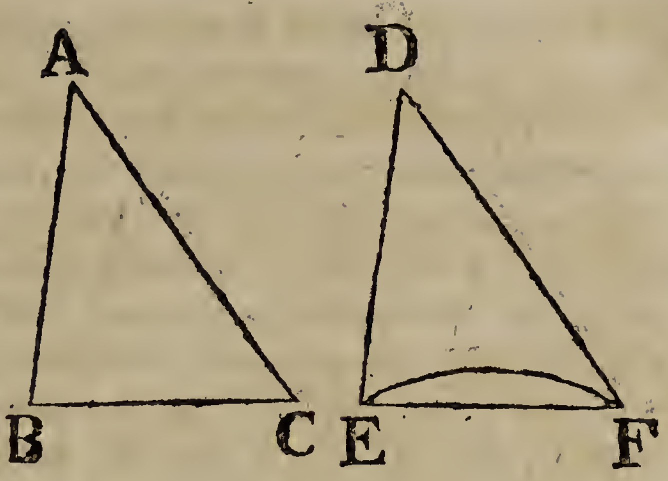
a. 2. 1. From the point A draw [a] the ſtraight line AD equal to C; and from the centre A, and at the diſ-
b. 3. Poſt. tance AD, deſcribe [b] the circle DEF; and becauſe A is the centre of the circle DEF, AE ſhall be equal to AD; but the ſtraight line C is likewiſe equal to AD; whence AE and C are each of them equal to AD; wherefore the ſtraight line AE is equal
c. 1. Ax. to [c] C, and from AB, the greater of two ſtraight lines, a part AE has been cut off equal to C the leſs. Which was to be done.

PROP. IV. THEOREM.

IF two triangles have two ſides of the one equal to two ſides of the other, each to each; and have likewiſe the angles contained by thoſe ſides equal to one another; they ſhall likewiſe have their baſes, or *third ſides*, equal; and the two triangles ſhall be equal; and their other angles ſhall be equal, each to each, viz, thoſe to which the equal ſides are oppoſite.

Let ABC, DEF be two triangles which have the two ſides AB, AC equal to the two ſides DE, DF, each to each, viz.

AB to DE, and AC to DF; and the angle BAC equal to the angle EDF, the baſe BC ſhall be equal to the baſe EF; and the triangle ABC to the triangle DEF; and the other angles, to which the equal ſides are oppoſite, ſhall be equal each to each, viz. the angle ABC to the angle DEF, and the angle ACB to DFE.



For, if the triangle ABC be applied to DEF, ſo that the point A may be on D, and the ſtraight line AB upon DE; the point B ſhall coincide with the point E, becauſe AB is equal to DE; and AB coinciding with DE, AC ſhall coincide with DF, becauſe the angle BAC is equal to the angle EDF; wherefore alſo the point C ſhall coincide with the point F, becauſe the ſtraight line AC is equal to DF: But the point B coincides with the point E; wherefore the baſe BC ſhall coincide with the baſe EF; becauſe the point B coinciding with E, and C with F, if the baſe BC does not coincide with the baſe EF, two ſtraight lines would incloſe a ſpace, which is impoſſible [a]. Therefore the baſe BC ſhall coincide with the baſe EF, and be equal to it. Wherefore the whole triangle ABC ſhall coincide with the whole triangle DEF, and be equal to it; and the other angles of the one ſhall coincide with the remaining angles of the other, and be equal to them, viz. the angle ABC to to the angle DEF, and the angle ACB to DFE. Therefore, if two triangles have two ſides of the one equal to two ſides of the other, each to each, and have likewiſe the angles contained by thoſe ſides equal to one another, their baſes ſhall likewiſe be equal, and the triangles be equal, and their other angles to which the equal ſides are oppoſite ſhall be equal, each to each. Which was to be demonſtrated.

[a] 10 Ax.

PROP. V. THEOR.

THE angles at the baſe of an Iſoſceles triangle are equal to one another; and, if the equal ſides be produced, the angles upon the other ſide of the baſe ſhall be equal.

Let ABC be an Iſoſceles triangle, of which the ſide AB is e-

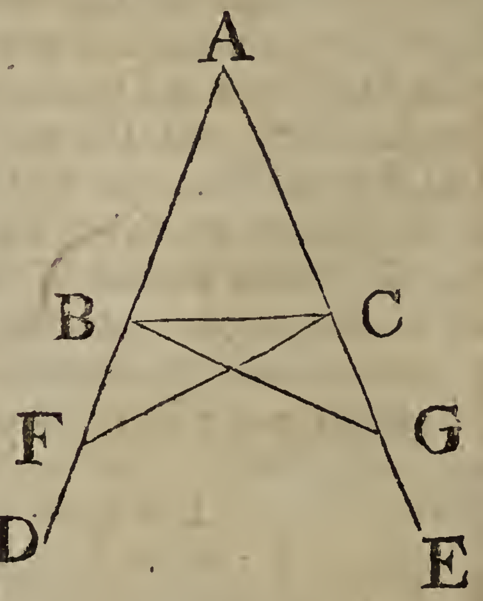
Book I. qual to AC, and let the ſtraight lines AB, AC be produced to D and E, the angle ABC ſhall be equal to the angle ACB, and the angle CBD to the angle BCE.

In BD take any point F, and from AE the greater, cut off

a 3. 1. AG equal [a] to AF, the leſs; and join FC, GB.

Becauſe AF is equal to AG, and AB to AC, the two ſides FA, AC are equal to the two GA, AB, each to each; and they contain the angle FAG common to the two triangles AFC, AGB; therefore the baſe FC is e-

b 4. 1 qual [b] to the baſe GB, and the triangle AFC to the triangle AGB; and the remaining angles of the one are equal [b] to the remaining angles of the other, each to each, to which the equal ſides are oppoſite; viz. the angle ACF to the angle ABG, and the angle AFC to the angle AGB: And becauſe the whole AF is equal to the whole AG, of which the parts AB, AC, are equal; the



c 3. Ax. remainder BF ſhall be equal [c] to the remainder CG; and FC was proved to be equal to GB; therefore the two ſides BF, FC are equal to the two CG, GB, each to each; and the angle BFC is equal to the angle CGB, and the baſe BC is common to the two triangles BFC, CGB; wherefore the triangles are equal [b], and their remaining angles, each to each, to which the equal ſides are oppoſite; therefore the angle FBC is equal to the angle GCB, and the angle BCF to the angle CBG: And, ſince it has been demonſtrated, that the whole angle ABG is equal to the whole ACF, the parts of which, the angles CBG, BCF are alſo equal; the remaining angle ABC is therefore equal to the remaining angle ACB, which are the angles at the baſe of the triangle ABC: And it has alſo been proved that the anlge FBC is equal to the angle GCB, which are the angles upon the other ſide of the baſe. Therefore the angles at the baſe, &c. Q. E. D.

Corollary. Hence every equilateral triangle is alſo equiangular.

PROP. VI. THEOR.

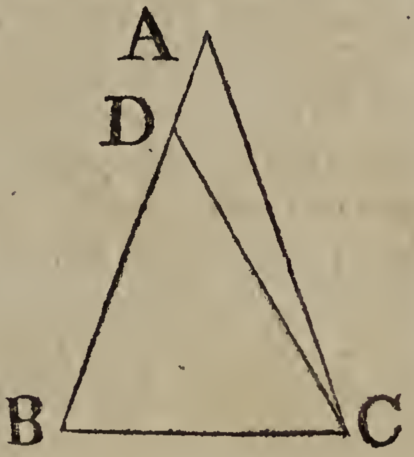
If two angles of a triangle be equal to one another, the ſides alſo which ſubtend, *or are oppoſite to*, the equal angles, ſhall be equal to one another.

Let ABC be a triangle having the angle ABC equal to the angle ACB; the ſide AB is alſo equal to the ſide AC.

For, if AB be not equal to AC, one of them is greater than the other: Let AB be the greater, and from it cut [a] off DB equal to AC, the leſs, and join DC; therefore, becauſe in the triangles DBC, ACB, DB is equal to AC, and BC common to both, the two ſides DB, BC are equal to the two AC, CB, each to each; and the angle DBC is equal to the angle ACB; therefore the baſe DC is equal to the baſe AB, and the triangle DBC is equal to the triangle [b] ACB, the leſs to the greater; which is abſurd. Therefore AB is not unequal to AC, that is, it is equal to it. Wherefore, if two angles, &c. Q. E. D.

a 3. 1.

b 4. 1.

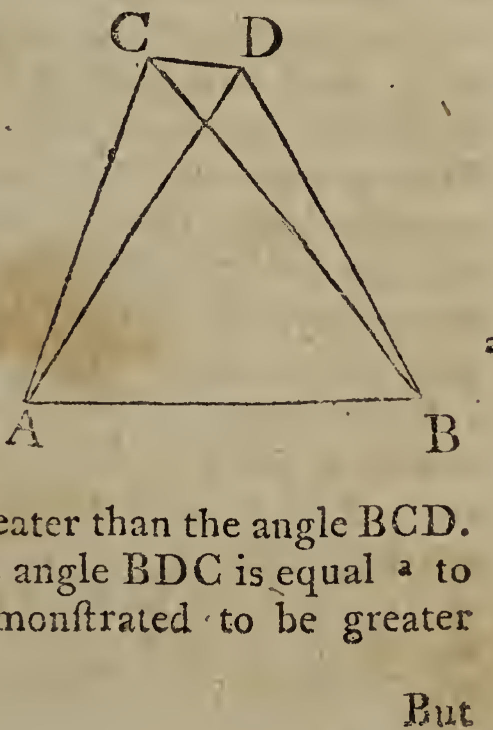


Cor. Hence every equiangular triangle is alſo equilateral.

PROP. VII. THEOR.

UPON the ſame baſe, and on the ſame ſide of it, there cannot be two triangles that have their ſides which are terminated in one extremity of the baſe equal to one another, and likewiſe thoſe which are terminated in the other extremity. See N.

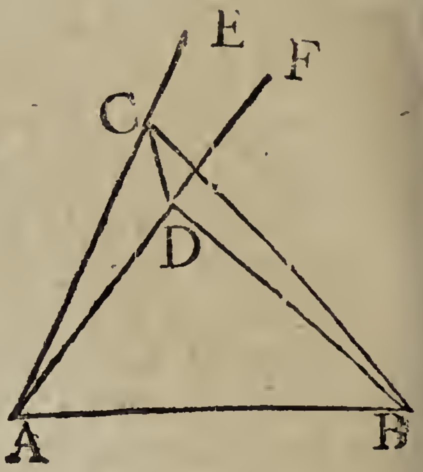
If it be poſſible, let there be two triangles ACB, ADB, upon the ſame baſe AB, and upon the ſame ſide of it, which have their ſides CA, DA, terminated in the extremity A of the baſe equal to one another, and likewiſe their ſides CB, DB, that are terminated in B.



Join CD; then, in the caſe in which the vertex of each of the triangles is without the other triangle, becauſe AC is equal to AD, the angle ACD is equal [a] to the angle ADC: But the angle ACD is greater than the angle BCD; therefore the angle ADC is greater alſo than BCD; much more then is the angle BDC greater than the angle BCD. Again, becauſe CB is equal to DB, the angle BDC is equal [a] to the angle BCD; but it has been demonſtrated to be greater than it; which is impoſſible.

a 5. 1.

Book. I. But if one of the vertices, as D, be within the other triangle ACB; produce AC, AD to E, F; therefore becauſe AC is equal to AD in the triangle ACD, the angles ECD, FDC upon the other ſide of the baſe CD are equal [a] to one another, but the angle ECD is greater, than the angle BCD; wherefore the angle FDC is likewiſe greater than BCD; much more then is the angle BDC greater than the angle BCD. Again, becauſe CB is equal to DB, the angle BDC is equal [a] to the angle BCD; but BDC has been proved to be greater than the ſame BCD; which is impoſſible. The caſe in which the vertex of one triangle is upon a ſide of the other, needs no demonſtration.



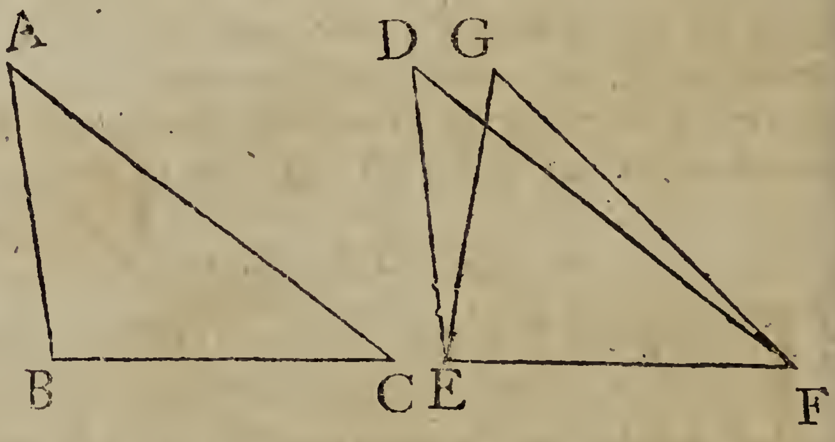
a 5. 1.

Therefore, upon the ſame baſe, and on the ſame ſide of it, there cannot be two triangles that have their ſides which are terminated in one extremity of the baſe equal to one another, and likewiſe thoſe which are terminated in the other extremity. Q. E. D.

PROP. VIII. THEOR.

IF two triangles have two ſides of the one equal to two ſides of the other, each to each, and have likewiſe their baſes equal; the angle which is contained by the two ſides of the one ſhall be equal to the angle contained by the two ſides equal to them, of the other.

Let ABC, DEF be two triangles having the two ſides AB, AC, equal to the two ſides DE, DF, each to each, viz. AB to DE, and AC to DF; and alſo the baſe BC equal to the baſe EF. The angle BAC is equal to the angle EDF.



For, if the triangle, ABC be applied to DEF, ſo that the point B be on E, and the ſtraight line BC upon EF; the point C ſhall alſo coincide with the point F. Becauſe

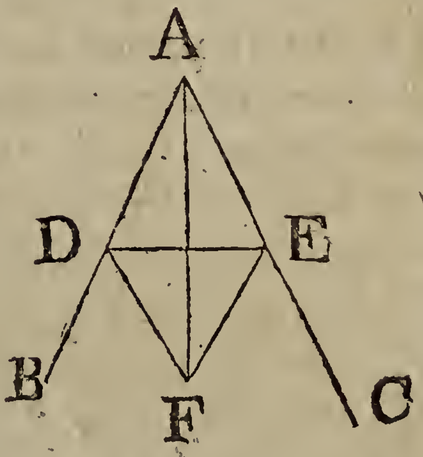
BC is equal to EF; therefore BC coinciding with EF, BA and AC ſhall coincide with ED and DF; for, if the baſe BC coincides with the baſe EF, but the ſides BA, CA do not coincide with the ſides ED, FD, but have a different ſituation as EG, FG; then, upon the ſame baſe EF, and upon the ſame ſide of it, there can be two triangles that have their ſides which are terminated in one extremity of the baſe equal to one another, and likewiſe their ſides terminated in the other extremity: But this is impoſſible [a]; therefore, if the baſe BC coincides with the baſe EF, the ſides BA, AC cannot but coincide with the ſides ED, DF; wherefore likewiſe the angle BAC coincides with the angle EDF, and is equal [b] to it. Therefore if two triangles, &c. Q. E. D. [a] 7. 1. [b] 8. Ax.

PROP. IX. PROB.

TO biſect a given rectilineal angle, that is, to divide it into two equal angles.

Let BAC be the given rectilineal angle, it is required to biſect it.

Take any point D in AB, and from AC cut [a] off AE equal to AD; join DE, and upon it deſcribe [b] an equilateral triangle DEF; then join AF; the ſtraight line AF biſects the angle BAC. [a] 3. 1. [b] 1. 1.



Becauſe AD is equal to AE, and AF is common to the two triangles DAF, EAF; the two ſides DA, AF, are equal to the two ſides EA, AF, each to each; and the baſe DF is equal to the baſe EF; therefore the angle DAF is equal [c] to the angle EAF; wherefore the given rectilineal angle BAC is biſected by the ſtraight line AF. Which was to be done. [c] 8. 1.

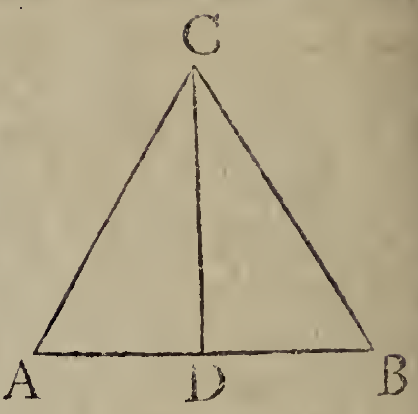
PROP. X. PROB.

TO biſect a given finite ſtraight line, that is, to divide it into two equal parts.

Let AB be the given ſtraight line; it is required to divide it into two equal parts.

Deſcribe [a] upon it an equilateral triangle ABC, and biſect [b] the angle ACB by the ſtraight line CD. AB is cut into two equal parts in the point D. [a] 1. 1. [b] 9. 1.

Book I. Becauſe AC is equal to CB, and CD common to the two triangles ACD, BCD; the two ſides AC, CD are equal to BC, CD, each to each; and the angle ACD is equal to the angle BCD; therefore the baſe AD is equal to the
c 4. 1. baſe [c] DB, and the ſtraight line AB is divided into two equal parts in the point D. Which was to be done.

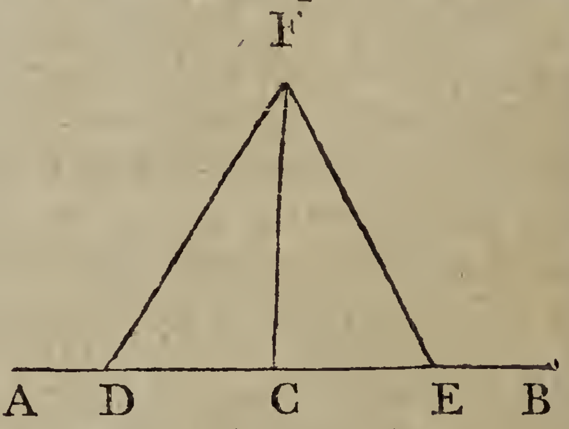


PROP. XI. PROB.

TO draw a ſtraight line at right angles to a given ſtraight line, from a given point in the ſame.

See N. Let AB be a given ſtraight line, and C a point given in it; it is required to draw a ſtraight line from the point C at right angles to AB.

a 3. 1. Take any point D in AC, and [a] make CE equal to CD, and
b 1. 1. upon DE deſcribe [b] the equilateral triangle DFE, and join FC; the ſtraight line FC drawn from the given point C is at right angles to the given ſtraight line AB.



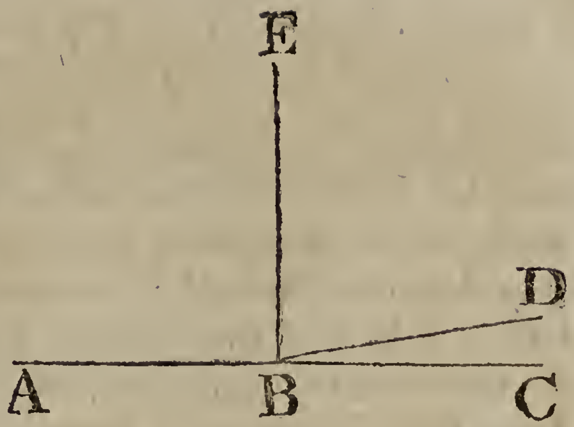
Becauſe DC is equal to CE and FC common to the two triangles DCF, ECF; the two ſides DC, CF, are equal to the two EC, CF, each to each; and the baſe DF is equal to the baſe EF; therefore the angle DCF
c 8. 1. is equal [c] to the angle ECF; and they are adjacent angles. But, when the adjacent angles which one ſtraight line makes with another ſtraight line are equal to one another, each of them
d 10. Def. 1. is called a right [d] angle; therefore each of the angles DCF, ECF, is a right angle. Wherefore, from the given point C, in the given ſtraight line AB, FC has been drawn at right angles to AB. Which was to be done.

Cor. By help of this problem, it may be demonſtrated, that two ſtraight lines cannot have a common ſegment.

If it be poſſible, let the two ſtraight lines ABC, ABD have the ſegment AB common to both of them. From the point B draw BE at right angles to AB; and becauſe ABC is a ſtraight line;

line, the angle CBE is equal[a] to the angle EBA; in the ſame manner, becauſe ABD is a ſtraight line, the angle DBE is equal to the angle EBA; wherefore the angle DBE is equal to the angle CBE, the leſs to the greater; which is impoſſible; therefore two ſtraight lines cannot have a common ſegment.

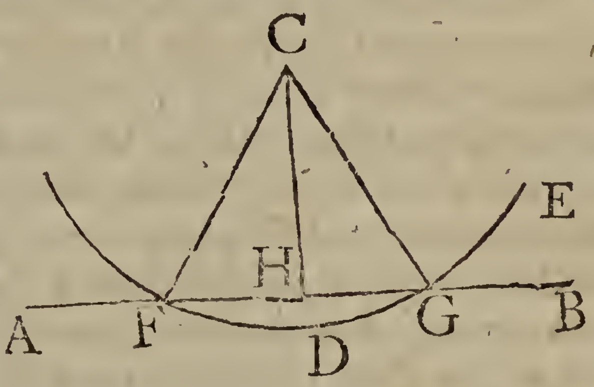
a 10. Def. I.



PROP. XII. PROB.

TO draw a ſtraight line perpendicular to a given ſtraight line of an unlimited length, from a given point without it.

Let AB be the given ſtraight line, which may be produced any length both ways, and let C be a point without it. It is required to draw a ſtraight line perpendicular to AB from the point C.



Take any point D upon the other ſide of AB, and from the centre C, at the diſtance CD, deſcribe[b] the circle EGF meeting AB in F G; and biſect,[c] FG in H, and join CF, CH, CG; the ſtraight line CH, drawn from the given point C, is perpendicular to the given ſtraight line AB.

b 3. Poſt.

c 10. I.

Becauſe FH is equal to HG, and HC common to the two triangles FHC, GHC, the two ſides FH, HC are equal to the two GH, HC, each to each; and the baſe CF is equal[d] to the baſe CG; therefore the angle CHF is equal[e] to the angle CHG; and they are adjacent angles; but when a ſtraight line ſtanding on a ſtraight line makes the adjacent angles equal to one another, each of them is a right angle; and the ſtraight line which ſtands upon the other is called a perpendicular to it; therefore from the given point C a perpendicular CH has been drawn to the given ſtraight line AB. Which was to be done.

d 15. Def. I.

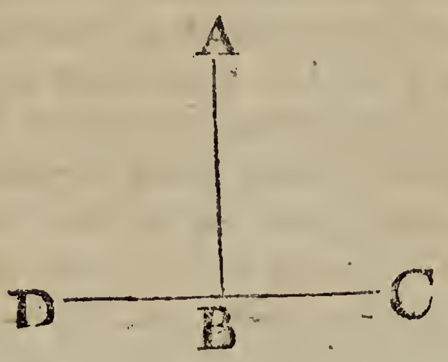
e 8. I.

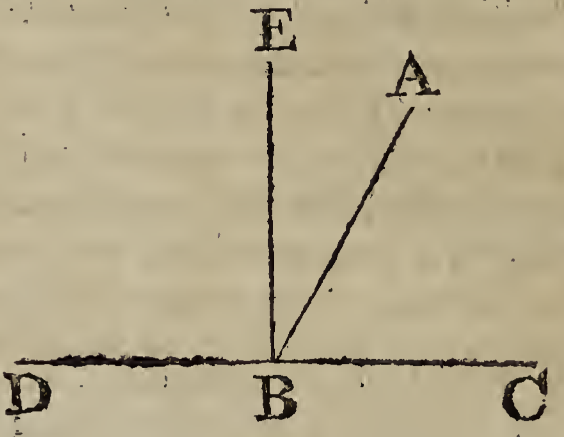
PROP. XIII. THEOR.

THE angles which one ſtraight line makes with another upon the one ſide of it, are either two right angles, or are together equal to two right angles.

Let the ſtraight line AB make with CD, upon one ſide of it, the angles CBA, ABD; theſe are either two right angles, or are together equal to two right angles.

For if the angle CBA be equal to ABD, each of them is a



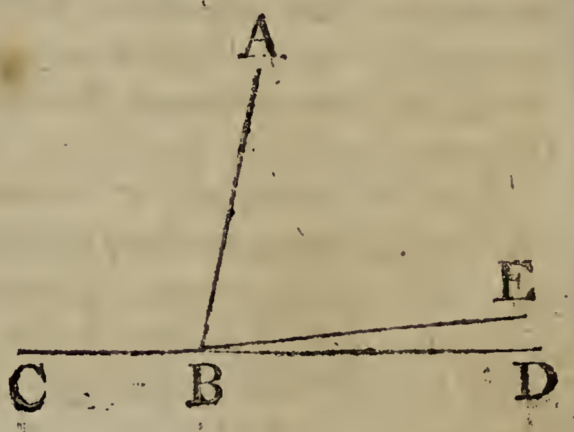


a Def. 10. right [a] angle; but, if not, from the point B draw BE at right
b 11. 1. angles [b] to CD; therefore the angles CBE, EBD are two right angles [a]; and becauſe CBE is equal to the two angles CBA, ABE together, add the angle EBD to each of theſe equals; there-
c 2. Ax. fore the angles CBE, EBD are equal [c] to the three angles CBA, ABE, EBD. Again, becauſe the angle DBA is equal to the two angles DBE, EBA, add to theſe equals the angle ABC, therefore the angles DBA, ABC are equal to the three angles DBE, EBA, ABC; but the angles CBE, EBD have been demonſtrated to be equal to the ſame three angles; and things
d 1. Ax. that are equal to the ſame are equal [d] to one another; therefore the angles CBE, EBD are equal to the angles DBA, ABC; but CBE, EBD are two right angles; therefore DBA, ABC are together equal to two right angles. Wherefore, when a ſtraight line, &c. Q. E. D.

PROP. XIV. THEOR.

IF, at a point in a ſtraight line, two other ſtraight lines, upon the oppoſite ſides of it, make the adjacent angles together equal to two right angles, theſe two ſtraight lines ſhall be in one and the ſame ſtraight line.

At the point B in the ſtraight line AB, let the two ſtraight lines BC, BD upon the oppoſite ſides of AB, make the adjacent angles ABC, ABD equal together to two right angles. BD is in the ſame ſtraight line with CB.



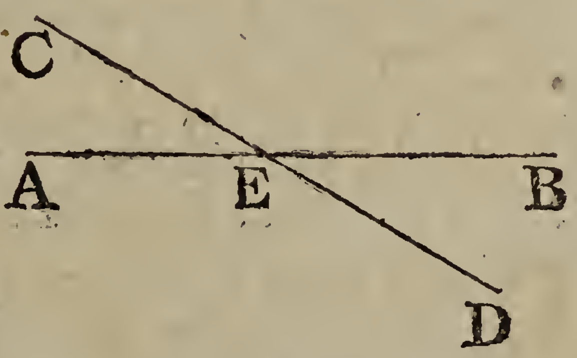
For, if BD be not in the ſame ſtraight line with CB, let BE be in the ſame ſtraight line with it; therefore, becauſe the ſtraight line AB makes angles with the ſtraight line CBE, upon one ſide of it, the angles ABC, ABE are together equal [a] to two right angles; but the angles ABC, ABD are likewiſe together equal to two right angles; therefore the angles CBA, ABE are equal to the angles CBA, ABD: Take away the common angle ABC, the remaining angle ABE is equal [b] to the remaining angle ABD, the leſs to the greater, which is impoſſible; therefore BE is not in the ſame ſtraight line with BC. And, in like manner, it may be demonſtrated, that no other can be in the ſame ſtraight line with it but BD, which therefore is in the ſame ſtraight line with CB. Wherefore, if at a point, &c. Q. E. D.

a 13. 1.
b 3. Ax.

PROP. XV. THEOR.

IF two ſtraight lines cut one another, the vertical, *or oppoſite*, angles ſhall be equal.

Let the two ſtraight lines AB, CD cut one another in the point E; the angle AEC ſhall be equal to the angle DEB, and CEB to AED.



Becauſe the ſtraight line AE makes with CD the angles CEA, AED, theſe angles are together equal [a] to two right angles. Again, becauſe the ſtraight line DE makes with AB the angles AED, DEB, theſe alſo are together equal [a] to two right angles; and CEA, AED have been demonſtrated to be equal to two right angles; wherefore the angles CEA, AED are equal to the angles AED, DEB. Take away the common angle AED, and the remaining angle CEA is equal [b] to the remaining angle DEB. In the ſame manner it can be demonſtrated that the angles CEB, AED are equal. Therefore, if two ſtraight lines, &c. Q. E. D.

a 13. 1.
b 3. Ax.

Cor. 1. From this it is manifeſt, that, if two ſtraight lines cut one another, the angles they make at the point where they cut, are together equal to four right angles.

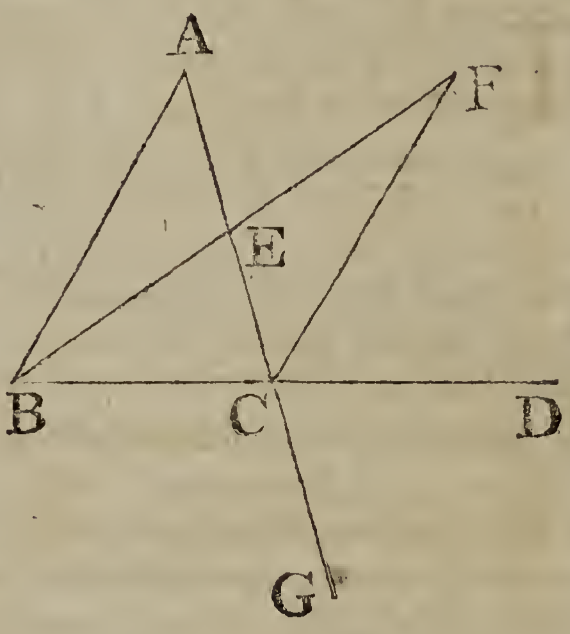
Cor. 2. And conſequently that all the angles made by any number of lines meeting in one point, are together equal to four right angles.

PROP.

PROP. XVI. THEOR.

IF one ſide of a triangle be produced, the exterior angle is greater than either of the interior oppoſite angles.

Let ABC be a triangle, and let its ſide BC be produced to D, the exterior angle ACD is greater than either of the interior oppoſite angles CBA, BAC.

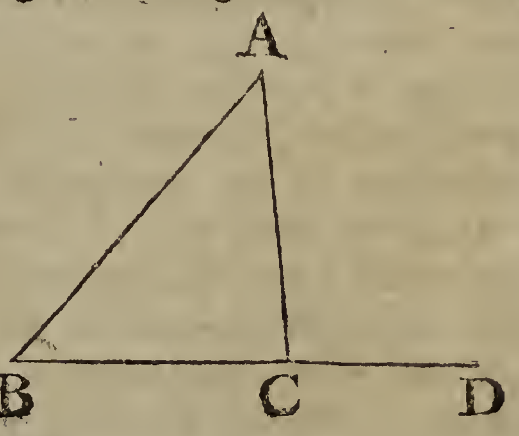


a 10. 1. Biſect [a] AC in E, join BE
and produce it to F, and
make EF equal to BE; join
alſo FC, and produce AC to
G.

Becauſe AE is equal to
EC, and BE to EF; AE,
EB are equal to CE, EF,
each to each; and the angle
b 15. 2. AEB is equal [b] to the angle
CEF, becauſe they are op-
poſite vertical angles; there-
c 4. 1. fore the baſe AB is equal [c]
to the baſe CF, and the tri-
angle AEB to the triangle CEF, and the remaining angles to
the remaining angles, each to each, to which the equal ſides
are oppoſite; wherefore the angle BAE is equal to the angle
ECF; but the angle ECD is greater than the angle ECF;
therefore the angle ACD is greater than BAE: In the ſame
manner, if the ſide BC be biſected, it may be demonſtrated that
d 15. 1 the angle BCG, that is [d], the angle ACD, is greater than the
angle ABC. Therefore, if one ſide, &c. Q. E. D.

PROP. XVII. THEOR.

ANY two angles of a triangle are together leſs than two right angles.



Let ABC be any triangle; any two of its angles together are leſs than two right angles.

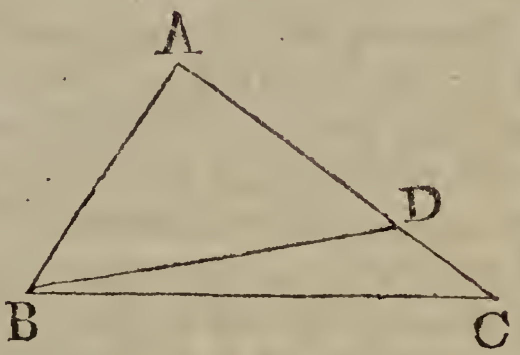
Produce BC to D; and be-
cauſe ACD is the exterior angle
of the triangle ABC, ACD is
a 16. 1. greater [a] than the interior and
oppoſite angle ABC; to each of

theſe add the angle ACB; therefore the angles ACD, ACB are greater than the angles ABC, ACB; but ACD, ACB are together equal [b] to two right angles; therefore the angles ABC, BCA are leſs than two right angles. In like manner, it may be demonſtrated, that BAC, ACB, as alſo CAB, ABC, are leſs than two right angles. Therefore any two angles, &c. Q. E. D. [b] 13. 1.

PROP. XVIII. THEOR.

THE greater ſide of every triangle is oppoſite to the greater angle.

Let ABC be a triangle, of which the ſide AC is greater than the ſide AB; the angle ABC is alſo greater than the angle BCA.



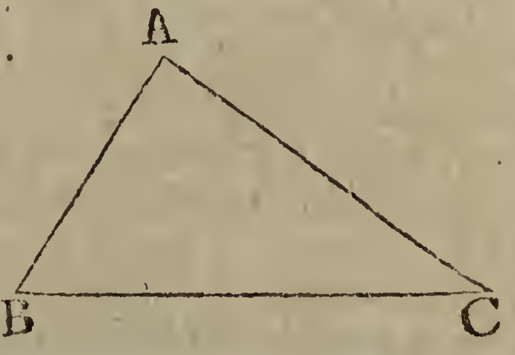
Becauſe AC is greater than AB, make [a] AD equal to AB, and join BD; and becauſe ADB is the exterior angle of the triangle BDC, it is greater [b] than the interior and oppoſite angle DCB; but ADB is equal [c] to ABD, becauſe the ſide AB is equal to the ſide AD; therefore the angle ABD is likewiſe greater than the angle ACB; wherefore much more is the angle ABC greater than ACB. Therefore the greater ſide, &c. Q. E. D.

[a] 3. 1.
[b] 16. 1.
[c] 5. 1.

PROP. XIX. THEOR.

THE greater angle of every triangle is ſubtended by the greater ſide, or *has the greater ſide oppoſite to it.*

Let ABC be a triangle, of which the angle ABC is greater than the angle BCA; the ſide AC is likewiſe greater than the ſide AB.



For, if it be not greater, AC muſt either be equal to AB, or leſs than it; it is not equal, becauſe then the angle ABC would be equal [a] to the angle ACB; but it is not; therefore AC is not equal to AB; neither is it leſs; becauſe then the angle

[a] 5. 1.

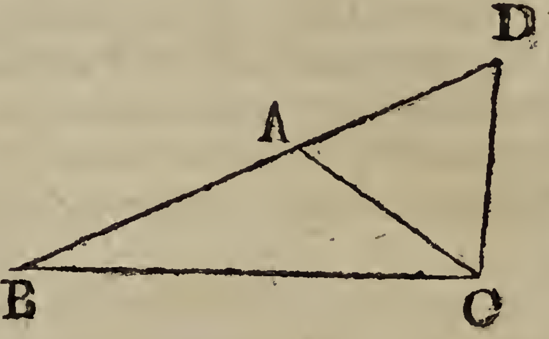
ABC would be leſs [b] than the angle ACB; but it is not; therefore the ſide AC is not leſs than AB; and it has been ſhewn
b 18. 1. that it is not equal to AB; therefore AC is greater than AB.
Wherefore the greater angle, &c. Q. E. D.

PROP. XX. THEOR.

See N. ANY two ſides of a triangle are together greater than the third ſide.

Let ABC be a triangle; any two ſides of it together are greater than the third ſide, viz. the ſides BA, AC greater than the ſide BC; and AB, BC greater than AC; and BC, CA greater than AB.

Produce BA to the point D,
a 3. 1. and make [a] AD equal to AC;
and join DC.



Becauſe DA is equal to AC, the angle ADC is likewiſe equal
b 5. 1. [b] to ACD; but the angle BCD is greater than the angle ACD; therefore the angle BCD is greater than the angle ADC; and becauſe the angle BCD of the triangle DCB is greater than its
c 19. 1. angle BDC, and that the greater [c] ſide is oppoſite to the greater angle; therefore the ſide DB is greater than the ſide BC; but DB is equal to BA and AC; therefore the ſides BA, AC are greater than BC. In the ſame manner it may be demonſtrated, that the ſides AB, BC are greater than CA, and BC, CA greater than AB. Therefore any two ſides, &c. Q. E. D.

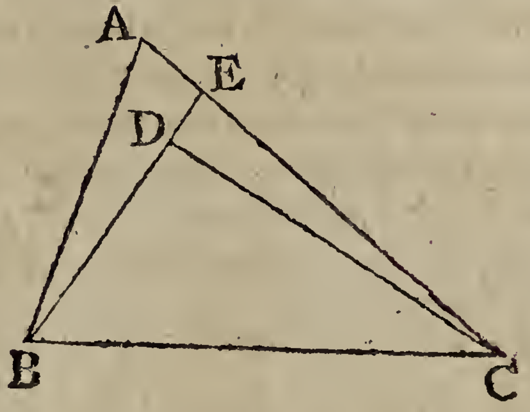
PROP. XXI. THEOR.

See N. IF, from the ends of the ſide of a triangle, there be drawn two ſtraight lines to a point within the triangle, theſe ſhall be leſs than the other two ſides of the triangle, but ſhall contain a greater angle.

Let the two ſtraight lines BD, CD be drawn from B, C, the ends of the ſide BC of the triangle ABC, to the point D within it; BD and DC are leſs than the other two ſides BA, AC of the triangle, but contain an angle BDC greater than the angle BAC.

Produce BD to E; and becauſe two ſides of a triangle are greater than the third ſide, the two ſides BA, AE of the triangle

angle ABE are greater than BE. To each of theſe add EC; therefore the ſides BA, AC are greater than BE, EC: Again, becauſe the two ſides CE, ED of the triangle CED are greater than CD, add DB to each of theſe; therefore the ſides CE, EB are greater than CD, DB; but it has been ſhewn that BA, AC are greater than BE, EC; much more then are BA, AC greater than BD, DC.



Again, becauſe the exterior angle of a triangle is greater than the interior and oppoſite angle, the exterior angle BDC of the triangle CDE is greater than CED; for the ſame reaſon, the exterior angle CEB of the triangle ABE is greater than BAC; and it has been demonſtrated that the angle BDC is greater than the angle CEB; much more then is the angle BDC greater than the angle BAC. Therefore, if from the ends of, &c. Q. E. D.

PROP. XXII. PROB.

See N.

TO make a triangle of which the ſides ſhall be equal to three given ſtraight lines, but any two whatever of theſe muſt be greater than the third [a].

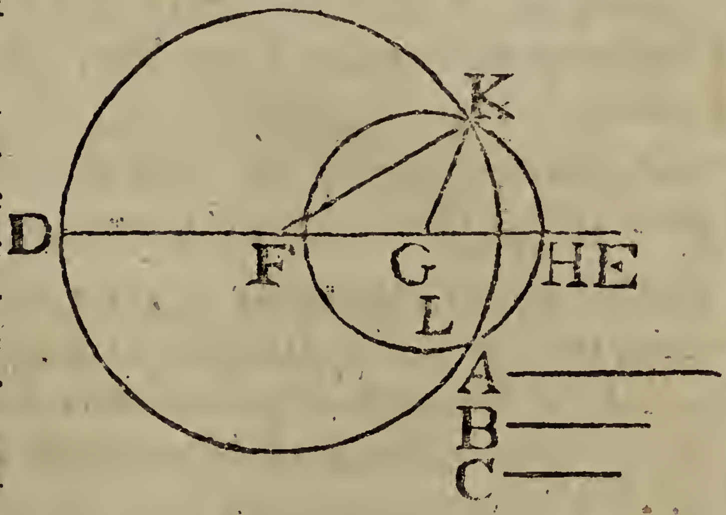
a 20. 1.

Let A, B, C be the three given ſtraight lines, of which any two whatever are greater than the third, viz. A and B greater than C; A and C greater than B; and B and C than A. It is required to make a triangle of which the ſides ſhall be equal to A, B, C, each to each.

Take a ſtraight line DE terminated at the point D, but unlimited towards E, and make [a] DF equal to A, FG to B, and GH equal to C; and from the centre F, at the diſtance FD, deſcribe [b] the circle DKL; and from the centre G, at the diſtance GH, deſcribe [b] another circle HLK; and join KF, KG; the triangle KFG has its ſides equal to the three ſtraight lines, A, B, C.

a 3. 1.

b. 3. Poſt.



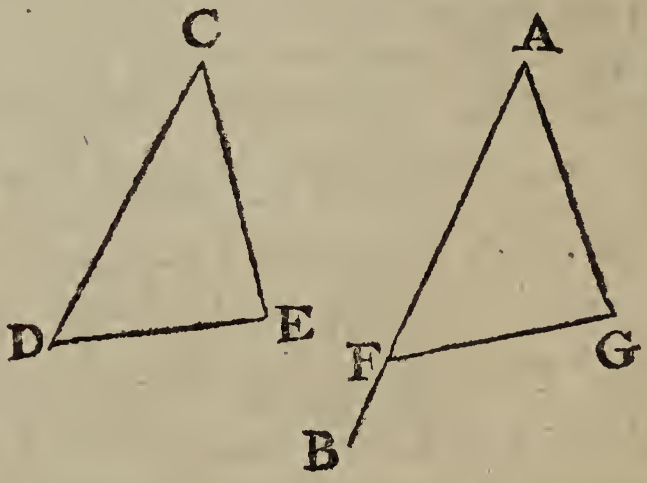
Becauſe the point F is the centre of the circle DKL, FD is equal

c. 15. Def. equal [c] to FK; but FD is equal to the ſtraight line A; therefore FK is equal to A: Again, becauſe G is the centre of the circle LKH, GH is equal [c] to GK; but GH is equal to C; therefore alſo GK is equal to C; and FG is equal to B; therefore the three ſtraight lines KF, FG, GK, are equal to the three A, B, C: And therefore the triangle KFG has its three ſides KF, FG, GK equal to the three given ſtraight lines, A, B, C. Which was to be done.

PROP. XXIII. PROB.

AT a given point in a given ſtraight line, to make a rectilineal angle equal to a given rectilineal angle.

Let AB be the given ſtraight line, and A the given point in it, and DCE the given rectilineal angle; it is required to make an angle at the given point A in the given ſtraight line AB, that ſhall be equal to the given rectilineal angle DCE.



Take in CD, CE any points D, E, and join DE; and make [a] the triangle AFG the ſides of which ſhall be equal to the three ſtraight lines CD, DE, CE, ſo that CD be equal to AF, CE to AG, and DE to FG; and becauſe DC, CE are equal to FA, AG, each to each, and the baſe DE to the baſe FG; the angle DCE is equal [b] to the angle FAG. Therefore, at the given point A in the given ſtraight line AB, the angle FAG is made equal to the given rectilineal angle DCE. Which was to be done.

a 22. 1.

b 8. 1.

PROP. XXIV. THEOR.

See N. IF two triangles have two ſides of the one equal to two ſides of the other, each to each, but the angle contained by the two ſides of one of them greater than the angle contained by the two ſides equal to them, of the other; the baſe of that which has the greater angle ſhall be greater than the baſe of the other.

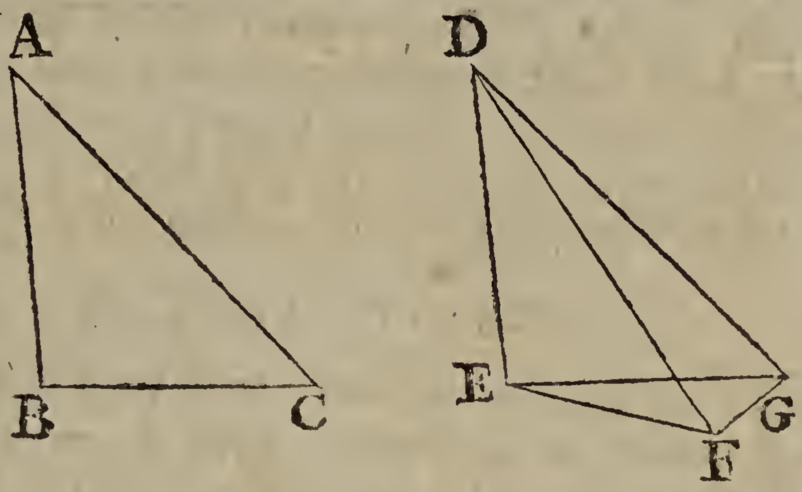
Let ABC, DEF be two triangles which have the two ſides AB, AC equal to the two DE, DF, each to each, viz. AB equal to DE, and AC to DF; but the angle BAC greater than the angle EDF; the baſe BC is alſo greater than the baſe EF.

Of the two ſides DE, DF, let DE be the ſide which is not greater than the other, and at the point D, in the ſtraight line DE, make [a] the angle EDG equal to the angle BAC; and make DG equal [b] to AC or DF, and join EG, GF.

[a] 23. 1.
[b] 3. 1.

Becauſe AB is equal to DE, and AC to DG, the two ſides BA, AC are equal to the two ED, DG, each to each, and the angle BAC is equal to the angle EDG; therefore the baſe BC is equal [c] to the baſe EG; and becauſe DG is equal to DF, the angle DFG is equal [d] to the angle DGF; but the angle DGF is greater than the angle EGF; therefore the angle DFG is greater than EGF; and much more is the angle EFG greater than the angle EGF; and becauſe the angle EFG of the triangle EFG is greater than its angle EGF, and that the greater [e] ſide is oppoſite to the greater angle; the ſide EG is therefore greater than the ſide EF; but EG is equal to BC; and therefore alſo BC is greater than EF. Therefore, if two triangles, &c. Q. E. D.

[c] 4. 1.
[d] 5. 1.
[e] 19. 1.



PROP. XXV. THEOR.

IF two triangles have two ſides of the one equal to two ſides of the other, each to each, but the baſe of the one greater than the baſe of the other; the angle alſo contained by the ſides of that which has the greater baſe, ſhall be greater than the angle contained by the ſides equal to them, of the other.

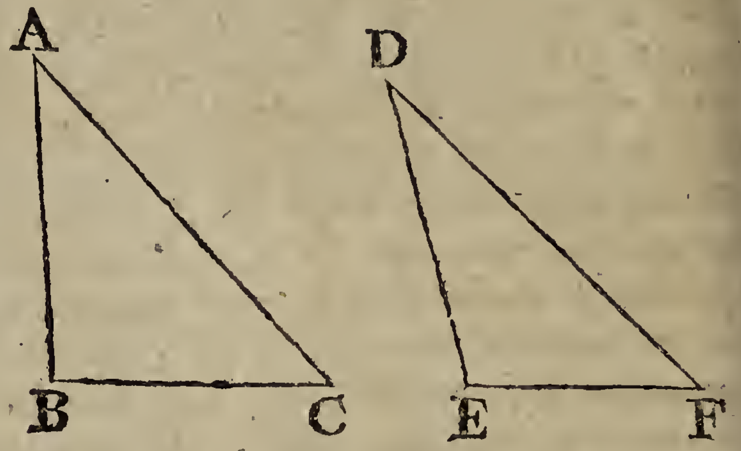
Let ABC, DEF be two triangles which have the two ſides AB, AC equal to the two ſides DE, DF, each to each, viz. AB equal to DE, and AC to DF; but the baſe CB is greater than the baſe EF; the angle BAC is likewiſe greater than the angle EDF.

For,

Book I. For, if it be not greater, it muſt either be equal to it, or leſs; but the angle BAC is not equal to the angle EDF, becauſe then the baſe BC would be equal [a] to EF; but it is not; therefore the angle BAC is not equal to the angle EDF; neither is it leſs; becauſe then the baſe BC would be leſs [b] than the baſe EF; but it is not; therefore the angle BAC is not leſs than the angle EDF; and it was ſhewn that it is not equal to it; therefore the angle BAC is greater than the angle EDF. Wherefore, if two triangles, &c. Q. E. D.

a 4. 1.

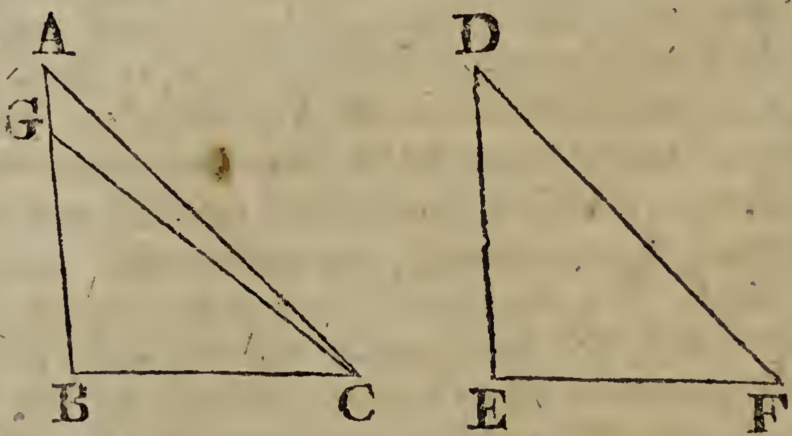
b 24. 1.



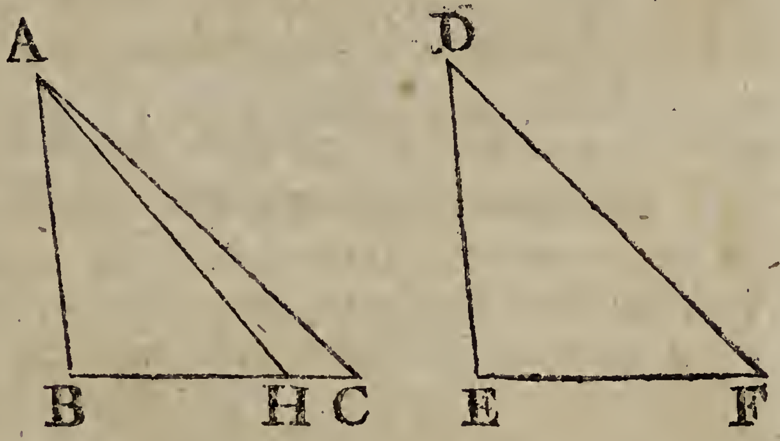
PROP. XXVI. THEOR.

IF two triangles have two angles of one equal to two angles of the other, each to each; and one ſide equal to one ſide, viz. either the ſides adjacent to the equal angles, or the ſides oppoſite to equal angles in each; then ſhall the other ſides be equal, each to each; and alſo the third angle of the one to the third angle of the other.

Let ABC, DEF be two triangles which have the angles ABC, BCA equal to the angles DEF, EFD, viz. ABC to DEF, and BCA to EFD; alſo one ſide equal to one ſide; and firſt let thoſe ſides be equal which are adjacent to the angles that are equal in the two triangles; viz. BC to EF; the other ſides ſhall be equal, each to each, viz. AB to DE, and AC to DF; and the third angle BAC to the third angle EDF.



For, if AB be not equal to DE, one of them muſt be the greater. Let AB be the greater of the two, and make BG equal to DE, and join GC; therefore, becauſe BG is equal to DE, and BC to EF, the two ſides GB, BC are equal to the two DE, EF, each to each; and the angle GBC is equal to the angle DEF; therefore the baſe GC is equal [a] to the baſe DF, and the triangle GBC to the triangle DEF, and the other angles to the other angles, each to each, to which the equal ſides are oppoſite; therefore the angle GCB is equal to the angle DFE; but DFE is, by the hypotheſis, equal to the angle BCA; wherefore alſo the angle BCG is equal to the angle BCA, the leſs to the greater, which is impoſſible; therefore AB is not unequal to DE, that is, it is equal to it; and BC is equal to EF; therefore the two AB, BC are equal to the two DE, EF, each to each; and the angle ABC is equal to the angle DEF; the baſe therefore AC is equal [a] to the baſe DF, and the third angle BAC to the third angle EDF.



Next, let the ſides which are oppoſite to equal angles in each triangle be equal to one another, viz. AB to DE; likewiſe in this caſe, the other ſides ſhall be equal, AC to DF, and BC to EF; and alſo the third angle BAC to the third EDF.

For, if BC be not equal to EF, let BC be the greater of them, and make BH equal to EF, and join AH; and becauſe BH is equal to EF, and AB to DE; the two AB, BH are equal to the two DE, EF, each to each; and they contain equal angles; therefore the baſe AH is equal to the baſe DF, and the triangle ABH to the triangle DEF, and the other angles ſhall be equal, each to each, to which the equal ſides are oppoſite; therefore the angle BHA is equal to the angle EFD; but EFD is equal to the angle BCA; therefore alſo the angle BHA is equal to the angle BCA, that is, the exterior angle BHA of the triangle AHC is equal to its interior and oppoſite angle BCA; which is impoſſible [b]; wherefore BC is not unequal to EF, that is, it is equal to it; and AB is equal to DE; therefore the two AB, BC are equal to the two DE, EF, each to each; and they contain equal angles; wherefore the baſe AC is equal to the baſe DF, and the third angle BAC to the third angle EDF. Therefore, if two triangles, &c. Q. E. D.

[a] 4. 1.

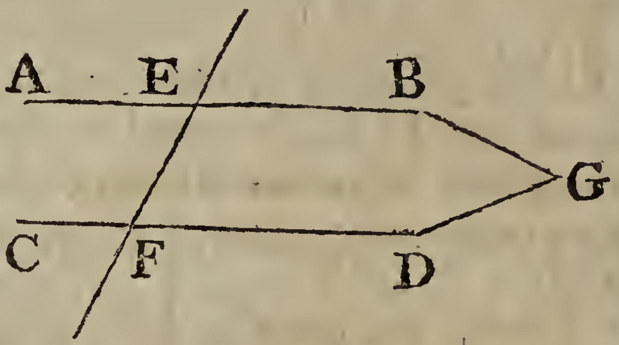
[b] 16. 1.

PROP. XXVII. THEOR.

IF a ſtraight line falling upon two other ſtraight lines makes the alternate angles equal to one another, theſe two ſtraight lines ſhall be parallel.

Let the ſtraight line EF, which falls upon the two ſtraight lines AB, CD make the alternate angles AEF, EFD equal to one another; AB is parallel to CD.

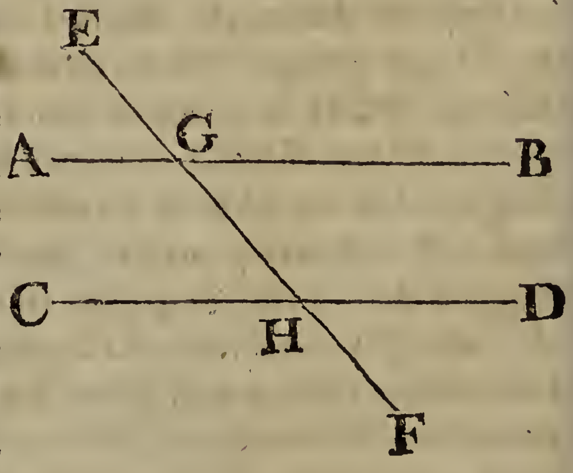
For, if it be not parallel, AB and CD being produced ſhall meet either towards B, D, or towards A, C; let them be produced and meet towards B, D in the point G; therefore GEF
a 16. 1. is a triangle, and its exterior angle AEF is greater [a] than the interior and oppoſite angle EFG; but it is alſo equal to it, which is impoſſible; therefore AB and CD being produced do not meet towards B, D. In like manner it may be demonſtrated that they do not meet towards A, C; but thoſe ſtraight lines which meet neither way, though produced ever ſo far, are pa-
b 35. Def. rallel [b] to one another. AB therefore is parallel to CD. Wherefore, if a ſtraight line, &c. Q. E. D.



PROP. XXVIII. THEOR.

IF a ſtraight line falling upon two other ſtraight lines makes the exterior angle equal to the interior and oppoſite upon the ſame ſide of the line; or makes the interior angles upon the ſame ſide together equal to two right angles; the two ſtraight lines ſhall be parallel to one another.

Let the ſtraight line EF, which falls upon the two ſtraight lines AB, CD, make the exterior angle EGB equal to the interior and oppoſite angle GHD upon the ſame ſide; or make the interior angles on the ſame ſide BGH, GHD together equal to two right angles; AB is parallel to CD.

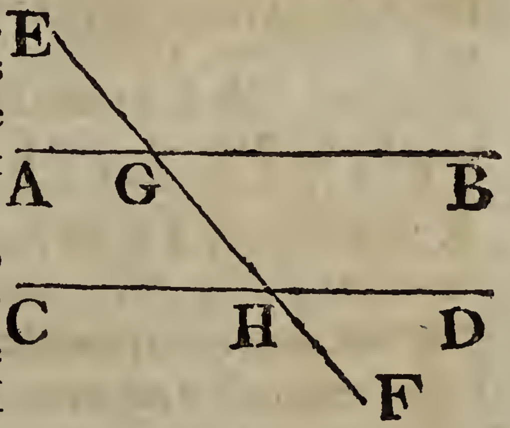


Becauſe the angle EGB is equal to the angle GHD, and the angle EGB equal [a] to the angle AGH, the angle AGH is equal to the angle GHD; and they are the alternate angles; therefore AB is parallel [b] to CD. a 15. 1. Again, becauſe the angles BGH, GHD b 27. 1. are equal [c] to two right angles; c By Hyp. and that AGH, BGH, are alſo equal [d] to two right angles; d 13. 1. the angles AGH, BGH are equal to the angles BGH, GHD: Take away the common angle BGH; therefore the remaining angle AGH is equal to the remaining angle GHD; and they are alternate angles; therefore AB is parallel to CD. Wherefore, if a ſtraight line, &c. Q. E. D.

PROP. XXIX. THEOR.

IF a ſtraight line fall upon two parallel ſtraight lines, it makes the alternate angles equal to one another; and the exterior angle equal to the interior and oppoſite upon the ſame ſide; and likewiſe the two interior angles upon the ſame ſide together equal to two right angles. See the notes on this propoſition.

Let the ſtraight line EF fall upon the parallel ſtraight lines AB, CD; the alternate angles AGH, GHD are equal to one another; and the exterior angle EGB is equal to the interior and oppoſite, upon the ſame ſide, GHD; and the two interior angles BGH, GHD upon the ſame ſide are together equal to two right angles.



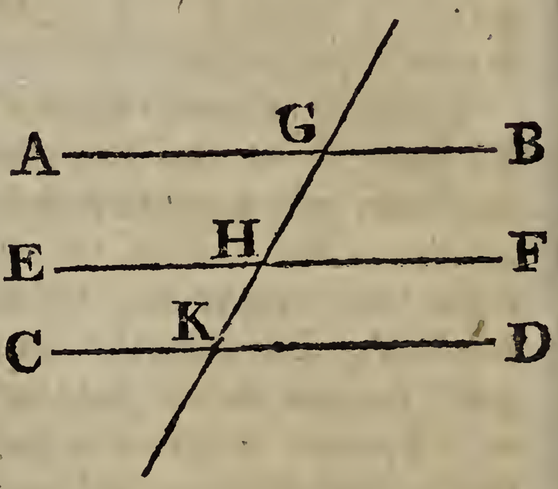
For, if AGH be not equal to GHD, one of them muſt be greater than the other; let AGH be the greater; and becauſe the angle AGH is greater than the angle GHD, add to each of them the angle BGH; therefore the angles AGH, BGH are greater than the angles BGH, GHD; but the angles AGH, BGH are equal [a] to two right angles; a 13. 1. therefore the angles BGH, GHD are leſs than two right angles; but thoſe ſtraight lines which, with another ſtraight line falling upon them, make the interior angles on the ſame ſide leſs than two right angles, do meet * together if continually produced; * 12. ax. See the notes on this propoſition. therefore the ſtraight lines AB, CD, if produced far enough, ſhall meet; but they never meet, ſince they are parallel by the hypotheſis; therefore the angle AGH is not unequal to the angle GHD, that is, it is equal to it; but the angle AGH is equal [b] to the angle EGB; b 15. 1. therefore likewiſe EGB is equal to GHD; add to each of theſe the angle EGH; therefore the angles EGB, BGH are equal to the angles BGH, GHD; but EGB, BGH are equal c 13. 1. c to two right angles; therefore alſo BGH, GHD are equal to two right angles. Wherefore, if a ſtraight, &c. Q. E. D.

PROP. XXX. THEOR.

STRAIGHT lines which are parallel to the ſame ſtraight line are parallel to one another.

Let AB, CD be each of them parallel to EF; AB is alſo parallel to CD.

Let the ſtraight line GHK cut AB, EF, CD; and becauſe GHK cuts the parallel ſtraight lines AB, EF, the angle AGH is a 29. 1. equal [a] to the angle GHF. Again, becauſe the ſtraight line GK cuts the parallel ſtraight lines EF, CD, the angle GHF, is equal [a] to the angle GKD; and it was ſhewn that the angle AGK is equal to the angle GHF; therefore alſo AGK is equal to GKD; and they are alternate angles; b 27. 1. therefore AB is parallel [b] to CD. Wherefore ſtraight lines, &c. Q. E. D.

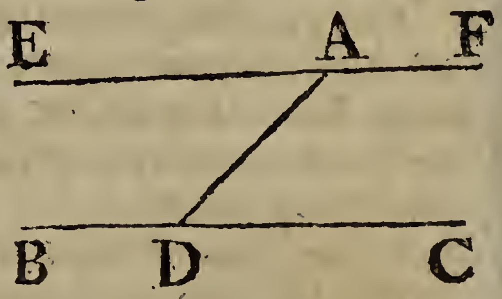


PROP. XXXI. PROB.

TO draw a ſtraight line through a given point parallel to a given ſtraight line.

Let A be the given point, and BC the given ſtraight line; it is required to draw a ſtraight line through the point A, parallel to the ſtraight line BC.

In BC take any point D, and join AD; and at the point A, in the a 23. 1. ſtraight line AD make [a] the angle DAE equal to the angle ADC; and produce the ſtraight line EA to F.



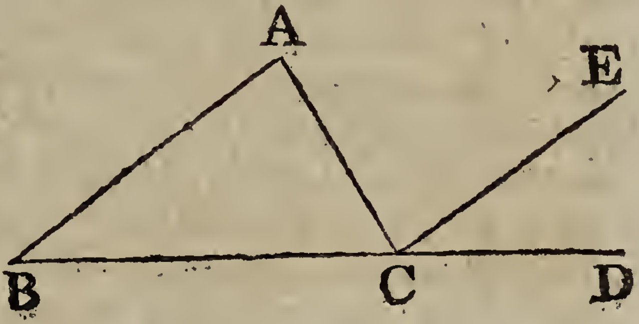
Becauſe the ſtraight line AD, which meets the two ſtraight lines BC, EF, makes the alternate angles EAD, ADC equal to b 27. 1. one another, EF is parallel [b] to BC. Therefore the ſtraight line

EAF is drawn through the given point A parallel to the given ſtraight line BC. Which was to be done.

PROP. XXXII. THEOR.

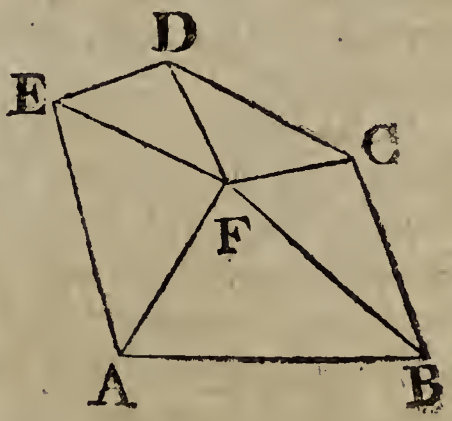
IF a ſide of any triangle be produced, the exterior angle is equal to the two interior and oppoſite angles; and the three interior angles of every triangle are equal to two right angles.

Let ABC be a triangle, and let one of its ſides BC be produced to D; the exterior angle ACD is equal to the two interior and oppoſite angles CAB, ABC and the three interior angles of the triangle, viz. ABC, BCA, CAB are together equal to two right angles.



Through the point C draw CE parallel [a] to the ſtraight line AB; and becauſe AB is parallel to CE and AC meets them, the alternate angles BAC, ACE are equal [b]. Again, becauſe AB is parallel to CE, and BD falls upon them, the exterior angle ECD is equal to the interior and oppoſite angle ABC; but the angle ACE was ſhewn to be equal to the angle BAC; therefore the whole exterior angle ACD is equal to the two interior and oppoſite angles CAB, ABC; to theſe equals add the angle ACB, and the angles ACD, ACB are equal to the three angles CBA, BAC, ACB; but the angles ACD, ACB are equal [c] to two right angles; therefore alſo the angles CBA, BAC, ACB are equal to two right angles. Wherefore if a ſide of a triangle, &c. Q. E. D.

a 31. 1.
b 29. 1.
c 13. 1.



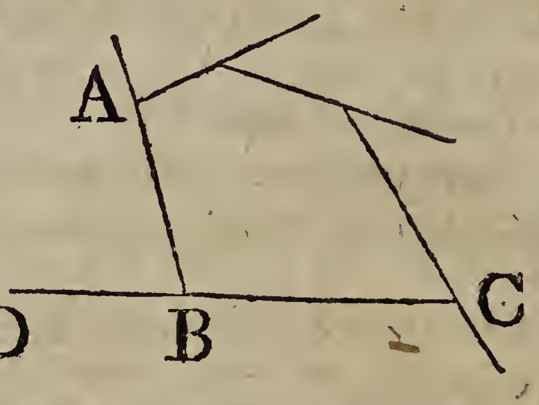
Cor. 1. All the interior angles of any rectilineal figure, together with four right angles, are equal to twice as many right angles as the figure has ſides.

For any rectilineal figure ABCDE can be divided into as many triangles as the figure has ſides, by drawing ſtraight lines from a point F within the figure to each of its angles. And, by the

Book I. the preceding propoſition, all the angles of theſe triangles are equal to twice as many right angles as there are triangles, that is, as there are ſides of the figure; and the ſame angles are equal to the angles of the figure, together with the angles at the point F, which is the common vertex of the triangles: that is [a], together with four right angles. Therefore all the angles of the figure, together with four right angles, are equal to twice as many right angles as the figure has ſides.

[a] 2. Cor. 15. 1.

Cor. 2. All the exterior angles of any rectilineal figure, are together equal to four right angles.

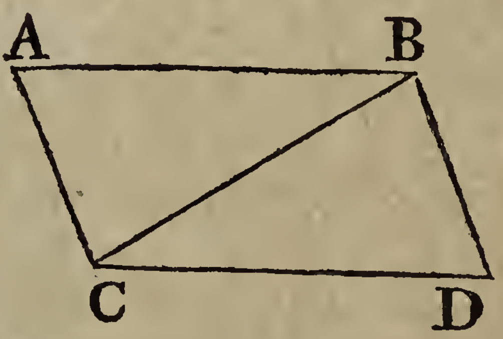


Becauſe every interior angle ABC, with its adjacent exterior ABD, is equal [b] to two right angles; therefore all the interior, together with all the exterior angles of the figure, are equal to twice as many right angles as there are ſides of the figure; that is, by the foregoing corollary, they are equal to all the interior angles of the figure, together with four right angles; therefore all the exterior angles are equal to four right angles.

[b] 13. 1.

PROP. XXXIII. THEOR.

THE ſtraight lines which join the extremities of two equal and parallel ſtraight lines, towards the ſame parts, are alſo themſelves equal and parallel.



Let AB, CD be equal and parallel ſtraight lines, and joined towards the ſame parts by the ſtraight lines AC, BD; AC, BD are alſo equal and parallel.

Join BC; and becauſe AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal [a]; and becauſe AB is equal to CD, and BC common to the two triangles ABC, DCB, the two ſides AB, BC are equal to the two DC, CB; and the angle ABC is equal to the angle BCD; therefore the baſe AC is equal [b] to the baſe BD, and the triangle ABC to the triangle BCD, and the other angles to the other angles [b], each to each, to which the equal ſides are oppoſite; therefore the angle

[a] 29. 1.

[b] 4. 1.

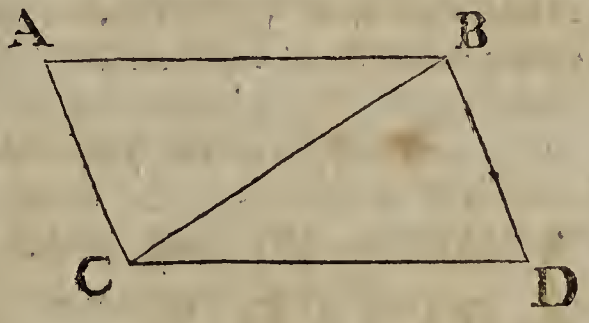
angle ACB is equal to the angle CBD; and becauſe the ſtraight line BC meets the two ſtraight lines AC, BD, and makes the alternate angles ACB, CBD equal to one another, AC is parallel [c] to BD; and it was ſhewn to be equal to it. Therefore, ſtraight lines, &c. Q. E. D. [c] 27. 1.

PROP. XXXIV. THEOR.

THE oppoſite ſides and angles of parallelograms are equal to one another, and the diameter biſects them, that is, divides them into two equal parts.

N. B. *A parallelogram is a four-ſided figure, of which the oppoſite ſides are parallel; and the diameter is the ſtraight line joining two of its oppoſite angles.*

Let ACDB be a parallelogram, of which BC is a diameter; the oppoſite ſides and angles of the figure are equal to one another; and the diameter BC biſects it.



Becauſe AB is parallel to CD, and BC meets them, the alternate angles ABC, BCD are equal [a] to one another; and becauſe AC is parallel to BD, and BC meets them, the alternate angles ACB, CBD are equal [a] to one another; wherefore the two triangles ABC, CBD have two angles ABC, BCA in one, equal to two angles BCD, CBD in the other, each to each, and one ſide BC common to the two triangles, which is adjacent to their equal angles; therefore their other ſides ſhall be equal, each to each, and the third angle of the one to the third angle of the other [b], viz. the ſide AB to the ſide CD, and AC to BD, and the angle BAC equal to the angle BDC: And becauſe the angle ABC is equal to the angle BCD, and the angle CBD to the angle ACB, the whole angle ABD is equal to the whole angle ACD: And the angle BAC has been ſhewn to be equal to the angle BDC; therefore the oppoſite ſides and angles of parallelograms are equal to one another; alſo, their diameter biſects them; for AB being equal to CD, and BC common, the two AB, BC are equal to the two DC, CB, each to each; and the angle ABC is equal

[a] 29. 1.

[b] 26. 1.

c 4. 1.

equal to the angle BCD; therefore the triangle ABC is equal [c] to the triangle BCD, and the diameter BC divides the parallelogram ACDB into two equal parts. Q. E. D.

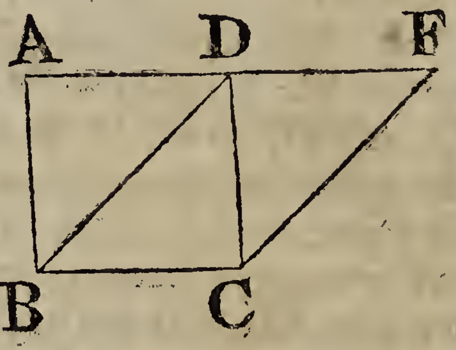
PROP. XXXV. THEOR.

See N.

PARALLELOGRAMS upon the ſame baſe and between the ſame parallels, are equal to one another.

See the 2d and 3d figures.

Let the parallelograms ABCD, EBCF be upon the ſame baſe BC, and between the ſame parallels AF, BC; the parallelogram ABCD ſhall be equal to the parallelogram EBCF.

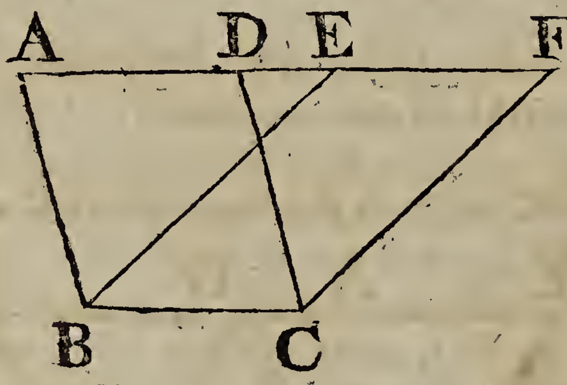


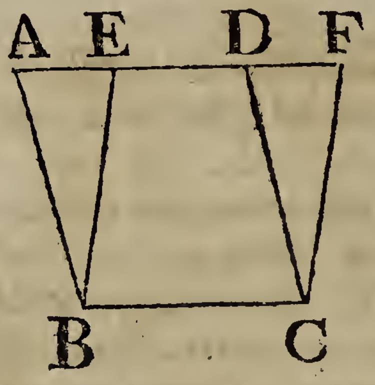
a 34. 1.

If the ſides AD, DF of the parallelograms ABCD, DBCF oppoſite to the baſe BC be terminated in the ſame point D; it is plain that each of the parallelograms is double [a] of the triangle BDC; and they are therefore equal to one another.

b 1. Ax.
c 2. or 3. Ax.

But, if the ſides AD, EF, oppoſite to the baſe BC of the parallelograms ABCD, EBCF, be not terminated in the ſame point; then, becauſe ABCD is a parallelogram, AD is equal [a] to BC; for the ſame reaſon EF is equal to BC; wherefore AD is equal [b] to EF; and DE is common; therefore the whole, or the remainder, AE is equal [c] to the whole, or the remainder DF; AB alſo is equal to DC; and the two EA, AB are therefore equal to





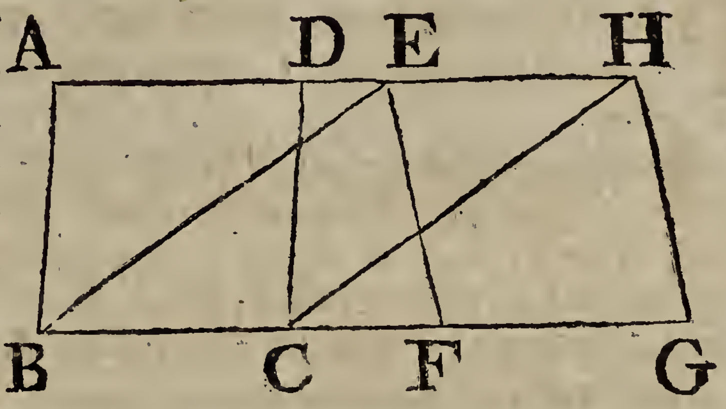
d 19. 1.
e 4. 1.
f 3. Ax.

the two FD, DC, each to each; and the exterior angle FDC is equal [d] to the interior EAB, therefore the baſe EB is equal to the baſe FC, and the triangle EAB equal [e] to the triangle FDC; take the triangle FDC from the trapezium ABCF, and from the ſame trapezium take the triangle EAB; the remainders therefore are equal [f], that is, the parallelogram ABCD is equal to the parallelogram EBCF. Therefore parallelograms upon the ſame baſe, &c. Q. E. D.

PROP.

PROP. XXXVI. THEOR.

PARALLELOGRAMS upon equal baſes, and between the ſame parallels, are equal to one another.



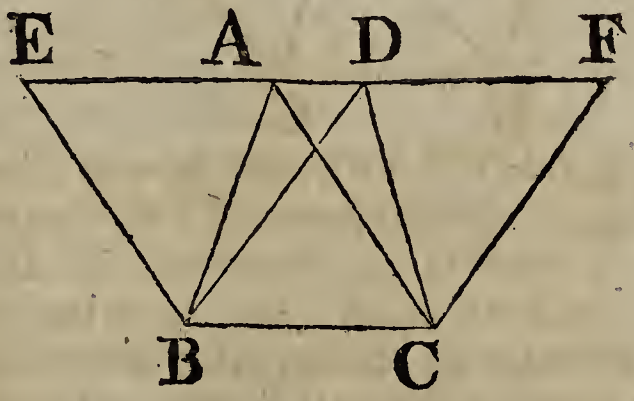
Let ABCD, EFGH be parallelograms upon equal baſes BC, FG, and between the ſame parallels AH, BG; the parallelogram ABCD is equal to EFGH.

Join BE, CH; and becauſe BC is equal to
FG, and FG to [a] EH, BC is equal to EH; and they are paral- a 34. 1.
lels, and joined towards the ſame parts by the ſtraight lines BE,
CH: But ſtraight lines which join equal and parallel ſtraight
lines towards the ſame parts, are themſelves equal and parallel [b]; b 33. 1.
therefore EB, CH are both equal and parallel, and EBCH is a
parallelogram; and it is equal [c] to ABCD, becauſe it is upon c 35. 1.
the ſame baſe BC, and between the ſame parallels BC, AD:
For the like reaſon, the parallelogram EFGH is equal to the
ſame EBCH: Therefore alſo the parallelogram ABCD is equal
to EFGH. Wherefore parallelograms, &c. Q. E. D.

PROP. XXXVII. THEOR.

TRIANGLES upon the ſame baſe, and between the ſame parallels, are equal to one another.

Let the triangles ABC, DBC be upon the ſame baſe BC and between the ſame parallels AD, BC: The triangle ABC is equal to the triangle DBC.



Produce AD both ways to the points E, F, and through
B draw [a] BE parallel to CA; a 31. 1.
and thro' C draw CF parallel to BD: Therefore each of the figures EBCA, DBCF
is a parallelogram; and EBCA is equal [b] to DBCF, becauſe b 35. 1.
they are upon the ſame baſe BC, and between the ſame parallels
BC, EF; and the triangle ABC is the half of the parallelo-
gram

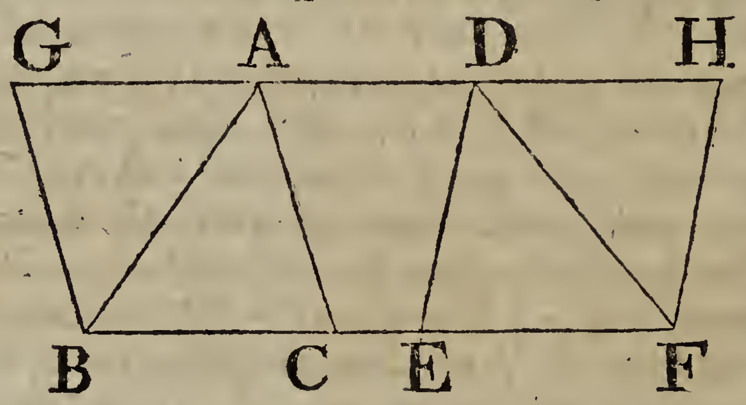
gram EBCA, becauſe the diameter AB beſects ᶜ it; and the triangle DBC is the half of the parallelogram DBCF, becauſe the
c 34. 1. diameter DC biſects it: But the halves of equal things are e-
d 7. Ax. qual ᵈ; therefore the triangle ABC is equal to the triangle DBC. Wherefore triangles, &c. Q. E. D.

PROP. XXXVIII. THEOR.

TRIANGLES upon equal baſes, and between the ſame parallels, are equal to one another.

Let the triangles ABC, DEF be upon equal baſes BC, EF, and between the ſame parallels BF, AD: The triangle ABC is equal to the triangle DEF.

Produce AD both ways to the points G, H, and through B
a 31. 1. draw BG parallel ᵃ to CA, and through F draw FH parallel to ED: Then each of the figures GBCA, DEFH is a parallelogram; and they
b 36. 1. are equal to ᵇ one another, becauſe they are upon equal baſes BC, EF, and between the ſame pa-
c 34. 1. rallels BF, GH; and the triangle ABC is the half ᶜ of the parallelogram GBCA, becauſe the diameter AB biſects it; and the triangle DEF is the half ᶜ of the parallelogram DEFH, becauſe the diameter DF biſects it: But the halves of equal things are
d 7. Ax. equal ᵈ; therefore the triangle ABC, is equal to the triangle DEF. Wherefore triangles, &c. Q. E. D.



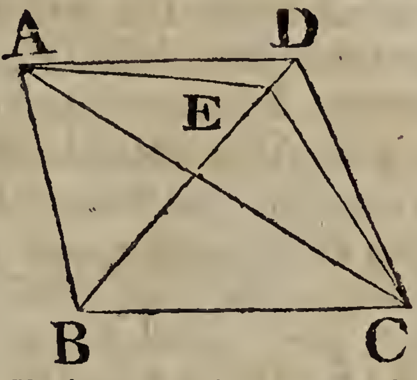
PROP. XXXIX. THEOR.

EQUAL triangles upon the ſame baſe, and upon the ſame ſide of it, are between the ſame parallels.

Let the equal triangles ABC, DBC be upon the ſame baſe BC, and upon the ſame ſide of it; they are between the ſame parallels.

Join AD; AD is parallel to BC; for, if it is not, through the
a 31. 1. point A draw ᵃ AE parallel to BC, and join EC: The triangle

ABC is equal [b] to the triangle EBC, becauſe it is upon the ſame baſe BC, and between the ſame parallels BC, AE: But the triangle ABC is equal to the triangle BDC; therefore alſo the triangle BDC is equal to the triangle EBC, the greater to the leſs, which is impoſſible: Therefore AE is not parallel to BC. In the ſame manner, it can be demonſtrated that no other line but AD is parallel to BC; AD is therefore parallel to it. Wherefore equal triangles upon, &c. Q. E. D. b 37. 1.

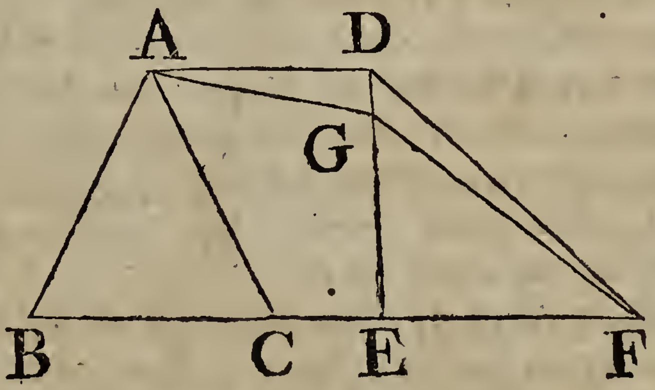


PROP. XL. THEOR.

EQUAL triangles upon equal baſes, in the ſame ſtraight line, and towards the ſame parts, are between the ſame parallels.

Let the equal triangles ABC, DEF be upon equal baſes BC, EF, in the ſame ſtraight line BF, and towards the ſame parts; they are between the ſame parallels.

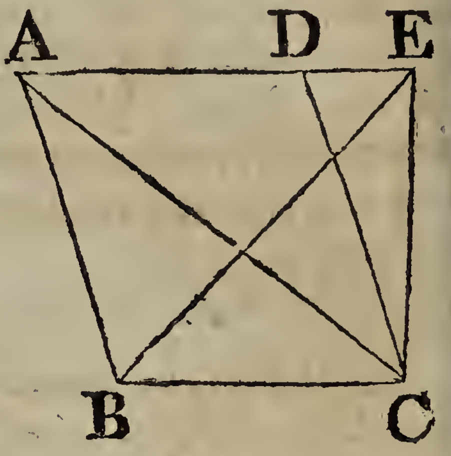
Join AD; AD is parallel to BC: For, if it is not, through A draw [a] AG parallel to BF, and join GF: a 31. 1. The triangle ABC is equal [b] b 38. 1. to the triangle GEF, becauſe they are upon equal baſes BC, EF, and between the ſame parallels BF, AG: But the triangle ABC is equal to the triangle DEF; therefore alſo the triangle DEF is equal to the triangle GEF, the greater to the leſs, which is impoſſible: Therefore AG is not parallel to BF: And in the ſame manner it can be demonſtrated that there is no other parallel to it but AD; AD is therefore parallel to BF. Wherefore equal triangles, &c. Q. E. D.



PROP. XLI. THEOR.

IF a parallelogram and triangle be upon the ſame baſe, and between the ſame parallels; the parallelogram ſhall be double of the triangle.

Let the parallelogram ABCD and the triangle EBC be upon the ſame baſe BC, and between the ſame parallels BC, AE; the parallelogram ABCD is double of the triangle EBC.

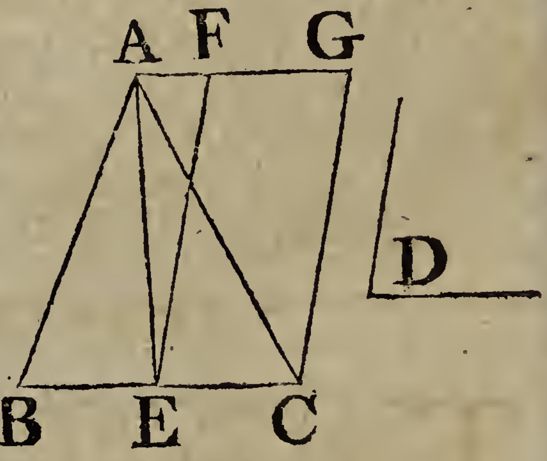


Join AC; then the triangle ABC
a 37. 1. is equal [a] to the triangle EBC, becauſe
they are upon the ſame baſe BC, and
between the ſame parallels BC, AE.
But the parallelogram ABCD is
b 34. 1. double [b] of the triangle ABC, becauſe
the diameter AC divides it into two
equal parts; wherefore ABCD is
alſo double of the triangle EBC.
Therefore, if a parallelogram, &c. Q. E. D.

PROP. XLII. PROB.

To deſcribe a parallelogram that ſhall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let ABC be the given triangle, and D the given rectilineal angle. It is required to deſcribe a parallelogram that ſhall be equal to the given triangle ABC, and have one of its angles equal to D.



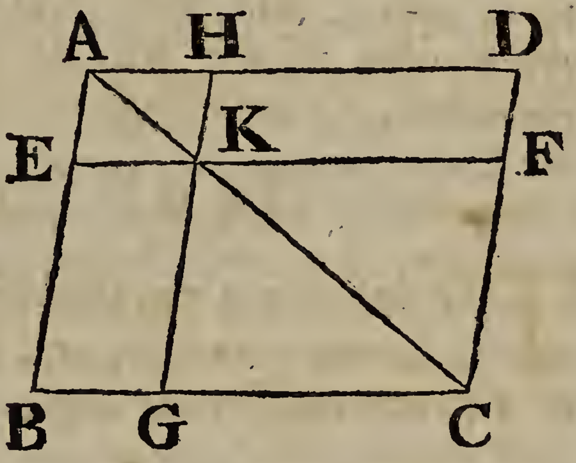
a 10. 1. Biſect [a] BC in E, join AE, and at the point E in the ſtraight
b 23. 1. line EC make [b] the angle CEF equal to D; and through A draw
c 31. 1. [c] AG parallel to EC, and thro'
C draw CG [c] parallel to EF:
Therefore FECG is a parallelogram: And becauſe BE is equal
to EC, the triangle ABE is like-
d 38. 1. wiſe equal [d] to the triangle AEC,
ſince they are upon equal baſes
BE, EC, and between the ſame
parallels BC, AG; therefore the
triangle ABC is double of the
triangle AEC. And the paral-
e 41. 1. lelogram FECG is likewiſe double [e] of the triangle AEC, becauſe it is upon the ſame baſe, and between the ſame parallels:
Therefore the parallelogram FECG is equal to the triangle
ABC, and it has one of its angles CEF equal to the given
angle D; wherefore there has been deſcribed a parallelogram

FECG equal to a given triangle ABC, having one of its angles CEF equal to the given angle D. Which was to be done.

PROP. XLIII. THEOR.

THE complements of the parallelograms which are about the diameter of any parallelogram, are equal to one another.

Let ABCD be a parallelogram, of which the diameter is AC, and EH, FG the parallelograms about AC, *that is, thro' which* AC *paſſes*, and BK, KD the other parallelograms which make up the whole figure ABCD, which are therefore called the complements: The complement BK is equal to the complement KD.



Becauſe ABCD is a parallelogram, and AC its diameter, the triangle ABC is equal [a] to the triangle ADC: And, becauſe EKHA is a parallelogram, the diameter of which is AK, the triangle AEK is equal to the triangle AHK: By the ſame reaſon, the triangle KGC is equal to the triangle KFC: Then, becauſe the triangle AEK is equal to the triangle AHK, and the triangle KGC to KFC; the triangle AEK, together with the triangle KGC is equal to the triangle AHK together with the triangle KFC: But the whole triangle ABC is equal to the whole ADC; therefore the remaining complement BK is equal to the remaining complement KD. Wherefore the complements, &c. Q. E. D.

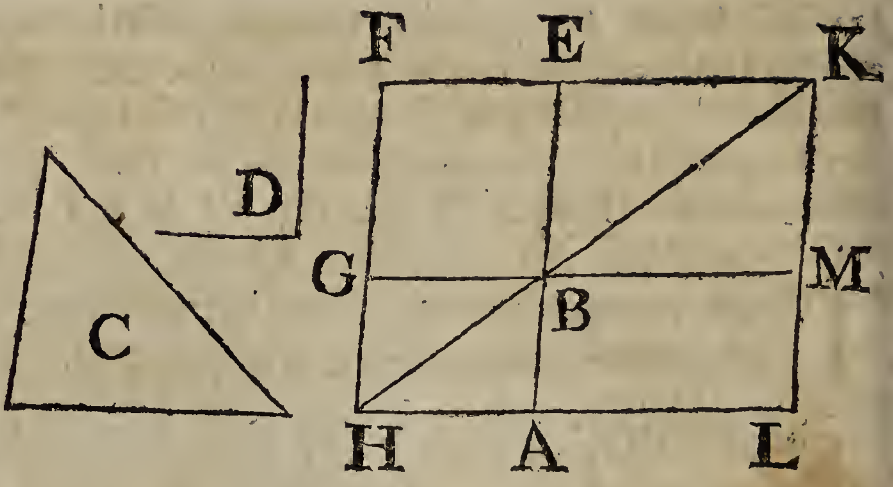
[a] 34. 1.

PROP. XLIV. PROB.

TO a given ſtraight line to apply a parallelogram, which ſhall be equal to a given triangle, and have one of its angles equal to a given rectilineal angle.

Let AB be the given ſtraight line, and C the given triangle, and D the given rectilineal angle. It is required to apply to the ſtraight line AB a parallelogram equal to the triangle C, and having an angle equal to D.

Make



a 42. I. Make [a] the parallelogram BEFG equal to the triangle C, and having the angle EBG equal to the angle D, ſo that BE be in the ſame ſtraight line
b 31. I. with AB, and produce FG to H; and thro' A draw [b] AH parallel to BG or EF, and join HB. Then becauſe the ſtraight line HF falls upon the parallels AH, EF, the angles AHF, HFE, are
c 29. I. together equal [c] to two right angles; wherefore the angles BHF, HFE are leſs than two right angles: But ſtraight lines which with another ſtraight line make the interior angles upon the
d 12. Ax. ſame ſide leſs than two right angles, do meet [d] if produced far enough: Therefore HB, FE ſhall meet, if produced; let them meet in K, and through K draw KL parallel to EA or FH, and produce HA, GB to the points L, M: Then HLKF is a parallelogram, of which the diameter is HK, and AG, ME are the parallelograms about HK; and LB, BF are the complements;
e 43. I. therefore LB is equal [e] to BF: But BF is equal to the triangle C; wherefore LB is equal to the triangle C; and becauſe the
f 15. I. angle GBE is equal [f] to the angle ABM, and likewiſe to the angle D; the angle ABM is equal to the angle D: Therefore the parallelogram LB is applied to the ſtraight line AB, is equal to the triangle C, and has the angle ABM equal to the angle D: Which was to be done.

PROP. XLV. PROB.

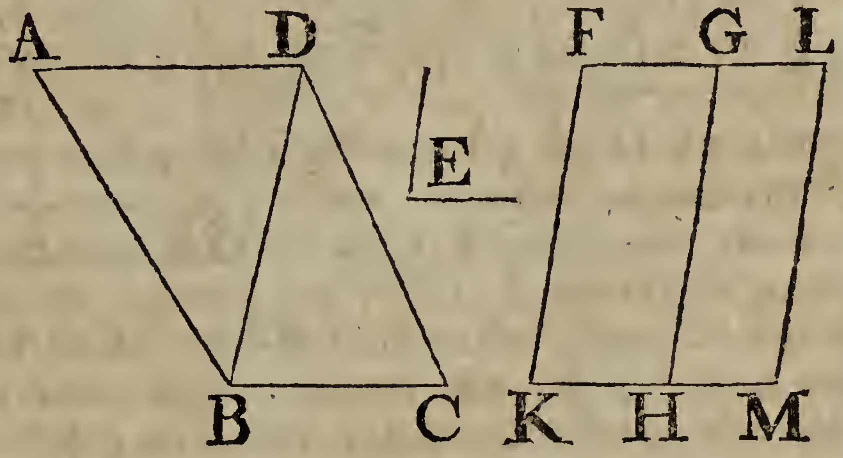
See N. TO deſcribe a parallelogram equal to a given rectilineal figure, and having an angle equal to a given rectilineal angle.

Let ABCD be the given rectilineal figure, and E the given rectilineal angle. It is required to deſcribe a parallelogram equal to ABCD, and having an angle equal to E.

a 42. I. Join DB, and deſcribe [a] the parallelogram FH equal to the triangle ADB, and having the angle HKF equal to the angle E;
b 44. I. and to the ſtraight line GH apply [b] the parallelogram GM equal

to the triangle DBC, having the angle GHM equal to the angle E: and becauſe the angle E is equal to each of the angles FKH, GHM, the angle FKH is equal to GHM; add to each of theſe the angle KHG; therefore the angles FKH, KHG are equal to the angles KHG, GHM; but FKH, KHG are equal[c] to two right angles: Therefore alſo KHG, GHM are equal to two right angles; and becauſe at the point H in the ſtraight line GH, the two ſtraight lines KH, HM, upon the oppoſite ſides of it make the adjacent angles equal to two right angles, KH is in the ſame ſtraight line[d] with HM; and becauſe the ſtraight line HG meets the parallels KM, FG, the alternate angles MHG, HGF are equal[c]: Add to each of theſe the angle HGL: Therefore the angles MHG, HGL, are equal to the angles HGF, HGL: But the angles MHG, HGL are equal[c] to two right angles; wherefore alſo the angles HGF, HGL are equal to two right angles, and FG is therefore in the ſame ſtraight line with GL: and becauſe KF is parallel to HG, and HG to ML; KF is parallel[e] to ML: and KM, FL are parallels; wherefore KFLM is a parallelogram; and becauſe the triangle ABD is equal to the parallelogram HF, and the triangle DBC to the parallelogram GM; the whole rectilineal figure ABCD is equal to the whole parallelogram KFLM; therefore the parallelogram KFLM has been deſcribed equal to the given rectilineal figure ABCD, having the angle FKM equal to the given angle E. Which was to be done.

c 29. 1.
d 14. 1.
e 30. 1.



Cor. From this it is manifeſt how to a given ſtraight line to apply a parallelogram, which ſhall have an angle equal to a given rectilineal angle, and ſhall be equal to a given rectilineal figure, viz. by applying[b] to the given ſtraight line a parallelogram equal to the firſt triangle ABD, and having an angle equal to the given angle.

b 44. 1.

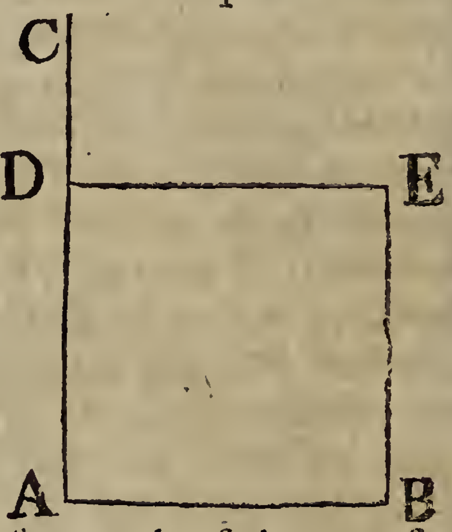
PROP.

PROP. XLVI. PROB.

TO deſcribe a ſquare upon a given ſtraight line.

Let AB be the given ſtraight line; it is required to deſcribe a ſquare upon AB.

a 11. 1. b 3. 1. c 31. 1. d 34. 1. From the point A draw [a] AC at right angles to AB; and make [b] AD equal to AB, and through the point D draw DE parallel [c] to AB, and through B draw BE parallel to AD; therefor ADEB is a parallelogram: whence AB is equal [d] to DE, and AD to BE: But BA is equal to AD; therefore the four ſtraight lines BA, AD, DE, EB are equal to one another, and the parallelogram ADEB is equilateral, likewiſe all its angles are right angles; becauſe the ſtraight line AD meeting the parallels AB, DE, the angles BAD, ADE are equal [e] to two right angles; but BAD is a right angle; therefore alſo ADE is a right angle; but the oppoſite angles of parallelograms are equal [d]; therefore each of the oppoſite angles ABE, BED is a right angle; wherefore the figure ADEB is rectangular, and it has been demonſtrated that it is equilateral; it is therefore a ſquare, and it is deſcribed upon the given ſtraight line AB: Which was to be done. e 29. 1.



Cor. Hence every parallelogram that has one right angle has all its angles right angles.

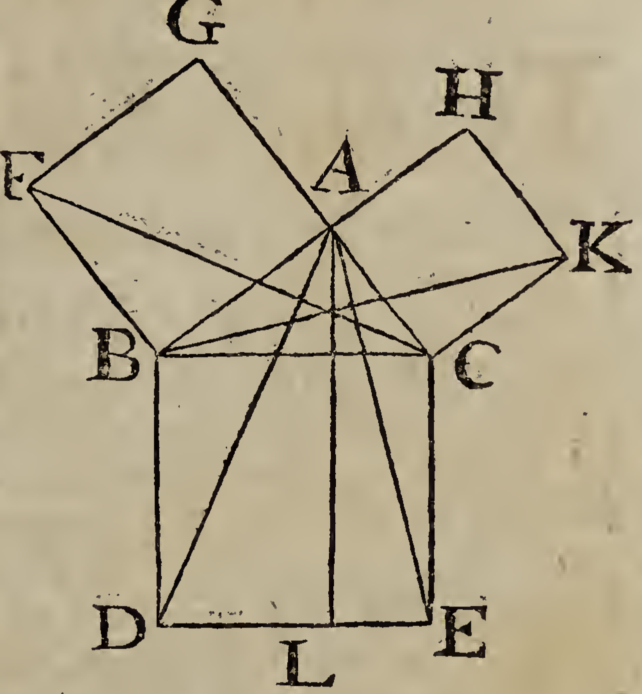
PROP. XLVII. THEOR.

IN any right angled triangle, the ſquare which is deſcribed upon the ſide ſubtending the right angle, is equal to the ſquares deſcribed upon the ſides which contain the right angle.

Let ABC be a right angled triangle having the right angle BAC; the ſquare deſcribed upon the ſide BC is equal to the ſquares deſcribed upon BA, AC.

a 46. 1. On BC deſcribe [a] the ſquare BDEC, and on BA, AC the

ſquares GB, HC ; and through A draw [b] AL parallel to BD, or
CE, and join AD ,FC ; then, becauſe each of the angles BAC,
BAG is aright angle [c], the b 31. 1. c 30. def.
two ſtraight lines AC, AG
upon the oppoſite ſides of AB,
make with it at the point A
the adjacent angles equal to
two right angles ; therefore
CA is in the ſame ſtraight
line [d] with AG ; for the ſame
reaſon, AB and AH are in
the ſame ſtraight line ; and d 14. 1.
becauſe the angle DBC is e-
qual to the angle FBA, each
of them being a right angle,
add to each the angle ABC,
and the whole angle DBA is
equal [e] to the whole FBC ; and becauſe the two ſides AB, BD e 2. Ax.
are equal to the two FB, BC, each to each, and the angle DBA
equal to the angle FBC ; therefore the baſe AD is equal [f] to f 4. 1.
the baſe FC, and the triangle ABD to the triangle FBC : Now
the parallelogram BL is double [g] of the triangle ABD, becauſe g 41. 1.
they are upon the ſame baſe BD, and between the ſame paral-
lels, BD, AL ; and the ſquare GB is double of the triangle
FBC, becauſe theſe alſo are upon the ſame baſe FB, and be-
tween the ſame parallels FB, GC. But the doubles of equals
are equal [h] to one another : Therefore the parallelogram BL h 6. A.
is equal to the ſquare GB : And, in the ſame manner, by join-
ing AE, BK, it is demonſtrated that the parallelogram CL is
equal to the ſquare HC : Therefore the whole ſquare BDEC is
equal to the two ſquares GB, HC ; and the ſquare BDEC is
deſcribed upon the ſtraight line BC, and the ſquares GB, HC
upon BA, AC : Wherefore the ſquare upon the ſide BC is e-
qual to the ſquares upon the ſides BA, AC. Therefore, in
any right angled triangle, &c. Q. E. D.

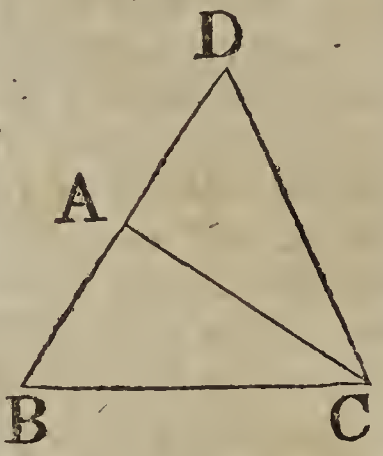


PROP. XLVIII. THEOR.

IF the ſquare deſcribed upon one of the ſides of a triangle, be equal to the ſquares deſcribed upon the other two ſides of it ; the angle contained by theſe two ſides is a right angle.

If the ſquare deſcribed upon BC, one of the ſides of the triangle ABC, be equal to the ſquares upon the other ſides BA, AC, the angle BAC is a right angle.

a 11. 1. From the point A draw [a] AD at right angles to AC, and make AD equal to BA, and join DC: Then, becauſe DA is equal to AB, the ſquare of DA is equal to the ſquare of AB: To each of theſe add the ſquare of AC; therefore the ſquares of DA, AC, are equal to the ſquares of
b 47. 1. BA, AC: But the ſquare of DC is equal [b] to the ſquares of DA, AC, becauſe DAC is a right angle; and the ſquare of BC, by hypotheſis, is equal to the ſquares of BA, AC; therefore the ſquare of DC is equal to the ſquare of BC; and therefore alſo the ſide DC is equal to the ſide BC. And becauſe the ſide DA is equal to AB, and AC common to the two triangles DAC, BAC, the two DA, AC are equal to the two BA, AC; and the baſe DC is equal to the baſe BC; there-
c 8. 1. fore the angle DAC is equal [c] to the angle BAC: But DAC is a right angle; therefore alſo BAC is a right angle. Therefore, if the ſquare, &c. Q. E. D.



THE
ELEMENTS
OF
EUCLID.

BOOK II.

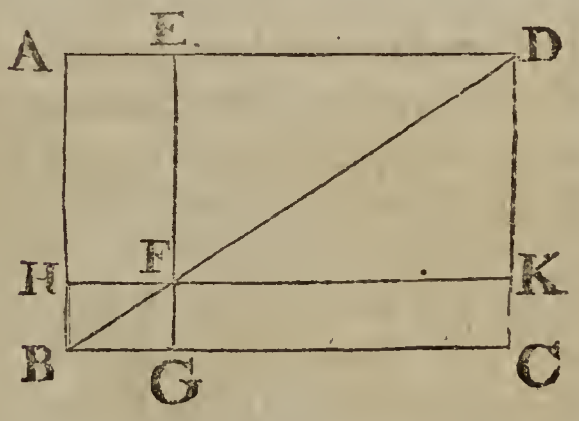
DEFINITIONS.

I.

EVERY right angled parallelogram is ſaid to be contained by any two of the ſtraight lines which contain one of the right angles.

II.

In every parallelogram, any of the parallelograms about a diameter, together with the two complements, is called a Gnomon. ‘ Thus the ‘ parallelogram HG, together ‘ with the complements AF, FC, is the ‘ gnomon, which is more ‘ briefly expreſſed by the ‘ letters AGK, or EHC, ‘ which are at the oppoſite ‘ angles of the parallelograms which make the gnomon.’

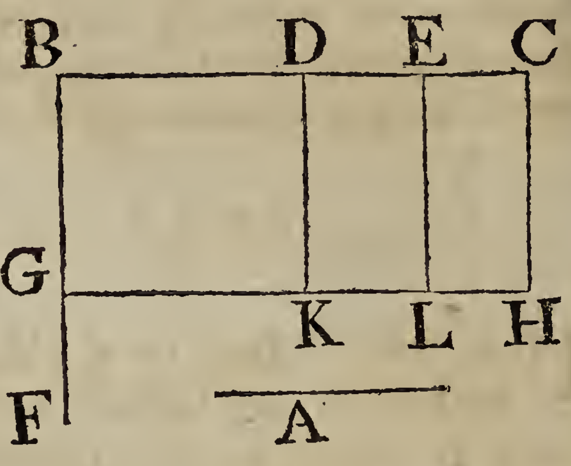


PROP. I. THEOR.

IF there be two ſtraight lines, one of which is divided into any number of parts; the rectangle contained by the two ſtraight lines, is equal to the rectangles contained by the undivided line, and the ſeveral parts of the divided line.

Book II.

Let A and BC be two ſtraight lines; and let BC be divided into any parts in the points D, E; the rectangle contained by the ſtraight lines A, BC is equal to the rectangle contained by A, BD, together with that contained by A, DE, and that contained by A, EC.



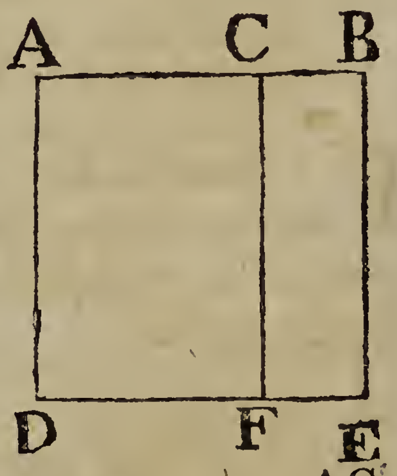
a 11. 1.
b 3. 1.
c 31. 1.
d 34. 1.

From the point B draw [a] BF at right angles to BC, and make BG equal [b] to A; and through G draw [c] GH parallel to BC; and through D, E, C, draw [c] DK, EL, CH parallel to BG; then the rectangle BH is equal to the rectangles BK, DL, EH; and BH is contained by A, BC, for it is contained by GB, BC, and GB is equal to A; and BK is contained by A, BD, for it is contained by GB, BD, of which GB is equal to A; and DL is contained by A, DE, becauſe DK, that is [d] BG, is equal to A; and in like manner the rectangle EH is contained by A, EC: Therefore the rectangle contained by A, BC is equal to the ſeveral rectangles contained by A, BD, and by A, DE; and alſo by A, EC. Wherefore, if there be two ſtraight lines, &c. Q. E. D.

PROP. II. THEOR.

IF a ſtraight line be divided into any two parts, the rectangles contained by the whole and each of the parts, are together equal to the ſquare of the whole line.

Let the ſtraight line AB be divided into any two parts in the point C; the rectangle contained by AB, BC, together with the rectangle * AB, AC, ſhall be equal to the ſquare of AB.



a 46. 1.
b 31. 1.

Upon AB deſcribe [a] the ſquare ADEB, and through C draw [b] CF, parallel to AD or BE; then AE is equal to the rectangles AF, CE; and AE is the ſquare of AB; and AF is the rectangle contained by BA, AC;

---

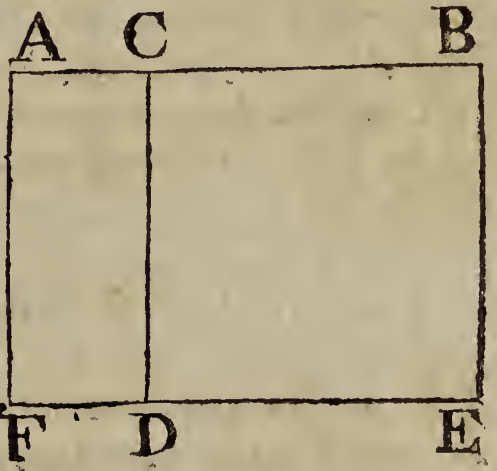
* N. B. To avoid repeating the word *contained* too frequently, the rectangle contained by two ſtraight lines AB, AC is ſometimes ſimply called the rectangle AB, AC.

AC; for it is contained by DA, AC, of which AD is equal to AB; and CE is contained by AB, BC, for BE is equal to AB; therefore the rectangle contained by AB, AC together with the rectangle AB, BC, is equal to the ſquare of AB. If therefore a ſtraight line, &c. Q. E. D.

PROP. III. THEOR.

IF a ſtraight line be divided into any two parts, the rectangle contained by the whole and one of the parts, is equal to the rectangle contained by the two parts, together with the ſquare of the foreſaid part.

Let the ſtraight line AB be divided into two parts in the point C; the rectangle AB, BC is equal to the rectangle AC, CB, together with the ſquare of BC.



Upon BC deſcribe [a] the ſquare CDEB, and produce ED to F, and through A draw [b] AF parallel to CD or BE; then the rectangle AE is equal to the rectangles AD, CE; and AE is the rectangle contained by AB, BC, for it is contained by AB, BE, of which BE is equal to BC; and AD is contained by AC, CB, for CD is equal to BC; and DB is the ſquare of BC; therefore the rectangle AB, BC is equal to the rectangle AC, CB together with the ſquare of BC. If therefore a ſtraight, &c. Q. E. D.

[a] 46. 1.

[b] 31. 1.

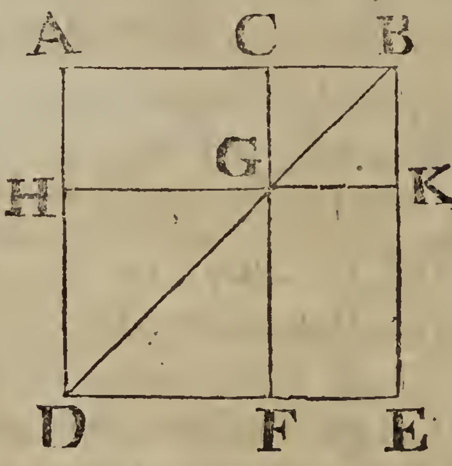
PROP. IV. THEOR.

IF a ſtraight line be divided into any two parts, the ſquare of the whole line is equal to the ſquares of the two parts, together with twice the rectangle contained by the parts.

Let the ſtraight line AB be divided into any two parts in C; the ſquare of AB is equal to the ſquares of AC, CB and to twice the rectangle contained by AC, CB.

a 46. 1.
b 31. 1.
c 29. 1.
d 5. 1.
e 6. 1.
f 34. 1.
g 43. 1.

Upon AB deſcribe [a] the ſquare ADEB, and join BD, and through C draw [b] CGF parallel to AD or BE, and through G draw HK parallel to AB or DE: And becauſe CF is parallel to AD, and BD falls upon them, the exterior angle BGC is equal [c] to the interior and oppoſite angle ADB; but ADB is equal [d] to the angle ABD, becauſe BA is equal to AD, being ſides of a ſquare; wherefore the angle CGB is equal to the angle GBC; and therefore the ſide BC is equal [e] to the ſide CG: But CB is equal [f] alſo to GK, and CG to BK; wherefore the figure CGKB is equilateral: It is likewiſe rectangular; for CG is parallel to BK, and CB meets them; the angles KBC, GCB are therefore equal to two right angles; and KBC is a right angle; wherefore GCB is a right angle; and therefore alſo the angles [f] CGK, GKB oppoſite to theſe, are right angles, and CGKB is rectangular: But it is alſo equilateral, as was demonſtrated; wherefore it is a ſquare, and it is upon the ſide CB: For the ſame reaſon HF alſo is a ſquare, and it is upon the ſide HG, which is equal to AC: Therefore HF, CK are the ſquares of AC, CB; and becauſe the complement AG is equal [g] to the complement GE, and that AG is the rectangle contained by AC, CB, for GC is equal to CB; therefore GE is alſo equal to the rectangle AC, CB; wherefore AG, GE are equal to twice the rectangle AC, CB: And HF, CK are the ſquares of AC, CB; wherefore the four figures HF, CK, AG, GE are equal to the ſquares of AC, CB, and to twice the rectangle AC, CB: But HF, CK, AG, GE make up the whole figure ADEB, which is the ſquare of AB: Therefore the ſquare of AB is equal to the ſquares of AC, CB and twice the rectangle AC, CB. Wherefore if a ſtraight line, &c. Q. E. D.



Cor. From the demonſtration, it is manifeſt that the parallelograms about the diameter of a ſquare are likewiſe ſquares.

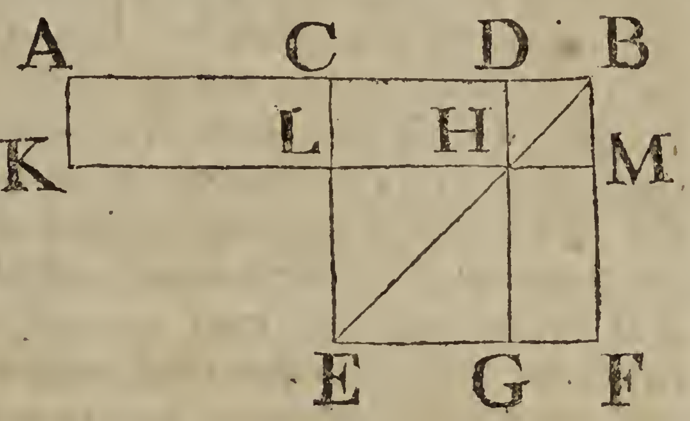
PROP.

PROP. V. THEOR.

IF a ſtraight line be divided into two equal parts, and alſo into two unequal parts; the rectangle contained by the unequal parts, together with the ſquare of the line between the points of ſection, is equal to the ſquare of half the line.

Let the ſtraight line AB be divided into two equal parts in the point C, and into two unequal parts at the point D; the rectangle AD, DB, together with the ſquare of CD, is equal to the ſquare of CB.

Upon CB deſcribe [a] the ſquare CEFB, join BE, and through a 46. 1. D draw [b] DHG parallel to CE or BF; and through H draw b 31. 1. KLM parallel to CB or EF; and alſo through A draw AK parallel to CL or BM: And becauſe the complement CH is equal [c] to the complement HF, to each of theſe add DM; c 43. 1. therefore the whole CM is equal to the whole DF; but CM is equal [d] to AL, d 36. 1. becauſe AC is equal to CB; therefore alſo AL is equal to DF. To each of theſe add CH, and the whole AH is equal to DF and CH: But AH is the rectangle contained by AD, DB, for DH, is equal [e] to DB; and DF together with CH is the gnomon CMG; e Cor. 4. 2. therefore the gnomon CMG is equal to the rectangle AD, DB: To each of theſe add LG, which is equal [e] to the ſquare of CD; therefore the gnomon CMG, together with LG, is equal to the rectangle AD, DB, together with the ſquare of CD: But the gnomon CMG and LG make up the whole figure CEFB, which is the ſquare of CB: Therefore the rectangle AD, DB, together with the ſquare of CD, is equal to the ſquare of CB. Wherefore, if a ſtraight line, &c. Q. E. D.



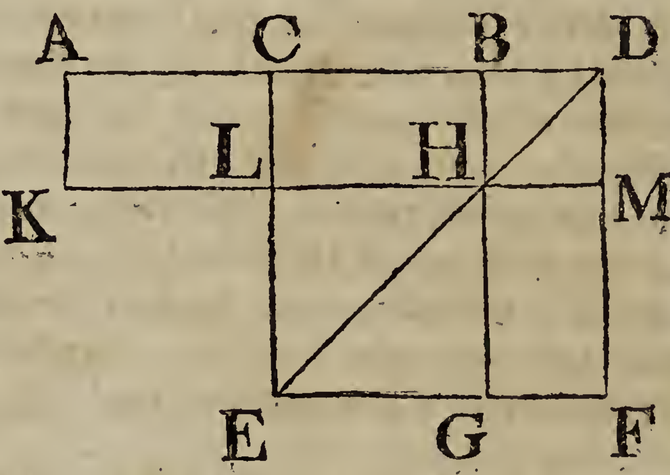
From this propoſition it is manifeſt, that the difference of the ſquares of two unequal lines AC, CD, is equal to the rectangle contained by their ſum and difference.

PROP. VI. THEOR.

IF a ſtraight line be biſected, and produced to any point; the rectangle contained by the whole line thus produced, and the part of it produced, together with the ſquare of half the line biſected, is equal to the ſquare of the ſtraight line which is made up of the half and the part produced.

Let the ſtraight line AB be biſected in C, and produced to the point D; the rectangle AD, DB, together with the ſquare of CB, is equal to the ſquare of CD.

a 46. 1. Upon CD deſcribe [a] the ſquare CEFD, join DE, and through
b 31. 1. B draw [b] BHG parallel to CE or DF, and through H draw KLM parallel to AD or EF, and alſo through A draw AK parallel to CL or DM: and becauſe AC is equal to CB,
c 36. 1. the rectangle AL is equal [c]
d 43. 1. to CH; but CH is equal [d] to HF; therefore alſo AL is equal to HF: To each of theſe add CM; therefore the whole AM is equal to the gnomon CMC: And AM is the rectangle con-



e Cor. 4. 2. tained by AD, DB, for DM is equal [e] to DB: Therefore the gnomon CMG is equal to the the rectangle AD, DB: Add to each of theſe LG, which is equal to the ſquare of CB, therefore the rectangle AD, DB, together with the ſquare of CB is equal to the gnomon CMG and the figure LG: But the gnomon CMG and LG make up the whole figure CEFD, which is the ſquare of CD; therefore the rectangle AD, DB together with the ſquare of CB, is equal to the ſquare of CD. Wherefore, if a ſtraight line, &c. Q. E. D.

PROP. VII. PROB.

IF a ſtraight line be divided into any two parts, the ſquares of the whole line, and of one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the ſquare of the other part.

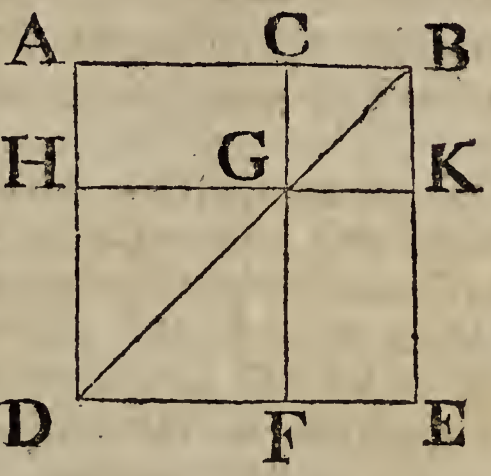
Let the ſtraight line AB be divided into any two parts in the point C; the ſquares of AB, BC are equal to twice the rectangle AB, BC together with the ſquare of AC.

Upon AB deſcribe [a] the ſquare ADEB, and conſtruct the figure as in the preceding propoſitions: and becauſe AG is equal [b] to GE, add to each of them CK; the whole AK is therefore equal to the whole CE; therefore AK, CE, are double of AK: But AK, CE are the gnomon AKF together with the ſquare CK; therefore the gnomon AKF, together with the ſquare CK, is double of AK: But twice the rectangle AB BC is double of AK, for BK is equal [c] to BC: Therefore the gnomon AKF, together with the ſquare CK, is equal to twice the rectangle AB, BC: To each of theſe equals add HF, which is equal to the ſquare of AC; therefore the gnomon AKF, together with the ſquares CK, HF, is equal to twice the rectangle AB, BC and the ſquare of AC: But the gnomon AKF, together with the ſquares CK, HF, make up the whole figure ADEB and CK, which are the ſquares of AB and BC: therefore the ſquares of AB and BC are equal to twice the rectangle AB, BC, together with the ſquare of AC. Wherefore if a ſtraight line &c. Q. E. D.

[a] 46. 1.

[b] 43. 1.

[c] Cor. 4. 2.



PROP. VIII. THEOR.

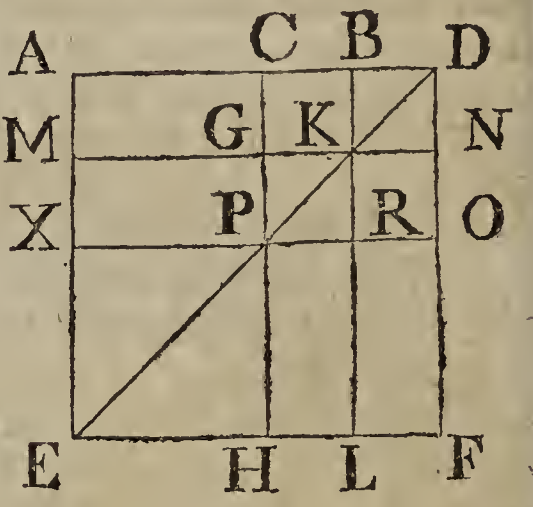
IF a ſtraight line be divided into any two parts, four times the rectangle contained by the whole line, and one of the parts, together with the ſquare of the other part, is equal to the ſquare of the ſtraight line which is made up of the whole and that part.

Let the ſtraight line AB be divided into any two parts in the point C; four times the rectangle AB, BC, together with the ſquare of AC, is equal to the ſquare of the ſtraight line made up of AB and BC together.

Produce AB to D, ſo that BD be equal to CB, and upon AD deſcribe the ſquare AEFD; and conſtruct two figures ſuch as in the preceding. Becauſe CB is equal to BD, and that CB is equal [a] to GK, and BD to KN; therefore GK is equal

[a] 34. 1.

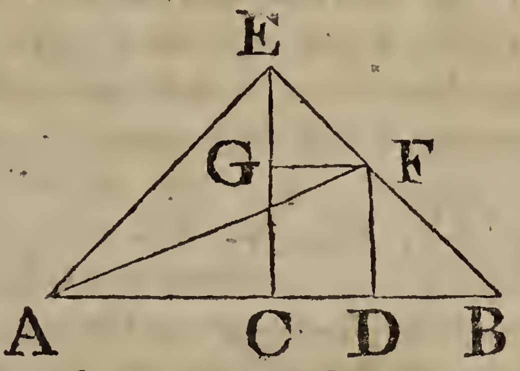
equal to KN: For the ſame reaſon, PR is equal to RO; and becauſe CB is equal to BD, and GK to KN, the rectangle b 36. 1. c 43. 1. CK is equal[b] to BN, and GR to RN: But CK is equal[c] to RN, becauſe they are the complements of the parallelogram CO; therefore alſo BN is equal to GR; and the four rectangles BN, CK, GR, RN are therefore equal to one another, and ſo are quadruple of one of them CK: Again, becauſe CB is equal to BD, and that BD is d Cor. 4. 2. equal[d] to BK, that is, to CG; and CB equal to GK, that[d] is, to GP; therefore CG is equal to GP: And becauſe CG is equal to GP, and PR to RO, the rectangle AG is equal to MP, and PL to e 43. 1. RF: But MP is equal[e] to PL, becauſe they are the complements of the parallelogram ML; wherefore AG is equal alſo to RF: Therefore the four rectangles AG, MP, PL, RF are equal to one another and ſo are quadruple of one of them AG. And it was demonſtrated, that the four CK, BN, GR, and RN are quadruple of CK: Therefore the eight rectangles which contain the gnomon AOH, are quadruple of AK: and becauſe AK is the rectangle contained by AB, BC, for BK is equal to BC, four times the rectangle AB, BC is quadruple of AK: But the gnomon AOH was demonſtrated to be quadruple of AK; therefore four times the rectangle AB, BC, is equal to the gnomon AOH. To Cor. 4. 2. each of theſe add XH, which is equal[f] to the ſquare of AC: Therefore four times the rectangle AB, BC together with the ſquare of AC, is equal to the gnomon AOH and the ſquare XH: But the gnomon AOH and XH make up the figure AEFD which is the ſquare of AD: Therefore four times the rectangle AB, BC, together with the ſquare of AC, is equal to the ſquare of AD, that is, of AB and BC added together in one ſtraight line. Wherefore, if a ſtraight line, &c. Q. E. D.



PROP. IX. THEOR.

IF a ſtraight line be divided into two equal, and alſo into two unequal parts; the ſquares of the two unequal parts are together double of the ſquare of half the line, and of the ſquare of the line between the points of ſection.

Let the ſtraight line AB be divided at the point C into two equal, and at D into two unequal parts: The ſquares of AD, DB are together double of the ſquares of AC, CD.



From the point C draw [a] CE at right angles to AB, and a 11. 1.
make it equal to AC or CB, and join EA, EB; through D draw
[b] DF parallel to CE, and through F draw FG parallel to AB; b 31. 1.
and join AF: Then, becauſe AC is equal to CE, the angle
EAC is equal [c] to the angle AEC; and becauſe the angle c 5. 1.
ACE is a right angle, the two others AEC, EAC together
make one right angle [d]; and they are equal to one another; d 32. 1.
each of them therefore is half of a right angle. For the ſame reaſon each of the angles CEB, EBC is half a right angle; and therefore the whole AEB is a right angle: And becauſe the angle GEF is half a right angle, and EGF a right angle, for it is
equal [e] to the interior and oppoſite angle ECB, the re- e 29. 1.
maning angle EFG is half a right angle; therefore the
angle GEF is equal to the angle EFG, and the ſide EG
equal [f] to the ſide GF: Again, becauſe the angle at B is f 6. 1.
half a right angle and FDB a right angle, for it is equal [e] to the interior and oppoſite angle ECB, the remaining angle BFD is half a right angle; therefore the angle at B is equal to the angle BFD, and the ſide DF to [f] the ſide DB: And becauſe AC is equal to CE, the ſquare of AC is equal to the ſquare of CE; therefore the ſquares of AC, CE are double of
the ſquare of AC: But the ſquare of EA is equal [g] to the g 47. 1.
ſquares of AC, CE, becauſe ACE is a right angle; therefore the ſquare of EA is double of the ſquare of AC: Again, becauſe EG is equal to GF, the ſquare of EG is equal to the ſquare of GF; therefore the ſquares of FG, GF are double of

the

Book II. the ſquare of GF; but the ſquare of EF is equal to the ſquares of EG, GF; therefore the ſquare of EF is double of the ſquare GF; and GF is equal [h] to CD; therefore the ſquare of EF is double of the ſquare of CD: But the ſquare of AE is likewiſe double of the ſquare of AC; therefore the ſquares of AE, EF are double of the ſquares of AC, CD: And the ſquare of AF is equal [i] to the ſquares of AE, EF, becauſe AEF is a right angle; therefore the ſquare of AF is double of the ſquares of AC, CD: But the ſquares of AD, DF are equal to the ſquare of AF, becauſe the angle ADF is a right angle; therefore the ſquares of AD, DF are double of the ſquares of AC, CD: And DF is equal to DB; therefore the ſquares of AD, DB are double of the ſquares of AC, CD. If therefore a ſtraight line, &c. Q. E. D.

h 34. 1.
i 47. 1.

PROP. X. THEOR.

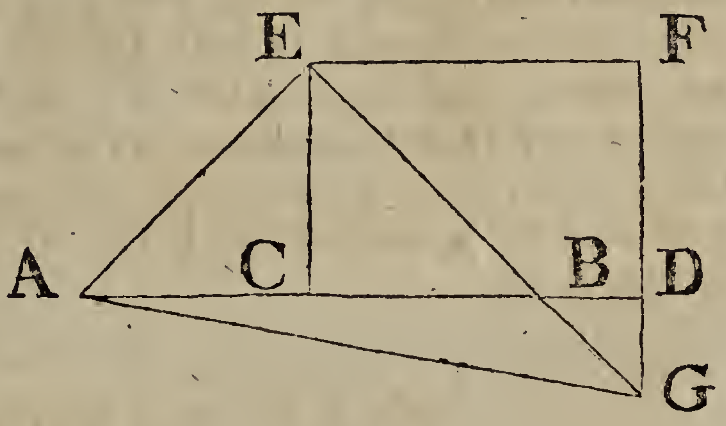
IF a ſtraight line be biſected, and produced to any point, the ſquare of the whole line thus porduced, and the ſquare of the part of it produced, are together double of the ſquare of half the line biſected, and of the ſquare of the line made up of the half and the part produced.

Let the ſtraight line AB be biſected in C, and produced to the point D; the ſquares of AD, DB are double of the ſquares of AC, CD.

From the point C draw [a] CE at right angles to AB: And make it equal to AC or CB, and join AE, EB; through E draw [b] EF parallel to AB, and through D draw DF parallel to CE: And becauſe the ſtraight line EF meets the parallels EC, FD, the angles CEF, EFD are equal [c] to two right angles; and therefore the angles BEF, EFD are leſs than two right angles: but ſtraight lines which with another ſtraight line make the interior angles upon the ſame ſide leſs than two right angles, do meet [d] if produced far enough: Therefore EB, FD ſhall meet, if produced towards B, D: Let them meet in G, and join AG: Then, becauſe AC is equal to CE, the angle CEA is equal [e] to the angle EAC; and the angle ACE is a right angle; therefore each of the angles CEA, EAC is half a right angle [f]: For the ſame reaſon, each

a 11. 1.
b 31. 1.
c 29. 1.
d 12. Ax.
e 5. 1.
f 32. 1.

each of the angles CEB, EBC is half a right angle; therefore
AEB is a right angle: And becauſe EBC is half a right angle,
DBG is alſo [f] half a right angle, for they are vertically oppo- f 15. 1.
ſite; but BDG is a right angle, becauſe it is equal [c] to the al- c 29. 1.
ternate angle DCE; therefore the remaining angle DGB is
half a right angle, and is therefore equal to the angle DBG;
wherefore alſo the ſide BD is equal [g] to the ſide DG: Again, g 6. 1.
becauſe EGF is half a
right angle, and that
the angle at F is a right
angle, becauſe it is e-
qual [h] to the oppoſite h 34. 1
angle ECD, the remain-
ing angle FEG is half a
right angle, and equal
to the angle EGF;
wherefore alſo the ſide
GF is equal [g] to the ſide FE. And becauſe EC is equal to
CA, the ſquare of EC is equal to the ſquare of CA; therefore
the ſquares of EC, CA are double of the ſquare of CA: But
the ſquare of EA is equal [i] to the ſquares EC, CA; there- i 47. 1.
fore the ſquare of EA is double of the ſquare of AC: Again,
becauſe GF is equal to FE, the ſquare of GF is equal to the
ſquare of FE; and therefore the ſquares of GF, FE are dou-
ble of the ſquare of EF: But the ſquare of EG is equal [i] to
the ſquares of GF, FE; therefore the ſquare of EG is double
of the ſquare of EF: And EF is equal to CD; wherefore the
ſquare of EG is double of the ſquare of CD: But it was demon-
ſtrated, that the ſquare of EA is double of the ſquare of AC;
therefore the ſquares of AE, EG are double of the ſquares of
AC, CD: And the ſquare of AG is equal to the ſquares of
AE, EG; therefore the ſquare of AG is double of the ſquares
of AC, CD: But the ſquares of AD, GD are equal [i] to the
ſquare of AG: therefore the ſquares of AD, DG are double
of the ſquares of AC, CD: But DG is equal to DB; therefore
the ſquares of AD, DB are double of the ſquares of AC, CD:
Wherefore, if a ſtraight line, &c. Q. E. D.



PROP.

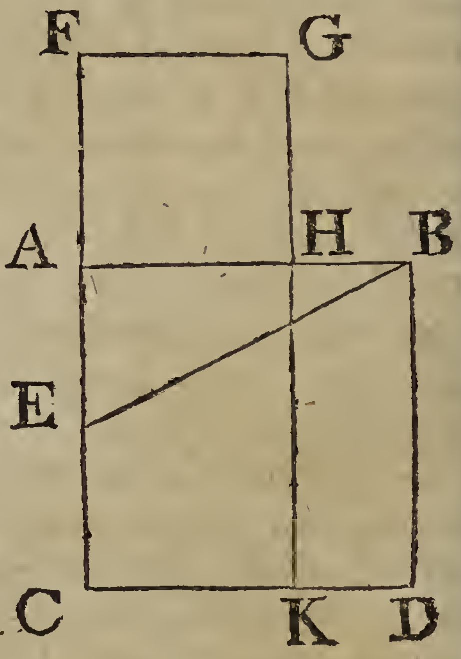
PROP. XI. PROB.

TO divide a given ſtraight line into two parts, ſo that the rectangle contained by the whole, and one of the parts, ſhall be equal to the ſquare of the other part.

Let AB be the given ſtraight line; it is required to divide it into two parts, ſo that the rectangle contained by the whole, and one of the parts, ſhall be equal to the ſquare of the other part.

a 46. 1. b 10. 1. c 3. 1. Upon AB deſcribe [a] the ſquare ABDC; biſect [b] AC in E, and join BE; produce CA to F, and make [c] EF equal to EB, and upon AF deſcribe [a] the ſquare FGHA; AB is divided in H, ſo that the rectangle AB, BH is equal to the ſquare of AH.

d 6. 2. e 47. 1. Produce GH to K: becauſe the ſtraight line AC is biſected in E, and produced to the point F, the rectangle CF, FA, together with the ſquare of AE, is equal [d] to the ſquare of EF: But EF is equal to EB; therefore the rectangle CF, FA, together with the ſquare of AE, is equal to the ſquare of EB: And the ſquares of BA, AE are equal [e] to the ſquare of EB, becauſe the angle EAB is a right angle; therefore the rectangle CF, FA together with the ſquare of AE, is equal to the ſquares of BA, AE: Take away the ſquare of AE, which is common to both, therefore the remaining rectangle CF, FA is equal to the ſquare of AB: and the figure FK is the rectangle contained by CF, FA, for AF is equal to FG; and AD is the ſquare of AB; therefore FK is equal to AD: Take away the common part AK, and the remainder FH is equal to the remainder HD: And HD is the rectangle contained by AB, BH, for AB is equal to BD; and FH is the ſquare of AH. Therefore the rectangle AB, BH is equal to the ſquare of AH: Wherefore the ſtraight line AB is divided in H ſo, that the rectangle AB, BH is equal to the ſquare of AH. Which was to be done.



PROP.

PROP. XII. THEOR.

IN obtuſe angled triangles, if a perpendicular be drawn from any of the acute angles to the oppoſite ſide produced, the ſquare of the ſide ſubtending the obtuſe angle is greater than the ſquares of the ſides containing the obtuſe angle, by twice the rectangle contained by the ſide upon which, when produced, the perpendicular falls, and the ſtraight line intercepted without the triangle between the perpendicular and the obtuſe angle.

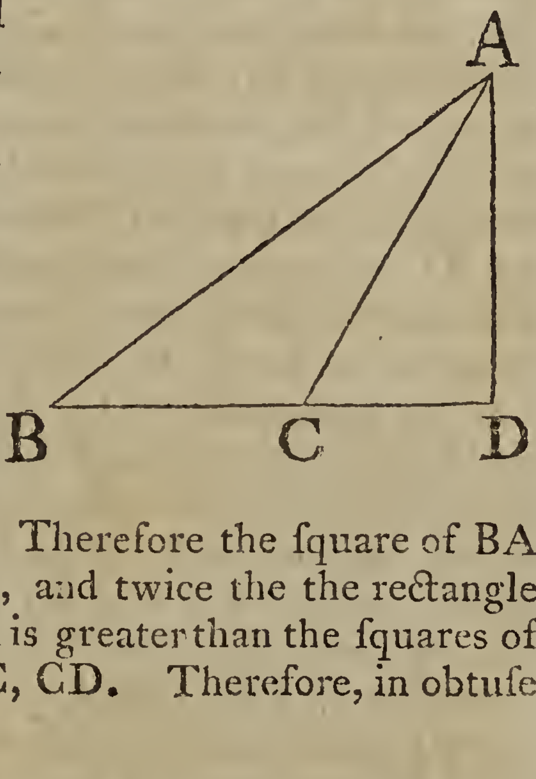
Let ABC be an obtuſe angled triangle, having the obtuſe angle ACB, and from the point A let AD be drawn [a] perpendicular to BC produced: The ſquare of AB is greater than the ſquares of AC, CB by twice the rectangle BC, CD.

a 12. 1.

Becauſe the ſtraight line BD is divided into two parts in the point C, the ſquare of BD is equal [b] to the ſquares of BC, CD, and twice the rectangle BC, CD: To each of theſe equals add the ſquare of DA; and the ſquares of DB, DA are equal to the ſquares of BC, CD, DA, and twice the rectangle BC, CD: But the ſquare of BA is equal [c] to the ſquares of BD, DA, becauſe the angle at D is a right angle; and the ſquare of CA is equal [c] to the ſquares of CD, DA: Therefore the ſquare of BA is equal to the ſquares of BC, CA, and twice the the rectangle BC, CD; that is, the ſquare of BA is greater than the ſquares of BC, CA, by twice the rectangle BC, CD. Therefore, in obtuſe angled triangles, &c. Q. E. D.

b 4. 2.

c 47. 1.

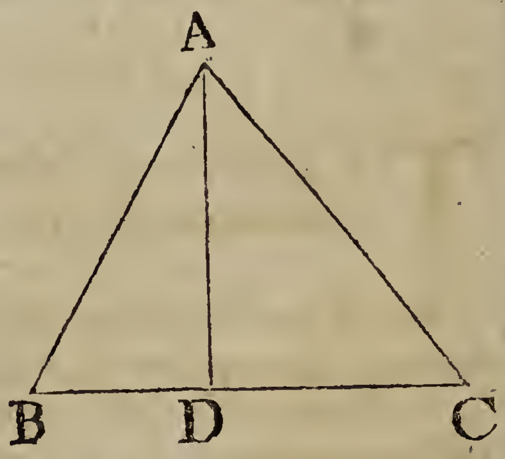


PROP. XIII. THEOR.

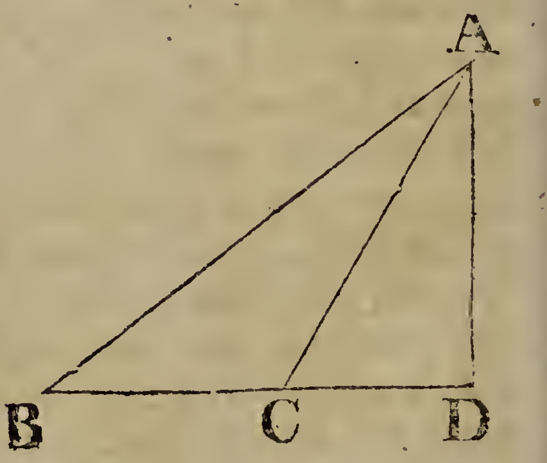
See N. IN every triangle, the ſquare of the ſide ſubtending any of the acute angles, is leſs than the ſquares of the ſides containing that angle, by twice the rectangle contained by either of theſe ſides, and the ſtraight line intercepted between the perpendicular let fall upon it from the oppoſite angle, and the acute angle.

Let ABC be any triangle, and the angle at B one of its acute angles, and upon BC, one of the ſides containing it, let
a 12. 1. fall the perpendicular [a] AD from the oppoſite angle: The ſquare of AC, oppoſite to the angle B, is leſs than the ſquares of CB, BA by twice the rectangle CB, BD.

First, Let AD fall within the triangle ABC; and becauſe the ſtraight line CB is divided into two parts in the point D, the ſquares of CB, BD are a-
b 7. 2. qual [b] to twice the rectangle contained by CB, BD, and the ſquare of DC: To each of theſe equals add the ſquare of AD; therefore the ſquares of CB, BD, DA, are equal to twice the rectangle CB, BD, and the ſquares of AD, DC: But the ſquare of AB is equal
c 47. 1. [c] to the ſquares BD, DA, becauſe the angle BDA is a right angle; and the ſquare of AC is equal to the ſquares of AD, DC: Therefore the ſquares of CB, BA are equal to the ſquare of AC, and twice the rectangle CB, BD, that is, the ſquare of AC alone is leſs than the ſquares of CB, BA by twice the rectangle CB, BD.

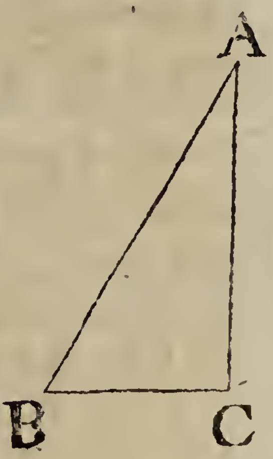


Secondly, Let AD fall without the triangle ABC: Then, becauſe the angle at D is a right angle, the angle ACB is greater
d 16. 1. [d] than a right angle; and there-
e 12. 2. fore the ſquare of AB is equal [e] to the ſquares of AC, CB, and twice the rectangle BC, CD: To theſe equals add the ſquare of BC, and the



ſquares of AB, BC are equal to the ſquare of AC, and twice the ſquare of BC, and twice the rectangle BC, CD : But becauſe BD is divided into two parts in C, the rectangle DB, BC is equal [f] to the rectangle BC, CD and the ſquare of BC : And the doubles of theſe are equal : Therefore the ſquares of AB, BC are equal to the ſquare of AC, and twice the rectangle DB, BC : Therefore the ſquare of AC alone is leſs than the ſquares of AB, BC by twice the rectangle DB, BC.

f 3. 2.



Laſtly, let the ſide AC be perpendicular to BC ; then is BC the ſtraight line between the perpendicular and the acute angle at B ; and it is manifeſt that the ſquares of AB, BC are equal [g] to the ſquare of AC and twice the ſquare of BC : Therefore, in every triangle, &c. Q. E. D.

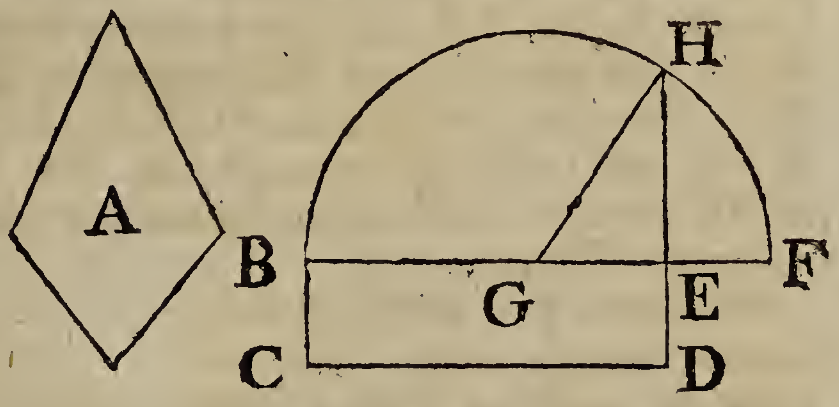
g 47. 1.

PROP. XIV. PROB.

TO deſcribe a ſquare that ſhall be equal to a given rectilineal figure.

See N.

Let A be the given rectilineal figure ; it is required to deſcribe a ſquare that ſhall be equal to A.



Deſcribe [a] the rectangular parallelogram BCDE equal to the rectilineal figure A. If then the ſides of it BE, ED are equal to one another, it is a ſquare, and what was required is now done : But if they are not equal, produce one of them BE to F, and make EF equal to ED and biſect BF in G : and from the centre G, at the diſtance GB, or GF, deſcribe the ſemicircle BHF, and produce DE to H, and join GH : Therefore becauſe the ſtraight line BF is divided into two equal parts in the point G, and into two unequal at E, the rectangle BE, EF, together with the ſquare of EG, is equal [b] to the ſquare of GF : But GF is equal to GH ; therefore the rectangle BE, EF,

a 45. 1.
b 5. 2.

together with the ſquare of EG, is equal to the ſquare of GH: But the ſquares of HE, EG are equal [c] to the ſquare of GH: Therefore the rectangle BE, EF, together with the ſquare of EG, is equal to the ſquares of HE, EG: Take away the ſquare of EG, which is common to both; and the remaining rectangle BE, EF is equal to the ſquare of EH: But the rectangle contained by BE, EF is the parallelogram BD, becauſe EF is equal to ED; therefore BD is equal to the ſquare of EH; but BD is equal to the rectilineal figure A; therefore the rectilineal figure A is equal to the ſquare of EH: Wherefore a ſquare has been made equal to the given rectilineal figure A, viz. the ſquare deſcribed upon EH. Which was to be done.

c 47. 1.

THE ELEMENTS OF EUCLID.

BOOK III.

DEFINITIONS.

I.

EQUAL circles are thoſe of which the diameters are equal, or from the centres of which the ſtraight lines to the circumferences are equal.

Book III.

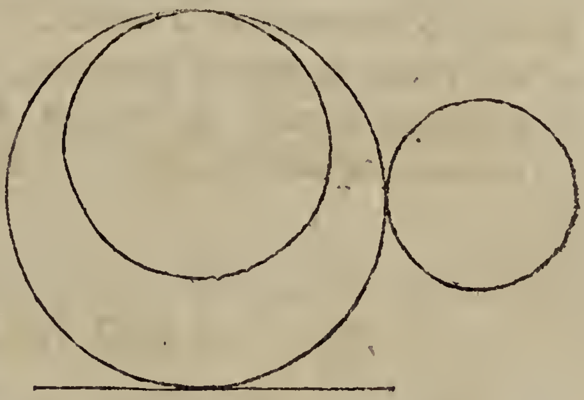
‘ This is not a definition but a theorem, the truth of which ‘ is evident; for, if the circles be applied to one another, ſo that ‘ their centres coincide, the circles muſt likewiſe coincide, ſince ‘ the ſtraight lines from the centres are equal.’

II.

A ſtraight line is ſaid to touch a circle, when it meets the circle, and being produced does not cut it.

III.

Circles are ſaid to touch one another, which meet but do not cut one another.

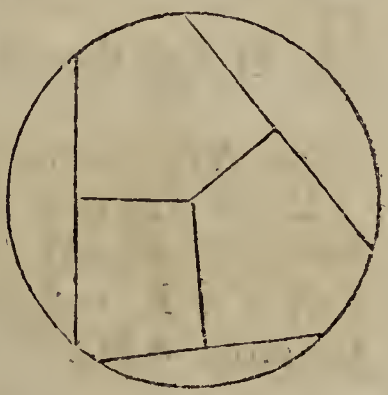


IV.

Straight lines are ſaid to be equally diſtant from the centre of a circle, when the perpendiculars drawn to them from the centre are equal.

V.

And the ſtraight line on which the greater perpendicular falls, is ſaid to be farther from the centre.



VI.

VI.

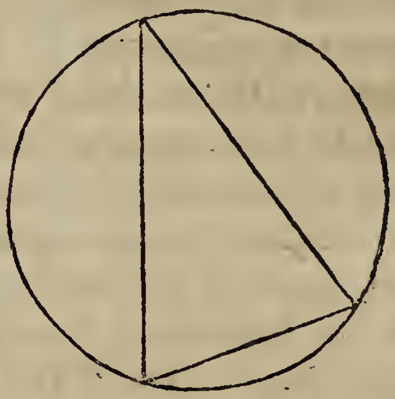
A ſegment of a circle is the figure contained by a ſtraight line and the circumference it cuts off.

VII.

“ The angle of a ſegment is that which is contained by the
“ ſtraight line and the circumference.”

VIII.

An angle in a ſegment is the angle contained by two ſtraight lines drawn from any point in the circumference of the ſegment, to the extremities of the ſtraight line which is the baſe of the ſegment.

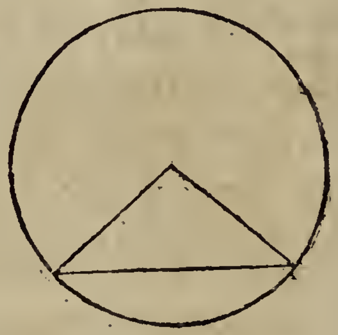


IX.

And an angle is ſaid to inſiſt or ſtand upon the circumference intercepted between the ſtraight lines that contain the angle.

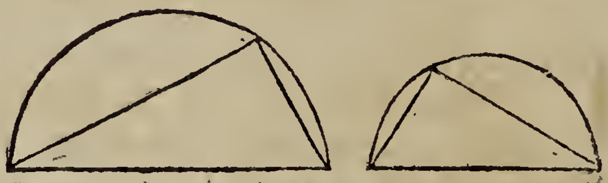
X.

The ſector of a circle is the figure contained by two ſtraight lines drawn from the centre, and the circumference between them.



XI.

Similar ſegments of a circle, are thoſe in which the angles are equal, or which contain equal angles.



PROP. I. PROB.

See N.

TO find the centre of a given circle.

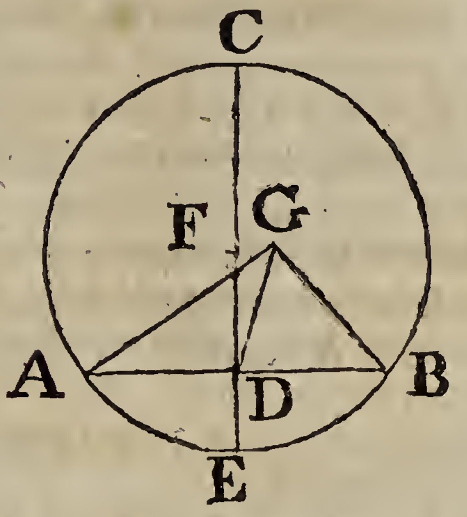
Let ABC be the given circle; it is required to find its centre. Draw within it any ſtraight line AB, and biſect [a] it in D; from the point D draw [b] DC at right angles to AB, and produce it to E, and biſect CE in F: The point F is the centre of the circle ABC.

a 10. 1.
b 11. 1.

For, if it be not, let, if poſſible, G be the centre, and join GA, GD, GB: Then, becauſe DA is equal to DB, and DG common to the two triangles ADG, BDG, the two ſides AD, DG are equal to the two BD, DG, each to each; and the baſe GA is equal to the baſe GB, becauſe they are drawn from the centre G *: Therefore the angle ADG is equal [c] to the angle GDB: But when a ſtraight line ſtanding upon another ſtraight line makes the adjacent angles equal to one another, each of the angles is a right angle [d]: Therefore the angle GDB is a right angle: But FDB is likewiſe a right angle; wherefore the angle FDB is equal to the angle GDB, the greater to the leſs, which is impoſſible: Therefore G is not the centre of the circle ABC: In the ſame manner it can be ſhown, that no other point but F is the centre; that is, F is the centre of the circle ABC: Which was to be found.

c 8. 1.

d 10. Def. 1.

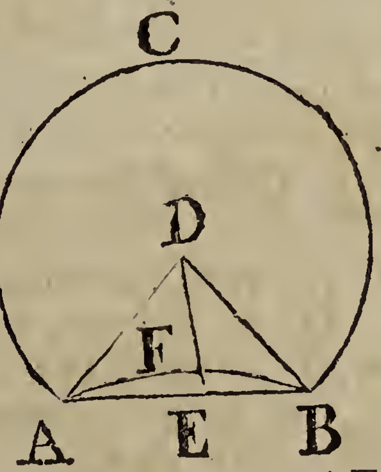


Cor. From this it is manifeſt, that if in a circle a ſtraight line biſect another at right angles, the centre of the circle is in the line which biſects another.

PROP. II. THEOR.

If any two points be taken in the circumference of a circle, the ſtraight line which joins them ſhall fall within the circle.

Let ABC be a circle, and A, B any two points in the circumference; the ſtraight line drawn from A to B ſhall fall within the circle.



For, if it do not, let it fall, if poſſible, without, as AEB; find [a] D the centre of the circle ABC, and join AD, DB, and produce DF, any ſtraight line meeting the circumference AB to E: Then becauſe DA is equal to DB, the angle DAB is equal [b] to the angle DBA; and becauſe AE, a ſide of the triangle

a 1. 3.

b 5. 1.

---

* N. B. Whenever the expreſſion "ſtraight lines from the centre," or "drawn "from the centre," occurs, it is to be underſtood that they are drawn to the circumference.

c 16. 1.
d 19. 1.

DAE, is produced to B, the angle DEB is greater [c] than the angle DAE; but DAE is equal to the angle DBE; therefore the angle DEB is greater than the angle DBE: But to the greater angle the greater ſide is oppoſite [d]; DB is therefore greater than DE: But DB is equal to DF; wherefore DF is greater than DE, the leſs than the greater, which is impoſſible: Therefore the ſtraight line drawn from A to B does not fall without the circle. In the ſame manner, it may be demonſtrated that it does not fall upon the circumference; it falls therefore within it. Wherefore, if any two points, &c. Q. E. D.

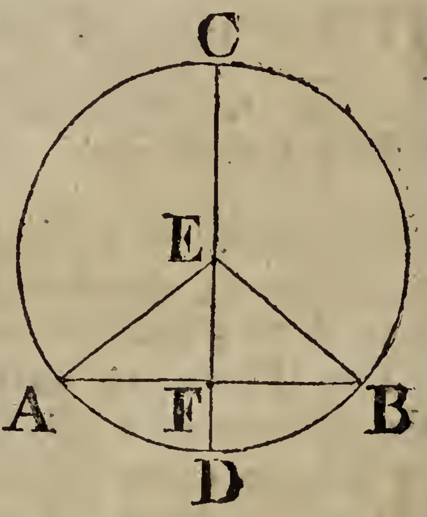
PROP. III. THEOR.

IF a ſtraight line drawn through the centre of a circle biſect a ſtraight line in it which does not paſs through the centre, it ſhall cut it at right angles; and, if it cuts it at right angles, it ſhall biſect it.

Let ABC be a circle; and let CD, a ſtraight line drawn through the centre, biſect any ſtraight line AB, which does not paſs through the centre, in the point F: It cuts it alſo at right angles.

a 1. 3.
b 8. 1.
c 10. Def. 1.

Take [a] E the centre of the circle, and join EA, EB. Then, becauſe AF is equal to FB, and FE common to the two triangles AFE, BFE, there are two ſides in the one equal to two ſides in the other, and the baſe EA is equal to the baſe EB; therefore the angle AFE is equal [b] to the angle BFE: But when a ſtraight line ſtanding upon another makes the adjacent angles equal to one another, each of them is a right [c] angle: Therefore each of the angles AFE, BFE is a right angle; wherefore the ſtraight line CD, drawn through the centre biſecting another AB that does not paſs through the centre, cuts the ſame at right angles.



But let CD cut AB at right angles; CD alſo biſects it, that is, AF is equal to FB.

d 5. 1.

The ſame conſtruction being made, becauſe EA, EB from the centre are equal to one another, the angle EAF is equal [d] to the angle EBF; and the right angle AFE is equal to the right angle BFE: Therefore, in the two triangles EAF, EBF, there are two angles in one equal to two angles in the other, and the ſide EF, which is oppoſite to one of the equal angles in each, is common to both; therefore the other ſides are equal [e]; AF therefore is equal to FB. Wherefore, if a ſtraight line, &c. Q. E. D.

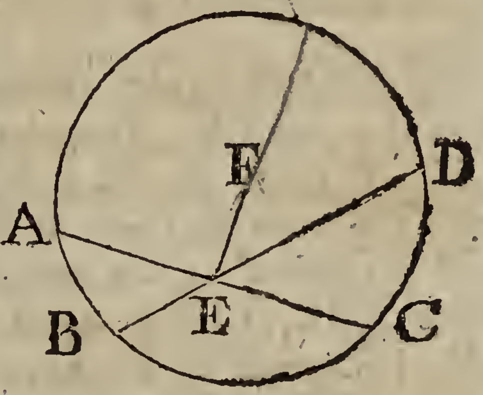
e 26. 1.

PROP. IV. THEOR.

IF in a circle two ſtraight lines cut one another which do not both paſs through the centre, they do not biſect each other.

Let ABCD be a circle, and AC, BD two ſtraight lines in it which cut one another in the point E, and do not both paſs through the centre: AC, BD do not biſect one another.

For, if it is poſſible, let AE be equal to EC, and BE to ED: If one of the lines paſs through the centre, it is plain that it cannot be biſected by the other which does not paſs through the centre: But if neither of them paſs through the centre, take [a] F the centre of the circle, and join EF: and becauſe FE, a ſtraight line through the centre, biſects another AC which does not paſs through the centre, it ſhall cut it at right [b] angles; wherefore FEA is a right angle: Again, becauſe the ſtraight line FE biſects the ſtraight line BD which does not paſs through the centre, it ſhall cut it at right [b] angles; wherefore FEB is a right angle: And FEA was ſhown to be a right angle; therefore FEA is equal to the angle FEB, the leſs to the greater, which is impoſſible: Therefore AC, BD do not biſect one another. Wherefore, if in a circle, &c. Q. E. D.



a 1. 3.

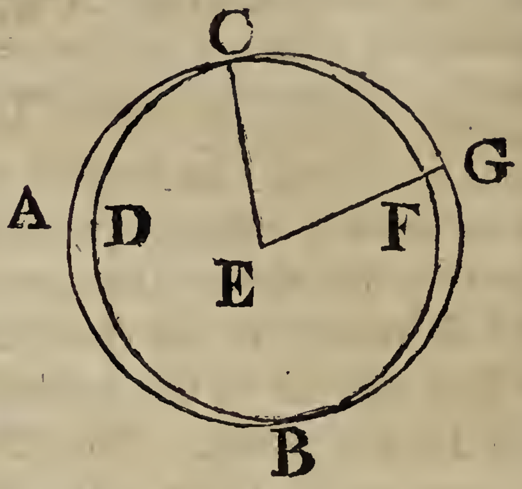
b 3. 3.

PROP. V. THEOR.

IF two circles cut one another, they ſhall not have the ſame centre.

Let the two circles ABC, CDG cut one another in the points B, C; they have not the ſame centre.

For, if it be poſſible, let E be their centre: Join EC, and draw any ſtraight line EFG meeting them in F and G: and becauſe E is the centre of the circle ABC, CE is equal to EF: Again, becauſe E is the centre of the circle CDG, CE is equal to EG: But CE was ſhown to be equal to EF; therefore EF is equal to EG, the leſs to the greater, which is impoſſible: Therefore E is not the centre of the circles ABC, CDG. Wherefore, if two circles, &c. Q. E. D.

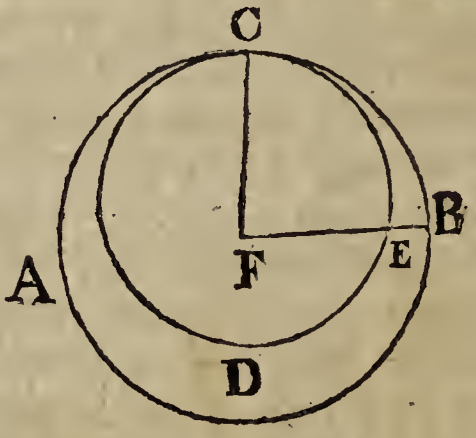


PROP. VI. THEOR.

IF two circles touch one another internally, they ſhall not have the ſame centre.

Let the two circles ABC, CDE, touch one another internally in the point C: They have not the ſame centre.

For, if they can, let it be F; join FC and draw any ſtraight line FEB meeting them in E and B; And becauſe F is the centre of the circle ABC, CF is equal to FB: Alſo, becauſe F is the centre of the circle CDE, CF is equal to FE: And CF was ſhewn equal to FB; therefore FE is equal to FB, the leſs to the greater, which is impoſſible; Wherefore F is not the centre of the circles ABC, CDE. Therefore, if two circles, &c. Q. E. D.

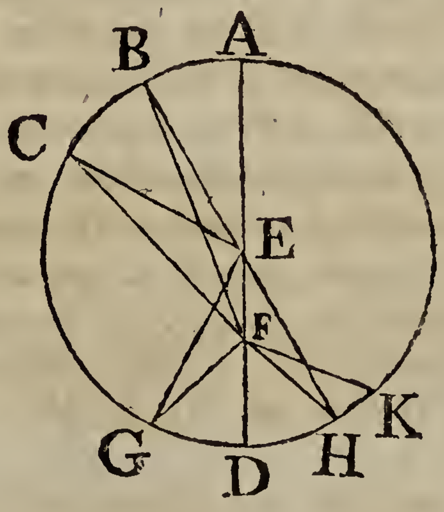


PROP. VII. THEOR.

IF any point be taken in the diameter of a circle which is not the centre, of all the ſtraight lines which can be drawn from it to the circumference, the greateſt is that in which the centre is, and the other part of that diameter is the leaſt; and, of any others, that which is nearer to the line which paſſes through the centre is always greater than one more remote: And from the ſame point there can be drawn only two ſtraight lines that are equal to one another, one upon each ſide of the ſhorteſt line.

Let ABCD be a circle, and AD its diameter, in which let any point F be taken which is not the centre: Let the centre be E; of all the ſtraight lines FB, FC, FG, &c. that can be drawn from F to the circumference, FA is the greateſt, and FD, the other part of the diameter BD, is the leaſt: And of the others, FB is greater than FC, and FC than FG.

Join BE, CE, GE; and becauſe two ſides of a triangle are greater [a] than the third, BE, EF are greater than BF; but AE is equal to EB; therefore AE, EF, that is AF, is greater than BF: Again, becauſe BE is equal to CE, and FE common to the triangles BEF, CEF, the two ſides BE, EF are equal to the two CE, EF; but the angle BEF is greater than the angle CEF; therefore the baſe BF is greater [b] than the baſe FC: For the ſame reaſon, CF is greater than GF: Again, becauſe GF, FE are greater [a] than EG, and EG is equal to ED; GF, FE are greater than ED: Take away the common part FE, and the remainder GF is greater than the remainder FD: Therefore FA is the greateſt, and FD the leaſt of all the ſtraight lines from F to the circumference; and BF is greater than CF, and CF than GF.



a 20. 1.

b 24. 1.

Alſo there can be drawn only two equal ſtraight lines from the point F to the circumference, one upon each ſide of the ſhorteſt

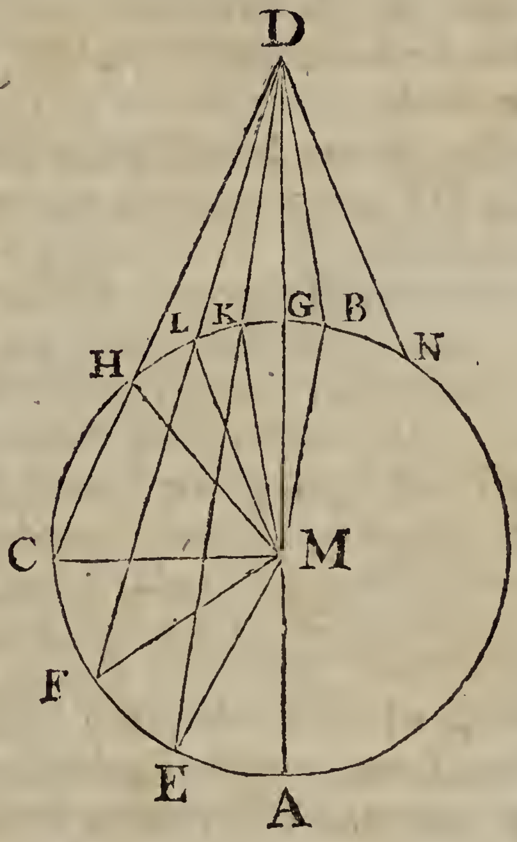
Book III. ſhorteſt line FD: At the point E in the ſtraight line EF, make c the angle FEH equal to the angle GEF, and join FH: Then c 23. 1. becauſe GE is equal to EH, and EF common to the two triangles GEF, HEF; the two ſides GE, EF are equal to the two HE, EF; and the angle GEF is equal to the angle HEF; there- d 4. 1. fore the baſe FG is equal [d] to the baſe FH: But, beſides FH, no other ſtraight line can be drawn from F to the circumference equal to FG: For, if there can, let it be FK; and becauſe FK is equal to FG, and FG to FH, FK is equal to FH; that is, a line nearer to that which paſſes through the centre, is equal to one which is more remote; which is impoſſible. Therefore, if any point be taken, &c. Q. E. D.

PROP. VIII. THEOR.

If any point be taken without a circle, and ſtraight lines be drawn from it to the circumference, whereof one paſſes through the centre; of thoſe which fall upon the concave circumference, the greateſt is that which paſſes through the centre; and of the reſt, that which is nearer to that through the centre is always greater than the more remote: But of thoſe which fall upon the convex circumference, the leaſt is that between the point without the circle, and the diameter; and of the reſt, that which is nearer to the leaſt is always leſs than the more remote: And only two equal ſtraight lines can be drawn from the point unto the circumference, one upon each ſide of the leaſt.

Let ABC be a circle, and D any point without it, from which let the ſtraight lines DA, DE, DF, DC be drawn to the circumference, whereof DA paſſes through the centre. Of thoſe which fall upon the concave part of the circumference AEFC, the greateſt is AD which paſſes through the centre; and the nearer to it is always greater than the more remote, viz. DE than DF, and DF than DC: But of thoſe which fall upon the convex circumference HLKG, the leaſt is DG between the point D and the diameter AG; and the nearer to it is always leſs than the more remote, viz. DK than DL, and DL than DH.

Take [a] M the centre of the circle ABC, and join ME, MF, MC, MK, ML, MH: And becauſe AM is equal to ME, add MD to each, therefore AD is equal to EM, MD; but EM, MD are greater [b] than ED; therefore alſo AD is greater than ED: Again, becauſe ME is equal to MF, and MD common to the triangles EMD, FMD; EM, MD are equal to FM, MD; but the angle EMD is greater than the angle FMD; therefore the baſe ED is greater [c] than the baſe FD: In like manner it may be ſhewn that FD is greater than CD: Therefore DA is the greateſt; and DE greater than DF, and DF than DC: And becauſe MK, KD are greater [b] than MD, and MK is equal to MG, the remainder KD is greater [d] than the remainder GD, that is GD is leſs than KD: And becauſe MK, DK are drawn to the point K within the triangle MLD from M, D, the extremities of its ſide MD, MK, KD are leſs [e] than ML, LD, whereof MK is equal to ML; therefore the remainder DK is leſs than the remainder DL: In like manner it may be ſhewn, that DL is leſs than DH: Therefore DG is the leaſt, and DK leſs than DL, and DL than DH: Alſo there can be drawn only two equal ſtraight lines from the point D to the circumference, one upon each ſide of the leaſt: At the point M, in the ſtraight line MD make the angle DMB equal to the angle DMK, and join DB: And becauſe MK is equal to MB, and MD common to the triangles KMD, BMD, the two ſides KM, MD are equal to the two BM, MD; and the angle KMD is equal to the angle BMD; therefore the baſe DK is equal [f] to the baſe DB: But, beſides DB, there can be no ſtraight line drawn from D to the circumference equal to DK: For, if there can, let it be DN; and becauſe DK is equal to DN, and alſo to DB; therefore DB is equal to DN, that is, the nearer to the leaſt equal to the more remote, which is impoſſible. If therefore, any point, &c. Q. E. D.



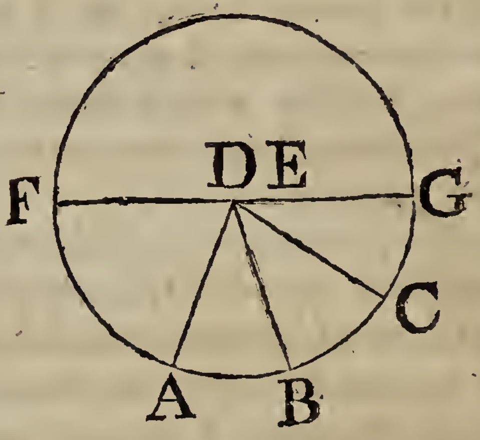
a 1. 3.
b 20. 1.
c 24. 1.
d 4. Ax.
e 21. 1.
f 4. 1.

PROP.

PROP. IX. PROB.

IF a point be taken within a circle, from which there fall more than two equal ſtraight lines to the circumference, that point is the centre of the circle.

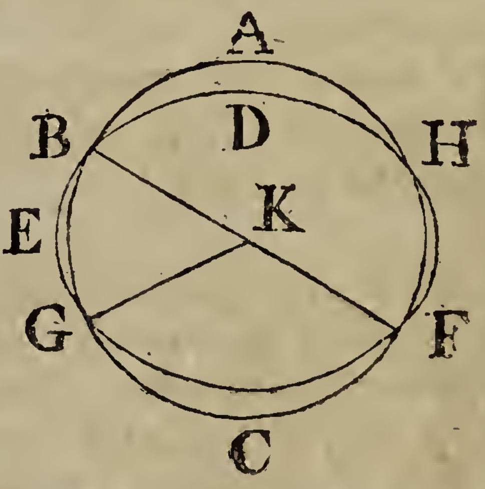
Let the point D be taken within the circle ABC, from which to the circumference there fall more than two equal ſtraight lines, viz. DA, DB, DC, the point D is the centre of the circle.



For, if not, let E be the centre, join DE and produce it to the circumference in F, G; then FG is a diameter of the circle ABC: And becauſe in FG, the diameter of the circle ABC, there is taken the point D which is not the centre, DG ſhall be the greateſt line from it to the circumference, and DC
a 7. 3. greater[a] than DB, and DB than DA: But they are likewiſe equal, which is impoſſible: Therefore E is not the centre of the circle ABC: In like manner, it may be demonſtrated, that no other point but D is the centre; D therefore is the centre. Wherefore, if a point be taken, &c. Q. E. D.

PROP. X. THEOR.

ONE circumference of a circle cannot cut another in more than two points.



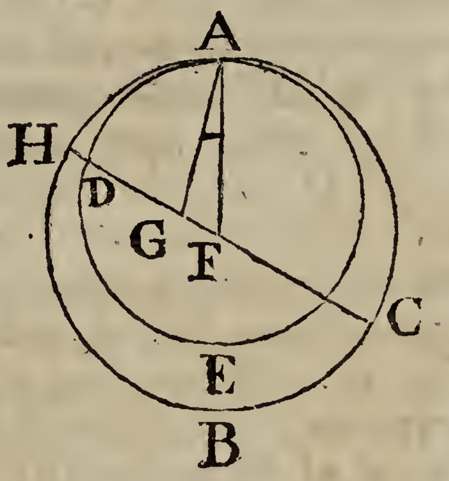
If it be poſſible, let the circumference FAB cut the circumference DEF in more than two points, viz. in B, G, F; take the centre K of the circle ABC, and join KB, KG, KF: And becauſe within the circle DEF there is taken the point K, from which to the circumference DEF fall more than two equal ſtraight
a 9. 3. lines KB, KG, KF, the point K is[a]

the centre of the circle DEF: But K is alſo the centre of the circle ABC; therefore the ſame point is the centre of two circles that cut one another, which is impoſſible [b]. Therefore one circumference of a circle cannot cut another in more than two points. Q. E. D. b 5. 3.

PROP. XI. THEOR.

IF two circles touch each other internally, the ſtraight line which joins their centres being produced ſhall paſs through the point of contact.

Let the two circles ABC, ADE, touch each other internally in the point A, and let F be the centre of the circle ABC, and G the centre of the circle ADE: The ſtraight line which joins the centres F, G, being produced, paſſes through the point A.



For, if not, let it fall otherwiſe, if poſſible, as FGDH, and join AF, AG: And becauſe AG, GF are greater [a] than FA, that is, than FH, for FA is equal to FH, both being from the ſame centre; take away the common part FG; therefore the remainder AG is greater than the remainder GH: But AG is equal to GD; therefore GD is greater than GH, the leſs than the greater, which is impoſſible. Therefore the ſtraight line which joins the points F, G cannot fall otherwiſe than upon the point A, that is, it muſt paſs through it. Therfore, if two circles &c. Q. E. D. a 20. 1.

PROP. XII. THEOR.

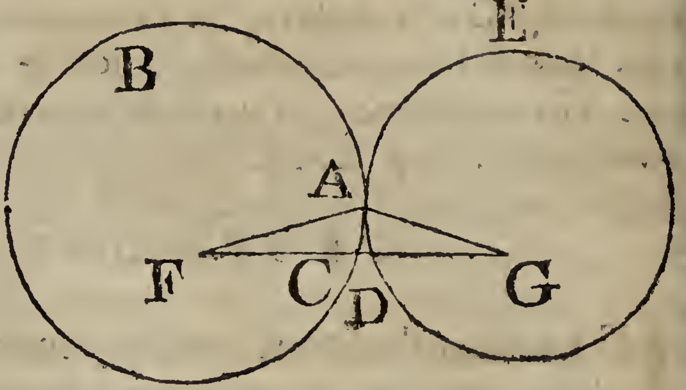
IF two circles touch each other externally, the ſtraight line which joins their centres ſhall paſs through the point of contact.

Let the two circles ABC, ADE touch each other externally in the point A; and let F be the centre of the circle ABC, and G the centre of ADE: The ſtraight line which joins the points F, G ſhall paſs through the point of contact A.

For, if not, let it paſs otherwiſe, if poſſible, as FCDG, and

join

Book III. join FA, AG: And becauſe F is the centre of the circle ABC, AF is equal to FC: Alſo becauſe G is the centre of the circle ADE, AG is equal to GD: Therefore FA, AG are equal to FC, DG; wherefore the whole FG is greater than FA, AG; But it is alſo



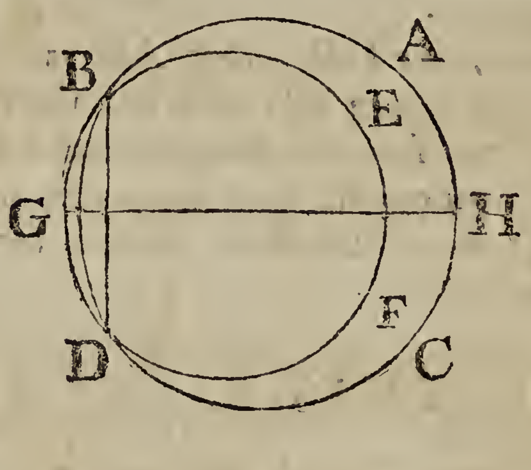
a 20. 1. leſs[a]; which is impoſſible: Therefore the ſtraight line which joins the points F, G ſhall not paſs otherwiſe than through the point of contact A, that is, it muſt paſs through it. Therefore, if two circles, &c. Q. E. D.

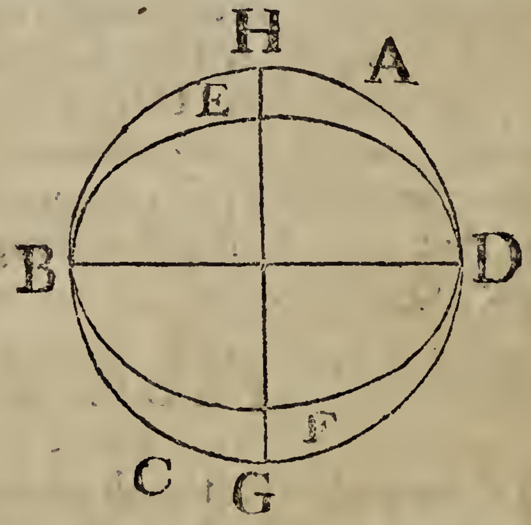
PROP. XII. THEOR.

See N.

ONE circle cannot touch another in more points than one, whether it touches it on the inſide or outſide.

For, if it be poſſible, let the circle EBF touch the circle ABC in more points than one, and firſt on the inſide, in the points
a 10. 11. 1. B, D; join BD, and draw[a] GH biſecting BD at right angles: Therefore, becauſe the points B, D are in the circumference of

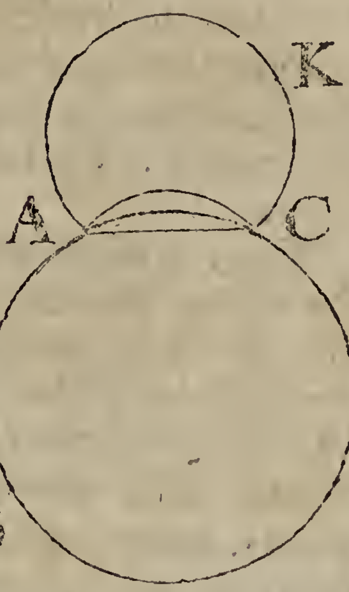




b 2. 3. each of the circles, the ſtraight line BD falls within each[b] of
c. Cor. 1. 3. them: And their centres are[c] in the ſtraight line GH which biſects BD at right angles: Therefore GH paſſes through the
d 11. 3. point of contact[d]; but it does not paſs through it, becauſe the points B, D are without the ſtraight line GH, which is abſurd: Therefore one circle cannot touch another on the inſide in more points than one.

Nor can two circles touch one another on the outſide in more than one point: For, if it be poſſible, let the circle ACK touch the circle ABC in the points A, C, and join AC: Therefore, becauſe the two points A, C are in the circumference of the circle ACK, the ſtraight line AC which joins them ſhall fall within [b] the circle ACK: And the circle ACK is without the circle ABC; and therefore the ſtraight line AC is without this laſt circle; but, becauſe the points A, C are in the circumference of the circle ABC, the ſtraight line AC muſt be within [b] the ſame circle, which is abſurd: Therefore one circle cannot touch another on the outſide in more than one point: And it has been ſhewn, that they cannot touch on the inſide in more points than one: Therefore, one circle, &c. Q. E. D.

b 2. 3.



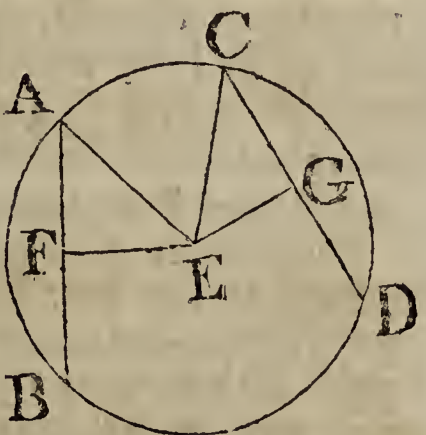
PROP. XIV. THEOR.

EQUAL ſtraight lines in a circle are equally diſtant from the centre; and thoſe which are equally diſtant from the centre, are equal to one another.

Let the ſtraight lines AB, CD, in the circle ABDC, be equal to one another; they are equally diſtant from the centre.

Take E the centre of the circle ABDC, and from it draw EF, EG perpendiculars to AB, CD: Then, becauſe the ſtraight line EF, paſſing through the centre, cuts the ſtraight line AB, which does not paſs through the centre, at right angles, it alſo biſects [a] it: Wherefore AF is equal to FB, and AB double of AF. For the ſame reaſon CD is double of CG: And AB is equal to CD; therefore AF is equal to CG: And becauſe AE is equal to EC, the ſquare of AE is equal to the ſquare of EC: But the ſquares of AF, FE are equal [b] to the ſquare of AE, becauſe the angle AFE is a right angle; and for the like reaſon, the ſquares of EG, GC are equal to the ſquare of EC: Therefore the ſquares of AF, FE are equal to the ſquares of CG, GE, of which the ſquare of AF is equal to the

a 3. 3.
b 47. 1.



the ſquare of CG, becauſe AF is equal to CG; therefore the remaining ſquare of FE is equal to the remaining ſquare of EG, and the ſtraight line EF is therefore equal to EG: But ſtraight lines in a circle are ſaid to be equally diſtant from the centre, when the perpendiculars drawn to them from the centre are equal [c]; Therefore AB, CD are equally diſtant from the centre.

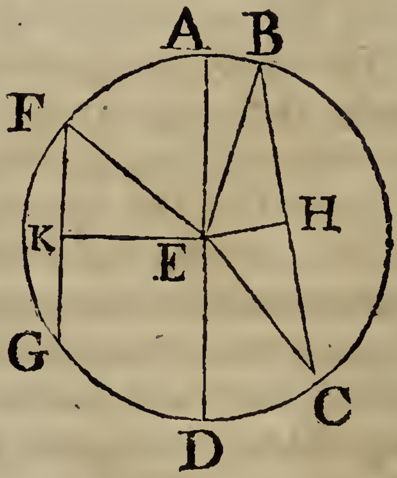
[c] 4. Def. 3.

Next, if the ſtraight lines AB, CD be equally diſtant from the centre, that is, if FE be equal to EG; AB is equal to CD: For, the ſame conſtruction being made, it may, as before, be demonſtrated, that AB is double of AF, and CD double of CG, and that the ſquares of EF, FA are equal to the ſquares of EG, GC; of which the ſquare of FE is equal to the ſquare of EG, becauſe FE is equal to EG; therefore the remaining ſquare of AF is equal to the remaining ſquare of CG; and the ſtraight line AF is therefore equal to CG: And AB is double of AF, and CD double of CG; wherefore AB is equal to CD. Therefore equal ſtraight lines, &c. Q. E. D.

PROP. XV. THEOR.

See N.

THE diameter is the greateſt ſtraight line in a circle; and, of all others, that which is nearer to the centre is always greater than one more remote; and the greater is nearer to the centre than the leſs.



Let ABCD be a circle, of which the diameter is AD, and the centre E; and let BC be nearer to the centre than FG; AD is greater than any ſtraight line BC which is not a diameter, and BC greater than FG.

From the centre draw EH, EK perpendiculars to BC, FG, and join EB, EC, EF; and becauſe AE is equal to EB, and ED to EC, AD is equal to EB, EC: But EB, EC, are greater [a] than BC; wherefore, alſo AD is greater than BC.

[a] 20. 1.

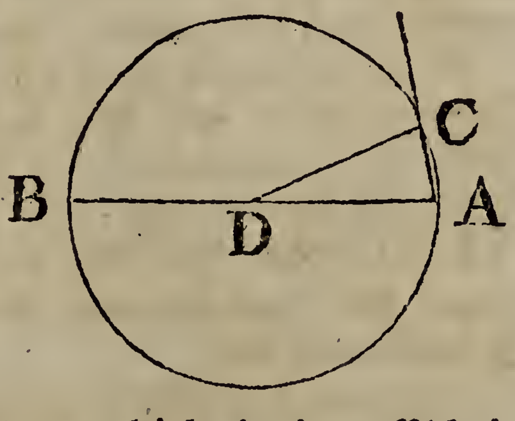
And, becauſe BC is nearer to the centre than FG, EH is leſs [b] than EK: But, as was demonſtrated in the preceding, BC is double of BH, and FG double of FK, and the ſquares of EH, HB are equal to the ſquares of EK, KF, of which the [b] 5. Def. 3. ſquare of EH is leſs than the ſquare of EK, becauſe EH is leſs than EK; therefore the ſquare of BH is greater than the ſquare of FK, and the ſtraight line BH greater than FK; and therefore BC is greater than FG.

Next, Let BC be greater than FG; BC is nearer to the centre than FG, that is, the ſame conſtruction being made, EH is leſs than EK: Becauſe BC is greater than FG, BH likewiſe is greater than KF: And the ſquares of BH, HE are equal to the ſquares of FK, KE, of which the ſquare of BH is greater than the ſquare of FK, becauſe BH is greater than FK; therefore the ſquare of EH is leſs than the ſquare of EK, and the ſtraight line EH leſs than EK. Wherefore the diameter, &c. Q. E. D.

PROP. XVI. THEOR.

THE ſtraight line drawn at right angles to the diameter of a circle, from the extremity of it, falls without the circle; and no ſtraight line can be drawn between that ſtraight line and the circumference from the extremity, ſo as not to cut the circle; or which is the ſame thing, no ſtraight line can make ſo great an acute angle with the diameter at its extremity, or ſo ſmall an angle with the ſtraight line which is at right angles to it, as not to cut the circle. See N.

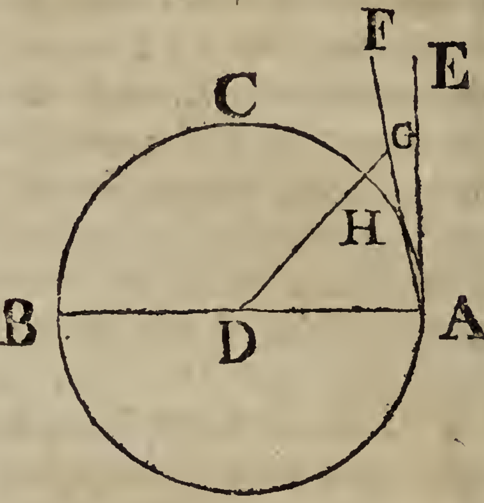
Let ABC be a circle, the centre of which is D, and the diameter AB: the ſtraight line drawn at right angles to AB from its extremity A, ſhall fall without the circle.



For, if it does not, let it fall, if poſſible, within the circle, as AC, and draw DC to the point C where it meets the circumference: And becauſe DA is equal to DC, the angle DAC is equal [a] to the angle ACD; but DAC is a right angle, therefore ACD is a right angle, and the angles DAC, ACD are therefore equal to two right angles; which is impoſſible [b]: [a] 5. 1. [b] 17. 1.

Book III. Therefore the ſtraight line drawn from A at right angles to BA does not fall within the circle: In the ſame manner, it may be demonſtrated that it does not fall upon the circumference; therefore it muſt fall without the circle, as AE.

And between the ſtraight line AE and the circumference no ſtraight line can be drawn from the point A which does not cut the circle: For, if poſſible, let FA be between them, and
c 12. 1. from the point D draw [c] DG perpendicular to FA, and let it meet the circumference in H: And becauſe AGD is a right
d 19. 1. angle, and DAG leſs [b] than a right angle: DA is greater [d] than DG: But DA is equal to DH; therefore DH is greater than DG, the leſs than the greater, which is impoſſible: Therefore no ſtraight line can be drawn from the point A between AE and the circumference, which does not cut the circle, or, which amounts to the ſame thing, however great an acute angle a ſtraight line makes with the diameter at the point A, or however ſmall an angle it makes with AE, the circumference paſſes between that ſtraight line and the perpendicular AE. ‘And this is all that is to be underſtood, ‘when, in the Greek text and tranſlations from it, the angle of ‘the ſemicircle is ſaid to be greater than any acute rectilineal ‘angle, and the remaining angle leſs than any rectilineal an‘gle.’



Cor. From this it is manifeſt that the ſtraight line which is drawn at right angles to the diameter of a circle from the extremity of it, touches the circle; and that it touches it only in one point, becauſe, if it did meet the circle in two, it would
e 2. 3. fall within it [e]. ‘Alſo it is evident that there can be but one ‘ſtraight line which touches the circle in the ſame point.’

PROP. XVII. PROB.

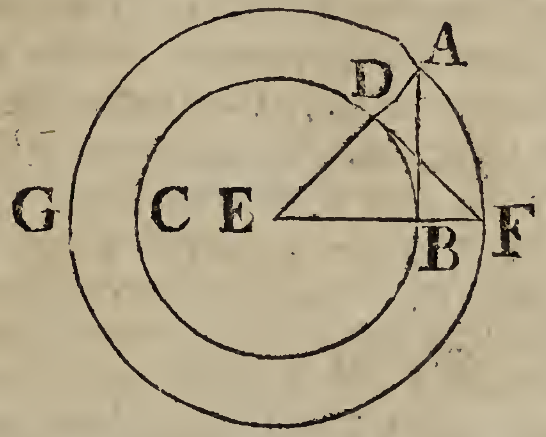
To draw a ſtraight line from a given point, either without or in the circumference, which ſhall touch a given circle.

Firſt, let A be a given point without the given circle BCD;

it is required to draw a ſtraight line from A which ſhall touch the circle.

Find [a] the centre E of the circle, and join AE; and from the centre E, at the diſtance EA, deſcribe the circle AFG; from the point D draw [b] DF at right angles to EA, and join EBF, AB. AB touches the circle BCD.

a 1. 3.
b 11. 1.



Becauſe E is the centre of the circles BCD, AFG, EA is equal to EF: And ED to EB; therefore the two ſides AE, EB are equal to the two FE, ED, and they contain the angle at E common to the two triangles AEB, FED; therefore the baſe DF is equal to the baſe AB, and the triangle FED to the triangle AEB, and the other angles to the other angles [c]: Therefore the angle EBA is equal to the angle EDF: But EDF is a right angle, wherefore EBA is a right angle: And EB is drawn from the centre: But a ſtraight line drawn from the extremity of a diameter, at right angles to it, touches the circle [d]: Therefore AB touches the circle; and it is drawn from the given point A. Which was to be done.

c 4. 1.
d Cor. 16. 3.

But, if the given point be in the circumference of the circle, as the point D, draw DE to the centre E, and DF at right angles to DE; DF touches the circle [d].

PROP. XVIII. THEOR.

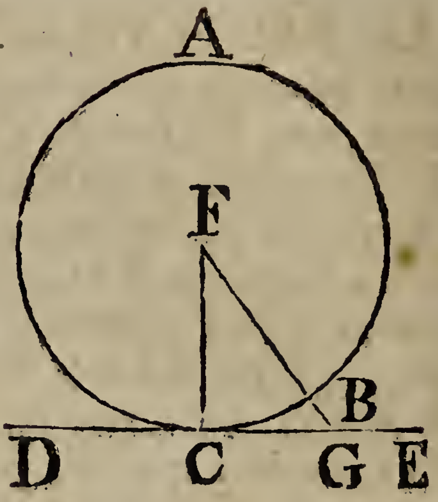
IF a ſtraight line touches a circle, the ſtraight line drawn from the centre to the point of contact, ſhall be perpendicular to the line touching the circle.

Let the ſtraight line DE touch the circle ABC in the point C; take the centre F, and draw the ſtraight line FC: FC is perpendicular to DE.

For, if it be not, from the point F draw FBG perpendicular to DE; and becauſe FGC is a right angle, GCF is [b] an acute angle; and to the greater angle the greateſt [c] ſide is oppoſite:

b 17. 1.
c 19. 1.

Therefore FC is greater than FG; but FC is equal to FB; therefore FB is greater than FG, the leſs than the greater, which is impoſſible: Wherefore FG is not perpendicular to DE: In the ſame manner it may be ſhewn, that no other is perpendicular to it beſides FC, that is, FC is perpendicular to DE. Therefore, if a ſtraight line, &c. Q. E. D.



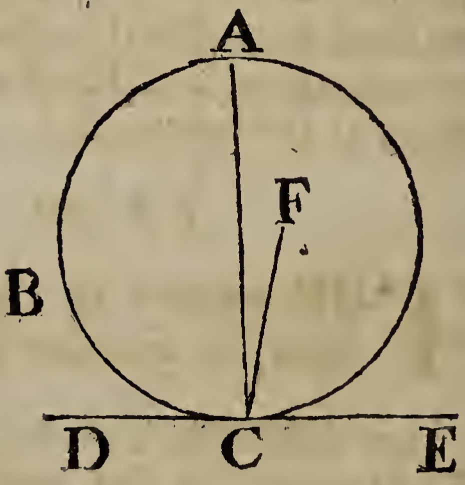
PROP. XIX. THEOR.

IF a ſtraight line touches a circle, and from the point of contact a ſtraight line be drawn at right angles to the touching line, the centre of the circle ſhall be in that line.

Let the ſtraight line DE touch the circle ABC in C, and from C let CA be drawn at right angles to DE; the centre of the circle is in CA.

For, if not, let F be the centre, if poſſible, and join CF: Becauſe DE touches the circle ABC, and FC is drawn from the centre to the point of contact, FC is perpendicular [a] to DE; therefore FCE is a right angle: But ACE is alſo a right angle; therefore the angle FCE is equal to the angle ACE, the leſs to the greater, which is impoſſible: Wherefore F is not the centre of the circle ABC: In the ſame manner, it may be ſhewn, that no other point which is not in CA, is the centre; that is, the centre is in CA. Therefore, if a ſtraight line, &c. Q. E. D.

a 18. 3.

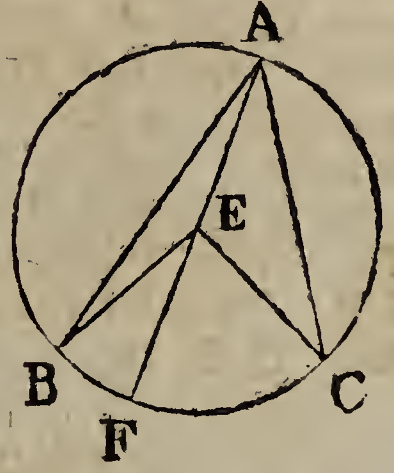


PROP. XX. THEOR.

See N.

THE angle at the centre of a circle is double of the angle at the circumference, upon the ſame baſe, that is, upon the ſame part of the circumference.

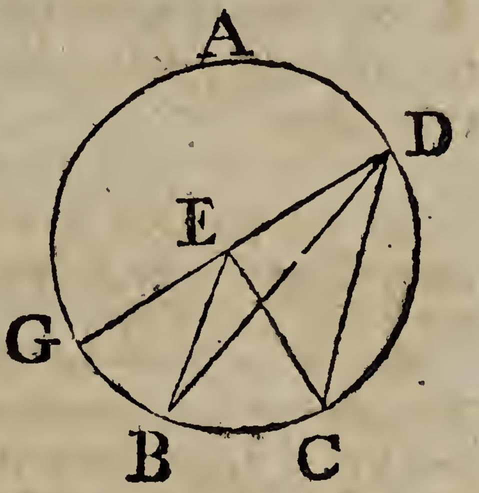
Let ABC be a circle, and BEC an angle at the centre, and BAC an angle at the circumference, which have the ſame circumference BC for their baſe; the angle BEC is double of the angle BAC.



Firſt, let E the centre of the circle be within the angle BAC, and join AE, and produce it to F: Becauſe EA is equal to EB, the angle EAB is equal [a] to the angle EBA; therefore the angles EAB, EBA are double of the angle EAB; but the angle BEF is equal [b] to the angles EAB, EBA; therefore alſo the angle BEF is double of the angle EAB: For the ſame reaſon, the angle FEC is double of the angle EAC: Therefore the whole angle BEC is double of the whole angle BAC.

[a] 5. 1.

[b] 32. 1.

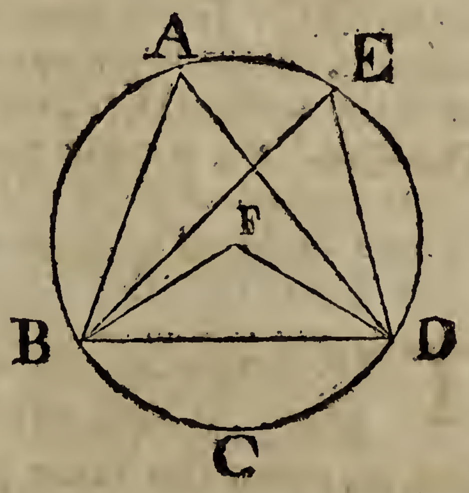


Again, let E the centre of the circle be without the angle BDC, and join DE and produce it to G. It may be demonſtrated, as in the firſt caſe, that the angle GEC is double of the angle GDC, and that GEB a part of the firſt is double of GDB a part of the other; therefore the remaining angle BEC is double of the remaining angle BDC. Therefore the angle at the centre, &c. Q. E. D.

PROP. XXI. THEOR.

THE angles in the ſame ſegment of a circle are equal to one another.

See N.



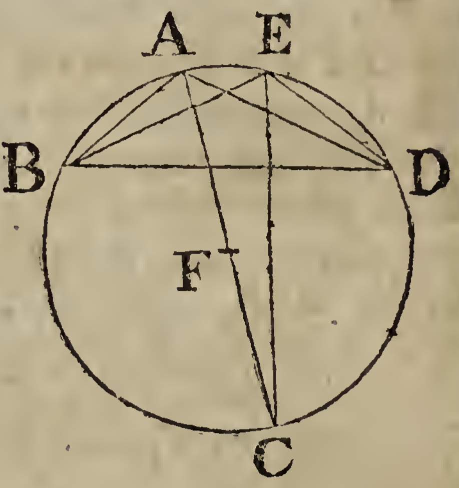
Let ABCD be a circle, and BAD, BED angles in the ſame ſegment BAED: The angles BAD, BED are equal to one another.

Take F the centre of the circle ABCD: And, firſt, let the ſegment BAED be greater than a ſemicircle, and join BF, FD: And becauſe the angle BFD is at the centre, and the angle BAD at the circumference, and that they have the ſame part of

Book III. the circumference, viz. BCD for their baſe; therefore the angle BFD is double [a] of the angle BAD: For the ſame reaſon, the angle BFD is double of the angle BED: Therefore the angle BAD is equal to the angle BED.

a 20. 3.

But, if the ſegment BAED be not greater than a ſemicircle, let BAD, BED be angles in it; theſe alſo are equal to one another: Draw AF to the centre, and produce it to C, and join CE: Therefore the ſegment BADC is greater than a ſemicircle; and the angles in it BAC, BEC are equal, by the firſt caſe: For the ſame reaſon, becauſe CBED is greater than a ſemicircle, the angles CAD, CED are equal: Therefore the whole angle BAD is equal to the whole angle BED. Wherefore the angles in the ſame ſegment, &c. Q. E. D.



PROP. XXII. THEOR.

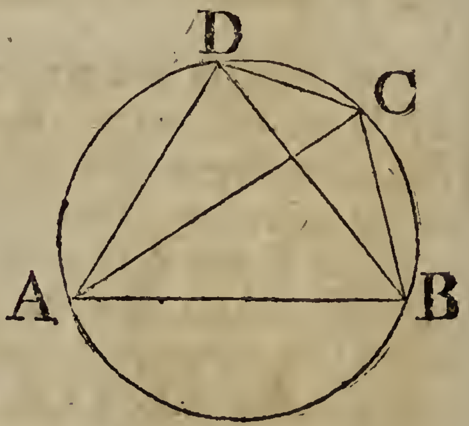
THE oppoſite angles of any quadrilateral figure deſcribed in a circle, are together equal to two right angles.

Let ABCD be a quadrilateral figure in the circle ABCD; any two of its oppoſite angles are together equal to two right angles.

Join AC, BD; and becauſe the three angles of every triangle are equal [a] to two right angles, the three angles of the triangle CAB, viz. the angles CAB, ABC, BCA are equal to two right angles: But the angle CAB is equal [b] to the angle CDB, becauſe they are in the ſame ſegment BADC, and the angle ACB is equal to the angle ADB, becauſe they are in the ſame ſegment ADCB: Therefore the whole angle ADC is equal to the angles CAB, ACB: To each of theſe equals add the angle ABC; therefore the angles ABC, CAB, BCA are equal to the angles ABC, ADC: But ABC, CAB, BCA are equal to two right angles; therefore alſo the angles ABC, ADC are equal to two right angles: In the ſame manner, the angles BAD,

a 32. 1.

b 21. 3.

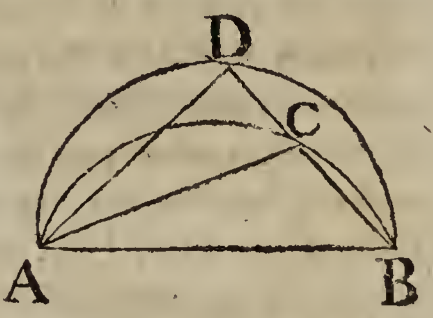


BAD, DCB may be ſhewn to be equal to two right angles. Therefore, the oppoſite angles, &c. Q. E. D.

PROP. XXIII. THEOR.

UPON the ſame ſtraight line, and upon the ſame ſide of it, there cannot be two ſimilar ſegments of circles, not coinciding with one another. See N.

If it be poſſible, let the two ſimilar ſegments of circles, viz. ACB, ADB, be upon the ſame ſide of the ſame ſtraight line AB, not coinciding with one another: Then, becauſe the circle ACB cuts the circle ADB in the two points A, B, they cannot cut one another in any other point [a]: One of the ſegments muſt therefore fall within the other; let ACB fall within ADB, and draw the ſtraight line BCD, and join CA, DA: And becauſe the ſegment ACB is ſimilar to the ſegment ADB, and that ſimilar ſegments of circles contain [b] equal angles; the angle ACB is equal to the angle ADB, the exterior to the interior, which is impoſſible [c]. Therefore, there cannot be two ſimilar ſegments of a circle upon the ſame ſide of the ſame line, which do not coincide. Q. E. D.



a 10. 3.

b 11. def. 3.

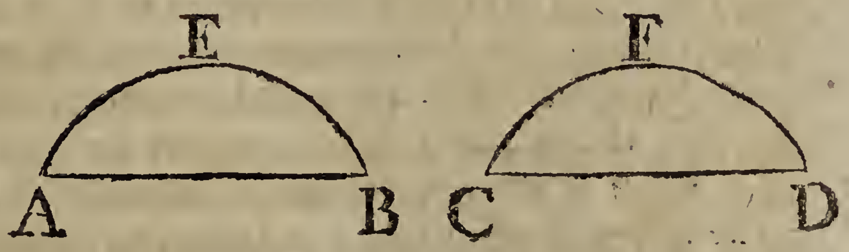
c 16. 1.

PROP. XXIV. THEOR.

SIMILAR ſegments of circles upon equal ſtraight lines, are equal to one another. See N.

Let AEB, CFD be ſimilar ſegments of circles upon the equal ſtraight lines AB, CD; the ſegment AEB is equal to the ſegment CFD.

For, if the ſegment AEB be applied to the ſegment CFD, ſo as the point A be on C, and the ſtraight line AB upon CD, the point B ſhall coincide with the point D, becauſe



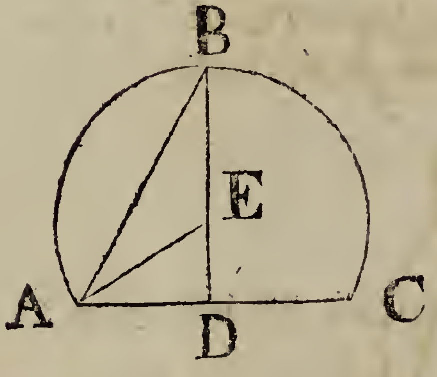
cauſe AB is equal to CD: Therefore the ſtraight line AB coinciding
with CD, the ſegment AEB muſt [a] coincide with the
a 23. 3. ſegment CFD, and therefore is equal to it. Wherefore ſimilar
ſegments, &c. Q. E. D.

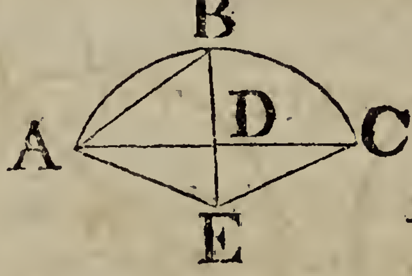
PROP. XXV. PROB.

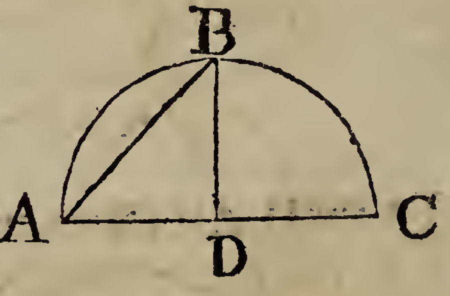
See N. A Segment of a circle being given, to deſcribe the circle of which it is the ſegment.

Let ABC be the given ſegment of a circle; it is required to deſcribe the circle of which it is the ſegment.

a 10. 1. Biſect [a] AC in D, and from the point D draw [b] DB at right
b 11. 1. angles to AC, and join AB: Firſt, let the angles ABD, BAD
c 6. 1. be equal to one another; then the ſtraight line BD is equal [c]
to DA, and therefore to DC; and becauſe the three ſtraight
lines DA, DB, DC, are all equal; D is the centre of the circle
d 9. 3. [d]: From the centre D, at the diſtance of any of the three
DA, DB, DC, deſcribe a circle; this ſhall paſs through the other
points; and the circle of which ABC is a ſegment is deſcribed:
And becauſe the centre D is in AC, the ſegment ABC is a ſemicircle:







But if the angles ABD, BAD are not equal to one
e 23. 1. another, at the point A, in the ſtraight line AB make [e] the angle
BAE equal to the angle ABD, and produce BD, if neceſſary, to
E, and join EC: And becauſe the angle ABE is equal to the angle
BAE, the ſtraight line BE is equal [c] to EA: And becauſe AD
is equal to DC, and DE common to the triangles ADE, CDE,
the two ſides AD, DE are equal to the two CD, DE, each to
each; and the angle ADE is equal to the angle CDE, for
each of them is a right angle; therefore the baſe AE is equal
f 4. 1. [f] to the baſe EC: But AE was ſhewn to be equal to EB, wherefore
alſo BE is equal to EC: And the three ſtraight lines AE,
EB

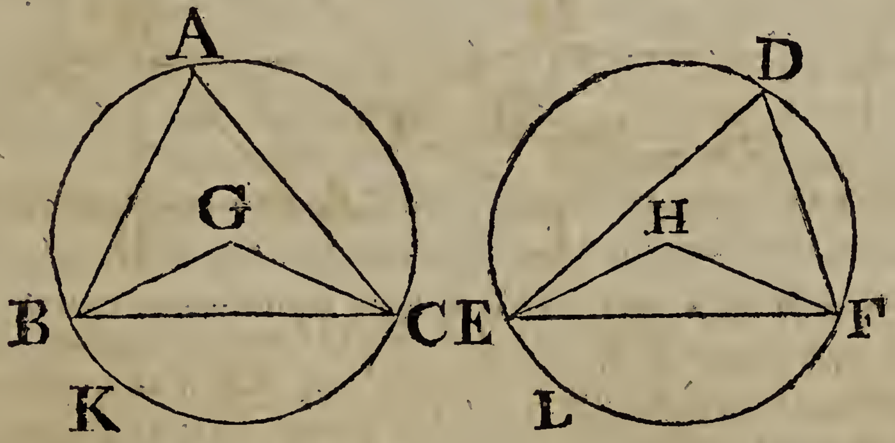
EB, EC are therefore equal to one another; wherefore [d] E is the centre of the circle. From the centre E, at the diſtance of any of the three AE, EB, EC, deſcribe a circle, this ſhall paſs d 9. 3. through the other points; and the circle of which ABC is a ſegment is deſcribed: And it is evident, that if the angle ABD be greater than the angle BAD, the centre E falls without the ſegment ABC, which therefore is leſs than a ſemicircle: But if the angle ABD be leſs than BAD, the centre E falls within the ſegment ABC, which is therefore greater than a ſemicircle: Wherefore a ſegment of a circle being given, the circle is deſcribed of which it is a ſegment. Which was to be done.

PROP. XXVI. THEOR.

IN equal circles, equal angles ſtand upon equal circumferences, whether they be at the centres or circumferences.

Let ABC, DEF be equal circles, and the equal angles BGC, EHF at their centres, and BAC, EDF at their circumferences: The circumference BKC is equal to the circumference ELF.

Join BC, EF; and becauſe the circles ABC, DEF are equal, the ſtraight lines drawn from their centres are equal: Therefore the two ſides BG, GC, are equal to the two EH, HF;



and the angle at G is equal to the angle at H; therefore the baſe BC is equal [a] to the baſe EF: And becauſe the angle at A a 4. 1. is equal to the angle at D, the ſegment BAC is ſimilar [b] to the b 11. def. 3. ſegment EDF; and they are upon equal ſtraight lines BC, EF; but ſimilar ſegments of circles upon equal ſtraight lines are equal [c] to one another, therefore the ſegment BAC is equal to c 24. 3. the ſegment EDF: But the whole circle ABC is equal to the whole

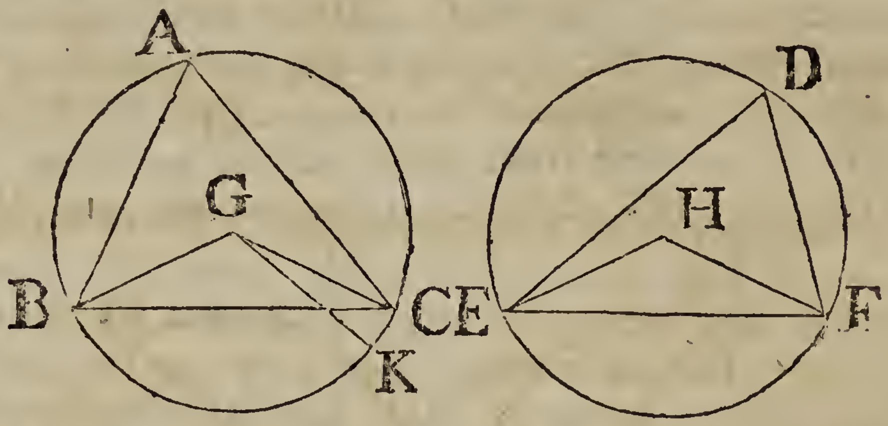
Book III. whole DEF; therefore the remaining ſegment BKC is equal to the remaining ſegment ELF, and the circumference BKC to the circumference ELF. Wherefore, in equal circles, &c. Q. E. D.

PROP. XXVII. THEOR.

IN equal circles, the angles which ſtand upon equal circumferences are equal to one another, whether they be at the centres or circumferences.

Let the angles BGC, EHF at the centres, and BAC, EDF at the circumferences of the equal circles ABC, DEF ſtand upon the equal circumferences BC, EF: The angle BGC is equal to the angle EHF, and the angle BAC to the angle EDF.

a 2. 3. If the angle BGC be equal to the angle EHF, it is manifeſt [a] that the angle BAC is alſo equal to EDF. But, if not, one of them is the greater: Let BGC be the greater, and at the



b 23. 1. point G, in the ſtraight line BG, make [b] the angle BGK equal to the angle EHF; but equal angles ſtand upon equal circumferences [c], when they are at the centre; therefore the circumference BK is equal to the circumference EF: But EF is equal to BC; therefore alſo BK is equal to BC, the leſs to the greater, which is impoſſible: Therefore the angle BGC is not unequal to the angle EHF; that is, it is equal to it: And the angle at A is half of the angle BGC, and the angle at D half of the angle EHF: Therefore the angle at A is equal to the angle at D. Wherefore, in equal circles, &c. Q. E. D.

c 26. 3.

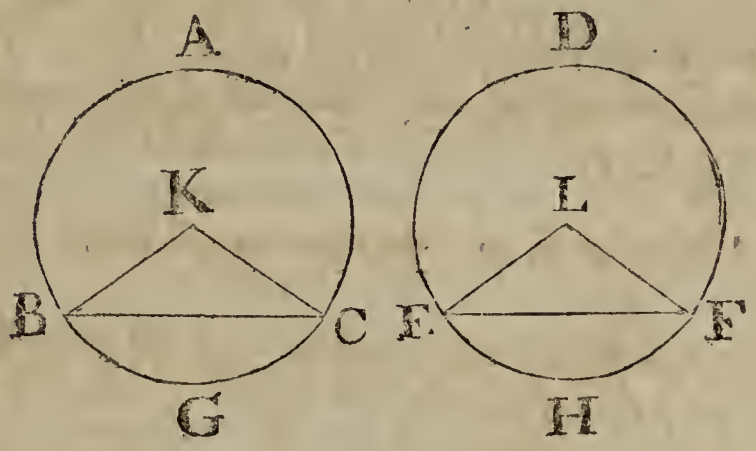
PROP.

PROP. XXVIII. THEOR.

IN equal circles, equal ſtraight lines cut off equal circumferences, the greater equal to the greater, and the leſs to the leſs.

Let ABC, DEF be equal circles, and BC, EF equal ſtraight lines in them, which cut off the two greater circumferences BAC, EDF, and the two leſs BGC, EHF: the greater BAC is equal to the greater EDF, and the leſs BGC to the leſs EHF.

Take [a] K, L the centres of the circles, and join BK, KC, EL, [a] 1. 3. LF: And becauſe the circles are equal, the ſtraight lines from their centres are equal; therefore BK, KC are equal to EL, LF; and the baſe BC is equal to the baſe EF; therefore the angle BKC is equal [b] to the angle ELF: But equal angles ſtand [b] 8. 1. upon equal [c] circumferences, when they are at the centres; [c] 26. 3. therefore the circumference BGC is equal to the circumference EHF. But the whole circle ABC is equal to the whole EDF; the remaining part therefore of the circumference, viz. BAC, is equal to the remaining part EDF. Therefore, in equal circles, &c. Q. E. D.



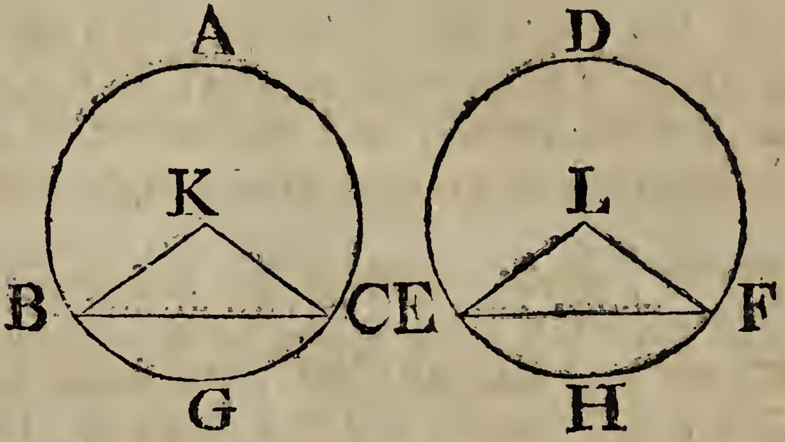
PROP. XXIX. THEOR.

IN equal circles equal circumferences are ſubtended by equal ſtraight lines.

Let ABC, DEF be equal circles, and let the circumferences BGC, EHF alſo be equal; and join BC, EF: The ſtraight line BC is equal to the ſtraight line EF.

Take

a 1. 3. Take[a] K, L the centres of the circles, and join BK, KC, EL, LF: And becauſe the circumference BGC is equal to the



b 27. 3. circumference EHF, the angle BKC is equal[b] to the angle ELF: And becauſe the circles ABC, DEF are equal, the ſtraight lines from their centres are equal: Therefore BK, KC are equal to EL, LF, and they contain equal angles: Therefore the c 4. 1. baſe BC is equal[c] to the baſe EF. Therefore, in equal circles, &c. Q. E. D.

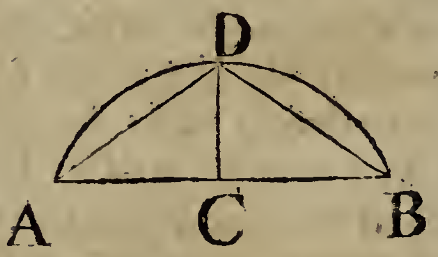
PROP. XXX. PROB.

TO biſect a given circumference, *that is, to divide it into two equal parts.*

Let ADB be the given circumference; it is required to biſect it.

a 10. 1. Join AB, and biſect[a] it in C; from the point C draw CD at right angles to AB, and join AD, DB: the circumference ADB is biſected in the point D.

Becauſe AC is equal to CB, and CD common to the triangles ACD, BCD, the two ſides AC, CD are equal to the two BC, CD; and the angle ACD is equal to the angle BCD, becauſe each of them is a right angle; therefore the baſe AD is equal b 4. 1. [b] to the baſe BD. But equal ſtraight c 28. 3. lines cut off equal[c] circumferences, the greater equal to the greater, and the leſs to the leſs, and AD, DB are each of them d Cor. 1. 3. leſs than a ſemicircle; becauſe DC paſſes through the centre[d]: Wherefore the circumference AD is equal to the circumference DB: Therefore the given circumference is biſected in D. Which was to be done.

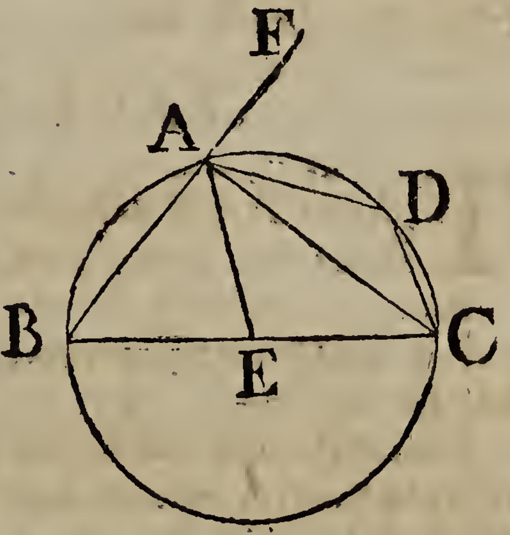


PROP. XXXI. THEOR.

IN a circle, the angle in a ſemicircle is a right angle; but the angle in a ſegment greater than a ſemicircle is leſs than a right angle; and the angle in a ſegment leſs than a ſemicircle is greater than a right angle.

Let ABCD be a circle, of which the diameter is BC, and centre E; and draw CA dividing the circle into the ſegments ABC, ADC, and join BA, AD, DC; the angle in the ſemicircle BAC is a right angle; and the angle in the ſegment ABC, which is greater than a ſemicircle, is leſs than a right angle; and the angle in the ſegment ADC, which is leſs than a ſemicircle, is greater than a right angle.

Join AE, and produce BA to F; and becauſe BE is equal
to EA, the angle EAB is equal [a] to EBA; alſo, becauſe AE a 5. 1.
is equal to EC, the angle EAC is equal to ECA; wherefore the whole angle BAC is equal to the two angles ABC, ACB: But FAC, the exterior angle of the triangle
ABC, is equal [b] to the two angles b 32. 1.
ABC, ACB; therefore the angle BAC is equal to the angle FAC, and each of them is therefore a
right [c] angle: Wherefore the angle c 10.Def. 1.
BAC in a ſemicircle is a right angle.



And becauſe the two angles ABC, BAC of the triangle
ABC are together leſs [d] than two right angles, and that BAC d 17. 1.
is a right angle, ABC muſt be leſs than a right angle; and therefore the angle in a ſegment ABC greater than a ſemicircle, is leſs than a right angle.

And becauſe ABCD is a quadrilateral figure in a circle, any
two of its oppoſite angles are equal [e] to two right angles; there- e 22. 3.
fore the angles ABC, ADC are equal to two right angles; and ABC is leſs than a right angle; wherefore the other ADC is greater than a right angle.

Beſides, it is manifeſt, that the circumference of the greater ſegment ABC falls without the right angle CAB, but the circumference of the leſs ſegment ADC falls within the right angle CAF. ‘ And this is all that is meant, when in the ‘ Greek

Book III. ‘ Greek text, and the tranſlations from it, the angle of the ‘ greater ſegment is ſaid to be greater, and the angle of the leſs ‘ ſegment is ſaid to be leſs, than a right angle.’

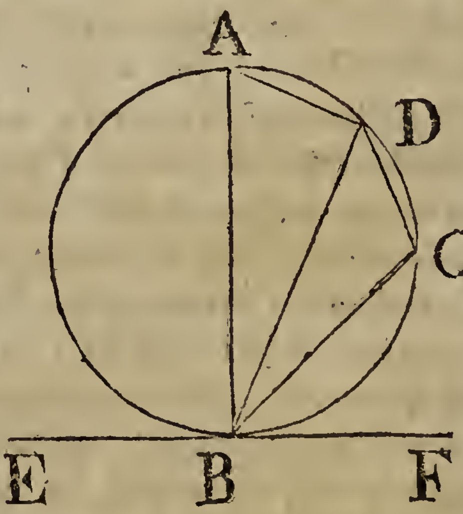
Cor. From this it is manifeſt, that if one angle of a triangle be equal to the other two, it is a right angle, becauſe the angle adjacent to it is equal to the ſame two; and when the adjacent angles are equal, they are right angles.

PROP. XXXII. THEOR.

IF a ſtraight line touches a circle, and from the point of contact a ſtraight line be drawn cutting the circle, the angles made by this line with the line touching the circle, ſhall be equal to the angles which are in the alternate ſegments of the circle.

Let the ſtraight line EF touch the circle ABCD in B, and from the point B let the ſtraight line BD be drawn cutting the circle: The angles which BD makes with the touching line EF ſhall be equal to the angles in the alternate ſegments of the circle: that is, the angle FBD is equal to the angle which is in the ſegment DAB, and the angle DBE to the angle in the ſegment BCD.

a 11. 1. From the point B draw [a] BA at right angles to EF, and take any point C in the circumference BD, and join AD, DC, CB; and becauſe the ſtraight line EF touches the circle ABCD in the point B, and BA is drawn at right angles to the touching line from the point of contact B, the b 19. 3. center of the circle is [b] in BA; therefore the angle ADB in a ſemi- c 31. 3. circle is a right [c] angle, and conſequently the other two angles BAD, d 32. 1. ABD are equal [d] to a right angle: But ABF is likewiſe a right angle; therefore the angle ABF is equal to the angles BAD, ABD: Take from theſe equals the common angle ABD; therefore the remaining angle DBF is equal to the angle BAD, which is in the alternate ſegment of the circle; and becauſe ABCD is a quadrilateral figure in a circle, the oppoſite e 22. 3. angles BAD, BCD are equal [e] to two right angles; therefore the

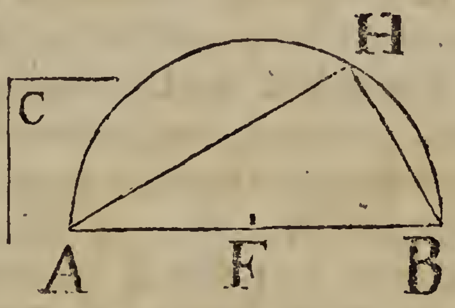


the angles DBF, DBE, being likewiſe equal [f] to two right angles, are equal to the angles BAD, BCD; and DBF has been proved equal to BAD: Therefore the remaining angle DBE is equal to the angle BCD in the alternate ſegment of the circle. Wherefore, if a ſtraight line, &c. Q. E. D. Book III. f 13. 1.

PROP. XXXIII. PROB.

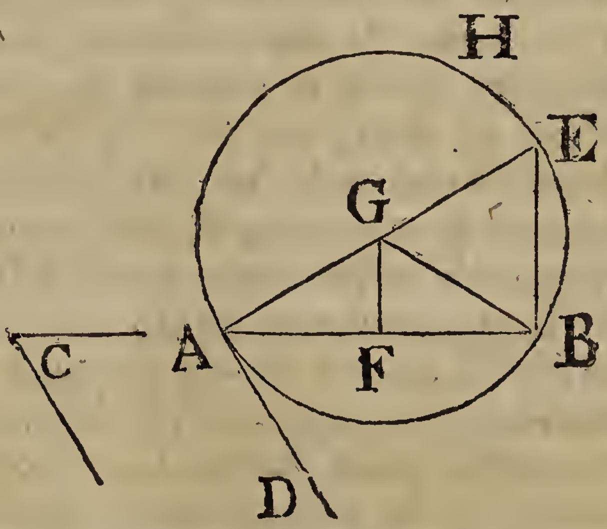
UPON a given ſtraight line to deſcribe a ſegment of a circle, containing an angle equal to a given rectilineal angle. See N.

Let AB be the given ſtraight line, and the angle at C the given rectilineal angle; it is required to deſcribe upon the given ſtraight line AB a ſegment of a circle, containing an angle equal to the angle C.



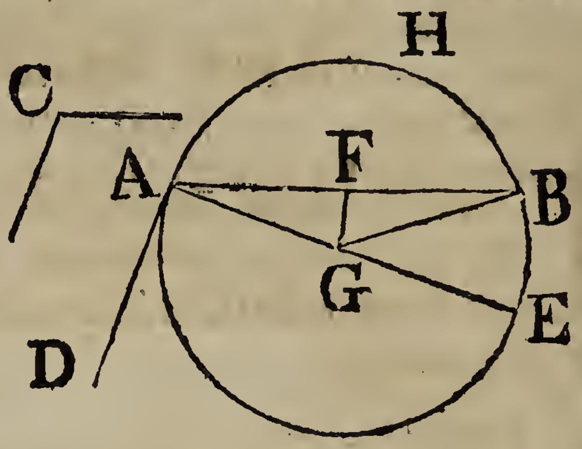
Firſt, let the angle at C be a right angle, and biſect [a] AB in F, and from the centre F, at the diſtance FB, deſcribe the ſemicircle AHB; therefore the angle AHB in a ſemicircle is [b] equal to the right angle at C. a 10. 1. b 31. 3.

But, if the angle C be not a right angle, at the point A, in the ſtraight line AB, make [c] the angle BAD equal to the angle C, and from the point A draw [d] AE at right angles to AD; biſect [a] AB in F, and from F draw [d] FG at right angles to AB, and join GB: And becauſe AF is equal to FB, and FG common to the triangles AFG, BFG, the two ſides AF, FG are equal to the two BF, FG; and the angle AFG is equal to the angle BFG; therefore the baſe AG is equal [e] to the baſe GB; and the circle deſcribed from the centre G, at the diſtance GA, ſhall paſs through the point B; let this be the circle AHB: And becauſe from the point A the extremity of the diameter AE, AD is drawn at right c 23. 1. d 11. 1. e 4. 1.



Book III.
f Cor. 16. 3.
g 32. 3.

right angles to AE, therefore AD [f] touches the circle; and becauſe AB drawn from the point of contact A cuts the circle, the angle DAB is equal to the angle in the alternate ſegment AHB [g]: But the angle DAB is equal to the angle C, therefore alſo the angle C is equal to the angle in the ſegment AHB: Wherefore, upon the given ſtraight line AB the ſegment AHB of a circle is deſcribed, which contains an angle equal to the given angle at C. Which was to be done.



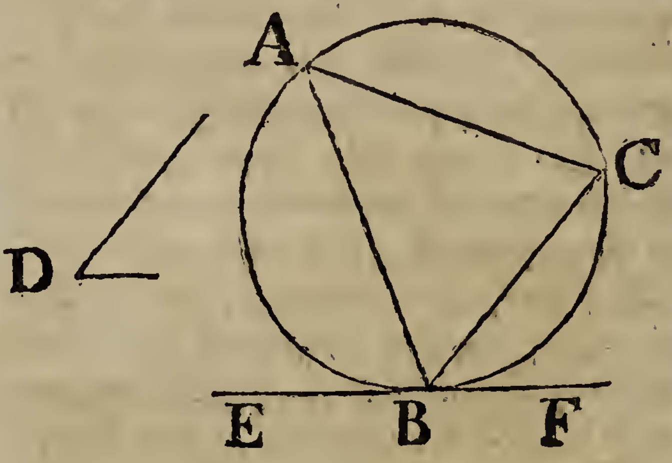
PROP. XXXIV. PROB.

TO cut off a ſegment from a given circle which ſhall contain an angle equal to a given rectilineal angle.

Let ABC be the given circle, and D the given rectilineal angle; it is required to cut off a ſegment from the circle ABC that ſhall contain an angle equal to the given angle D.

a 17. 3.
b 23. 1.
c 32. 3.

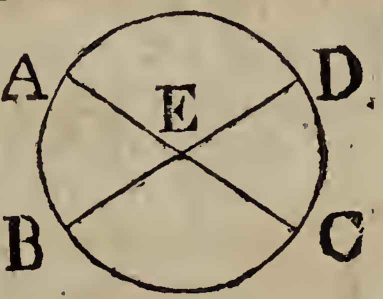
Draw [a] the ſtraight line EF touching the circle ABC in the point B, and at the point B, in the ſtraight line BF make [b] the angle FBC equal to the angle D: Therefore, becauſe the ſtraight line EF touches the circle ABC, and BC is drawn from the point of contact B, the angle FBC is equal [c] to the angle in the alternate ſegment BAC of the circle: But the angle FBC is equal to the angle D; therefore the angle in the ſegment BAC is equal to the angle D: Wherefore the ſegment BAC is cut off from the given circle ABC containing an angle equal to the given angle D: Which was to be done.



PROP. XXXV. THEOR.

IF two ſtraight lines within a circle cut one another, the rectangle contained by the ſegments of one of them is equal to the rectangle contained by the ſegments of the other. See N.

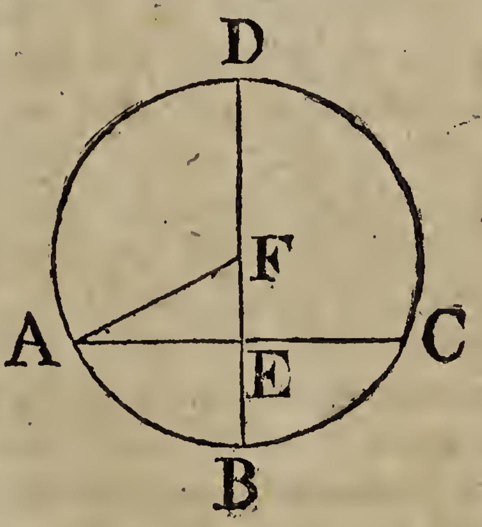
Let the two ſtraight lines AC, BD, within the circle ABCD, cut one another in the point E: the rectangle contained by AE, EC is equal to the rectangle contained by BE, ED.



If AC, BD paſs each of them through the centre, ſo that E is the centre; it is evident, that AE, EC, BE, ED, being all equal, the rectangle AE, EC is likewiſe equal to the rectangle BE, ED.

But let one of them BD paſs through the centre, and cut the other AC which does not paſs through the centre, at right angles, in the point E: Then, if BD be biſected in F, F is the centre of the circle ABCD; join AF: And becauſe BD, which paſſes through the centre, cuts the ſtraight line AC, which does not paſs through the centre, at right angles in E, AE, EC are equal [a] to one another: And becauſe the ſtraight line BD is cut into two equal parts in the point F, and into two unequal in the point E, the rectangle BE, ED together with the ſquare of EF, is equal [b] to the ſquare of FB; that is, to the ſquare of FA; but the ſquares of AE, EF are equal [c] to the ſquare of FA; therefore the rectangle BE, ED, together with the ſquare of EF, is equal to the ſquares of AE, EF: Take away the common ſquare of EF, and the remaining rectangle BE, ED is equal to the remaining ſquare of AE; that is, to the rectangle AE, EC.

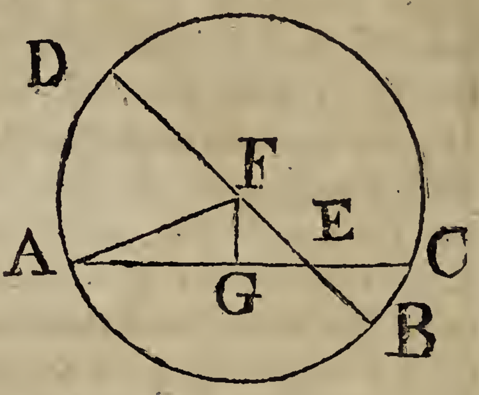
a 3. 3.
b 5. 2.
c 47. 1



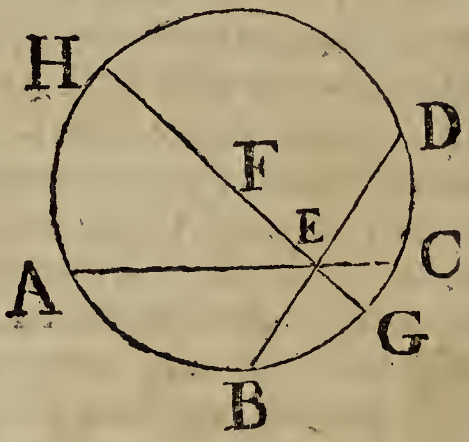
Next, Let BD which paſſes through the centre, cut the other AC, which does not paſs through the centre, in E, but not at right angles: Then, as before, if BD be biſected in F, F is the centre of the circle. Join AF, and from F draw [d] FG per- d 12. 1.

pendicular to AC; therefore AG is equal [a] to GC; wherefore the rectangle AE, EC, together with the ſquare of EG, is equal [b] to the ſquare of AG: To each of theſe equals add the ſquare of GF; therefore the rectangle AE, EC, together with the ſquares of EG, GF, is equal to the ſquares of AG, GF: But the ſquares of EG, GF are equal [c] to the ſquare of EF; and the ſquares of AG, GF are equal to the ſquare of AF: Therefore the rectangle AE, EC, together with the ſquare of EF, is equal to the ſquare of AF; that is, to the ſquare of FB: But the ſquare of FB is equal [b] to the rectangle BE, ED, together with the ſquare of EF; therefore the rectangle AE, EC, together with the ſquare of EF, is equal to the rectangle BE, ED, together with the ſquare of EF: Take away the common ſquare of EF, and the remaining rectangle AE, EC is therefore equal to the remaining rectangle BE, ED.

[a] 3. 3.
[b] 5. 2.
[c] 47. 1.



Laſtly, Let neither of the ſtraight lines AC, BD paſs through the centre: Take the centre F, and through E, the interſection of the ſtraight lines AC, DB, draw the diameter GEFH: And becauſe the rectangle AE, EC is equal, as has been ſhewn, to the rectangle GE, EH; and, for the ſame reaſon, the rectangle BE, ED is equal to the ſame rectangle GE, EH; therefore the rectangle AE, EC is equal to the rectangle BE, ED. Wherefore, if two ſtraight lines, &c. Q. E. D.

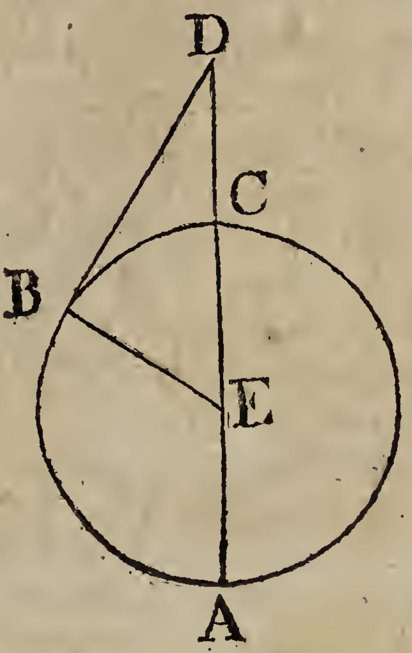


PROP. XXXVI. THEOR.

IF from any point without a circle two ſtraight lines be drawn, one of which cuts the circle, and the other touches it; the rectangle contained by the whole line which cuts the circle, and the part of it without the circle, ſhall be equal to the ſquare of the line which touches it.

Let D be any point without the circle ABC, and DCA, DB two ſtraight lines drawn from it, of which DCA cuts the circle, and DB touches the ſame: The rectangle AD, DC is equal to the ſquare of DB.

Either DCA paſſes through the centre, or it does not; firſt, let it paſs through the centre E, and join EB; therefore the angle EBD is a right [a] angle: And becauſe the ſtraight line AC is biſected in E, and produced to the point D, the rectangle AD, DC, together with the ſquare of EC, is equal [b] to the ſquare of ED, and CE is equal to EB: Therefore the rectangle AD, DC, together with the ſquare of EB, is equal to the ſquare of ED: But the ſquare of ED is equal [c] to the ſquares of EB, BD, becauſe EBD is a right angle: Therefore the rectangle AD, DC, together with the ſquare of EB, is equal to the ſquares of EB, BD: Take away the common ſquare of EB; therefore the remaining rectangle AD, DC is equal to the ſquare of the tangent DB.

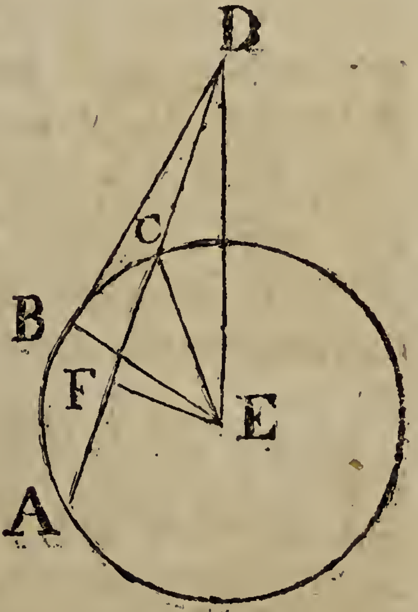


a 18. 3.

b 6. 2.

c 47. 1.

But if DCA does not paſs through the centre of the circle ABC, take [d] the centre E, and draw EF perpendicular [e] to AC, and join EB, EC, ED: And becauſe the ſtraight line EF, which paſſes through the centre, cuts the ſtraight line AC, which does not paſs through the centre, at right angles, it ſhall likewiſe biſect [f] it; therefore AF is equal to FC: And becauſe the ſtraight line AC is biſected in F, and produced to D, the rectangle AD, DC, together with the ſquare of FC, is equal [b] to the ſquare of FD: To each of theſe equals add the ſquare of FE; therefore the rectangle AD, DC, together with the ſquares of CF, FE, is equal to the ſquares of DF, FE: But the ſquare of ED is equal [c] to the ſquares of DF, FE, becauſe EFD is a right angle; and the ſquare of EC is equal to the ſquares of CF, FE; therefore the rectangle AD, DC, together with the ſquare of EC, is equal to the ſquare of ED: And CE is equal to EB; therefore the rectangle AD, DC, together with the ſquare of EB, is equal to the ſquare of ED:



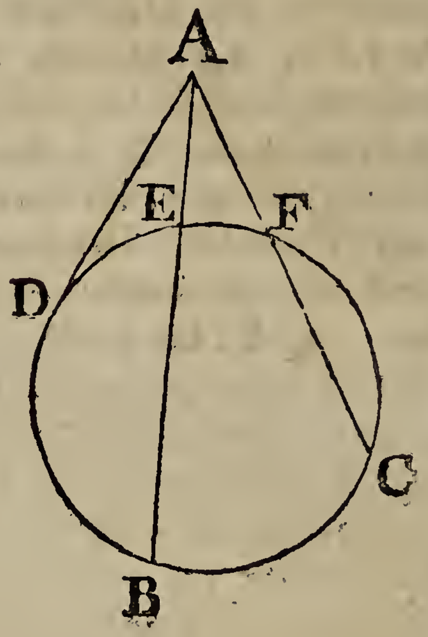
d 1. 3.

e 12. 1.

f 3. 3.

Book III. But the ſquares of EB, BD are equal to the ſquare [c] of ED, becauſe EBD is a right angle; therefore the rectangle AD, DC, together with the ſquare of EB, is equal to the ſquares of EB, BD: Take away the common ſquare of EB; therefore the remaining rectangle AD, DC is equal to the ſquare of DB. Wherefore, if from any point, &c. Q. E. D.

c 47. 1.



Cor. If from any point without a circle, there be drawn two ſtraight lines cutting it, as AB, AC, the rectangles contained by the whole lines and the parts of them without the circle, are equal to one another, viz. the rectangle BA, AE to the rectangle CA, AF: For each of them is equal to the ſquare of the ſtraight line AD which touches the circle.

PROP. XXXVII. THEOR.

See N.

If from a point without a circle there be drawn two ſtraight lines, one of which cuts the circle, and the other meets it; if the rectangle contained by the whole line which cuts the circle, and the part of it without the circle be equal to the ſquare of the line which meets it, the line which meets ſhall touch the circle.

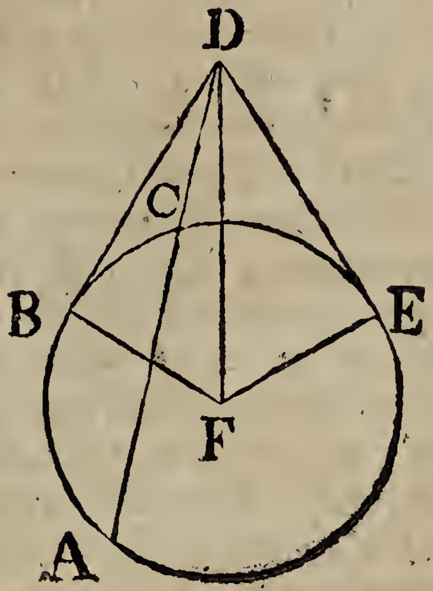
Let any point D be taken without the circle ABC, and from it let two ſtraight lines DCA and DB be drawn, of which DCA cuts the circle, and DB meets it; if the rectangle AD, DC be equal to the ſquare of DB; DB touches the circle.

a 17. 3. Draw [a] the ſtraight line DE touching the circle ABC, find
b 18. 3. its centre F, and join FE, FB, FD; then FED is a right [b] angle: And becauſe DE touches the circle ABC, and DCA cuts
c 36. 3. it, the rectangle AD, DC is equal [c] to the ſquare of DE: But the rectangle AD, DC is, by hypotheſis, equal to the ſquare of DB: Therefore the ſquare of DE is equal to the ſquare of DB; and the ſtraight line DE equal to the ſtraight line DB: And

FE is equal to FB, wherefore DE, EF are equal to DB, BF; and the baſe FD is common to the two triangles DEF, DBF; therefore the angle DEF is equal [d] to the angle DBF; but DEF is a right angle, therefore alſo DBF is a right angle: And FB, if produced, is a diameter, and the ſtraight line which is drawn at right angles to a diameter, from the extremity of it, touches [e] the circle: Therefore DB touches the circle ABC. Wherefore, if from a point, &c. Q. E. D.

d 8. 1.

e 16. 3.



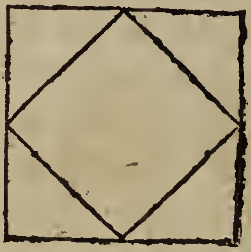
THE
ELEMENTS
OF
EUCLID.

BOOK IV.

DEFINITIONS.

I.

See N. A RECTILINEAL figure is ſaid to be inſcribed in another rectilineal figure, when all the angles of the inſcribed figure are upon the ſides of the figure in which it is inſcribed, each upon each.

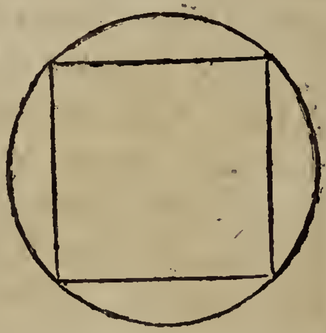


II.

In like manner, a figure is ſaid to be deſcribed about another figure, when all the ſides of the circumſcribed figure paſs through the angular points of the figure about which it is deſcribed, each through each.

III.

A rectilineal figure is ſaid to be inſcribed in a circle, when all the angles of the inſcribed figure are upon the circumference of the circle.



IV.

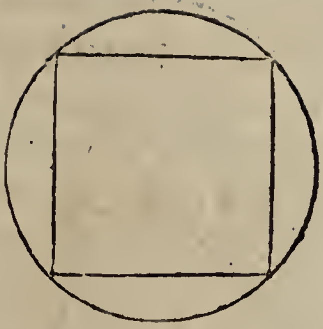
A rectilineal figure is ſaid to be deſcribed about a circle, when each ſide of the circumſcribed figure touches the circumference of the circle.

V.

In like manner, a circle is ſaid to be inſcribed in a rectilineal figure, when the circumference of the circle touches each ſide of the figure.

VI.

A circle is ſaid to be deſcribed about a rectilineal figure, when the circumference of the circle paſſes through all the angular points of the figure about which it is deſcribed.



VII.

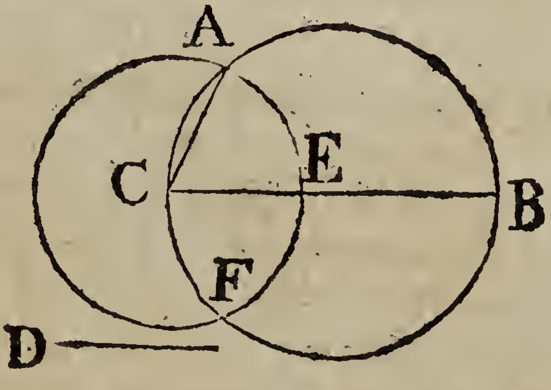
A ſtraight line is ſaid to be placed in a circle, when the extremities of it are in the circumference of the circle.

PROP. I. PROB.

IN a given circle to place a ſtraight line, equal to a given ſtraight line not greater than the diameter of the circle.

Let ABC be the given circle, and D the given ſtraight line, not greater than the diameter of the circle.

Draw BC the diameter of the circle ABC; then, if BC is equal to D, the thing required is done; for in the circle ABC a ſtraight line BC is placed equal to D: But, if it is not, BC is greater than D; make CE equal [a] to D, and from the centre C, at the diſtance CE, deſcribe the circle AEF, and join CA: Therefore, becauſe C is the centre of the circle AEF, CA is equal to CE; but D is equal to CE; therefore D is equal to CA: Wherefore, in the circle ABC, a ſtraight line is placed equal to the given ſtraight line D, which is not greater than the diameter of the circle. Which was to be done.



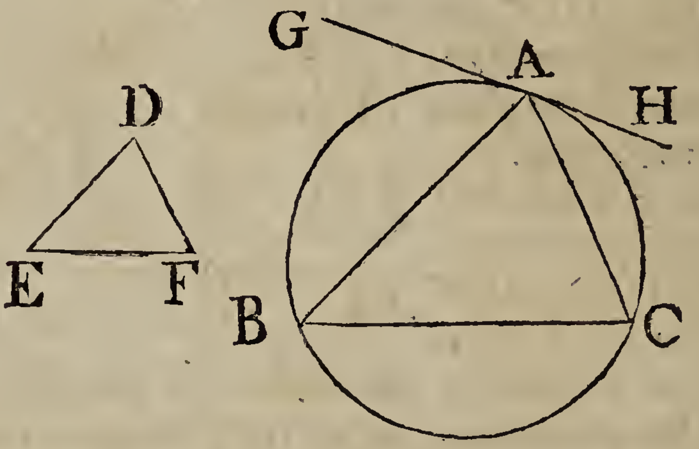
[a] 3. 1.

PROP. II. PROB.

IN a given circle to inſcribe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle; it is required to inſcribe in the circle ABC a triangle equiangular to the triangle DEF.

a 17. 3. Draw [a] the ſtraight line GAH touching the circle in the
b 23. 1. point A, and at the point A, in the ſtraight line AH, make [b]
the angle HAC equal to the angle DEF; and at the point A, in the ſtraight line AG, make the angle GAB equal to the angle DFE, and join BC: Therefore, becauſe HAG touches the circle ABC, and AC is drawn from the point of contact, the angle HAC is e-
c 32. 3. qual [c] to the angle ABC in the alternate ſegment of the circle: But HAC is equal to the angle DEF; therefore alſo the angle ABC is equal to DEF: For the ſame reaſon, the angle ACB is equal to the angle
d 32. 1. DFE; therefore the remaining angle BAC is equal [d] to the remaining angle EDF: Wherefore the triangle ABC is equiangular to the triangle DEF, and it is inſcribed in the circle ABC. Which was to be done.



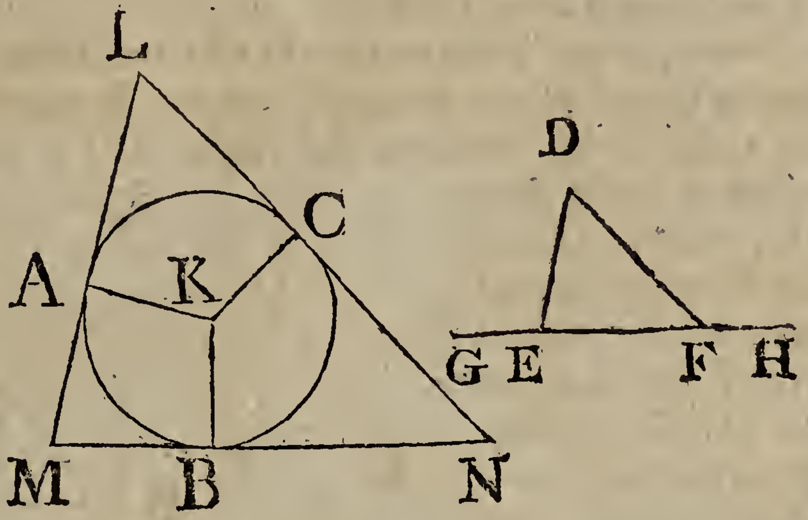
PROP. III. PROB.

ABOUT a given circle to deſcribe a triangle equiangular to a given triangle.

Let ABC be the given circle, and DEF the given triangle; it is required to deſcribe a triangle about the circle ABC equiangular to the triangle DEF.

Produce EF both ways to the points G, H, and find the centre K of the circle ABC, and from it draw any ſtraight line
a 23. 1. KB; at the point K, in the ſtraight line KB, make [a] the angle BKA equal to the angle DEG, and the angle BKC equal to the angle DFH; and through the points A, B, C, draw the
b 17. 3. ſtraight lines LAM, MBN, NCL, touching [b] the circle ABC: Therefore, becauſe LM, MN, NL touch the circle ABC in the points A, B, C, to which from the centre are drawn KA, KB,
c 18. 3. KC, the angles at the points A, B, C, are right [c] angles: And becauſe the four angles of the quadrilateral figure AMBK are

equal



equal to four right angles, for it can be divided into two triangles: and that two of them KAM, KBM are right angles, the other two AKB, AMB are equal to two right angles: But the angles DEG, DEF are likewiſe equal [d] to two right angles; therefore the angles AKB, AMB are equal to the angles DEG, DEF, of which AKB is equal to DEG; wherefore the remaining angle AMB is equal to the remaining angle DEF: In like manner, the angle LNM may be demonſtrated to be equal to DFE; and therefore the remaining angle MLN is equal [e] to the remaining angle EDF: Wherefore the triangle LMN is equiangular to the triangle DEF: And it is deſcribed about the circle ABC. Which was to be done.

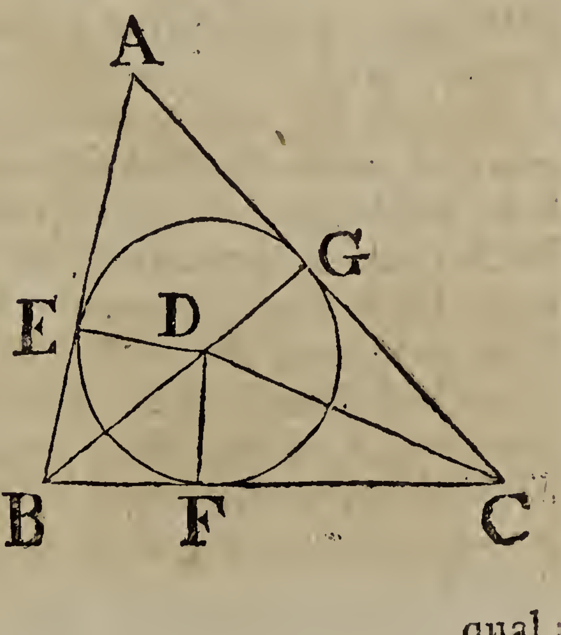
d 13. 1.
e 32. 1.

PROP. IV. PROB.

TO inſcribe a circle in a given triangle.

See N.

Let the given triangle be ABC; it is required to inſcribe a circle in ABC.



Biſect [a] the angles ABC, BCA by the ſtraight lines BD, CD meeting one another in the point D, from which draw [b] DE, DF, DG perpendiculars to AB, BC, CA: And becauſe the angle EBD is equal to the angle FBD, for the angle ABC is biſected by BD, and that the right angle BED is equal to the right angle BFD, the two triangles EBD, FBD have two angles of the one equal to two angles of the other, and the ſide BD, which is oppoſite to one of the equal angles in each, is common to both; therefore their other ſides ſhall be e-

a 9. 1.
b 12. 1.

qual c; wherefore DE is equal to DF: For the ſame reaſon, DG is equal to DF; therefore the three ſtraight lines DE, DF, DG are equal to one another, and the circle deſcribed from the centre D, at the diſtance of any of them, ſhall paſs through the extremities of the other two, and touch the ſtraight lines AB, BC, CA, becauſe the angles at the points E, F, G are right angles, and the ſtraight line which is drawn from the extremity of a diameter at right angles to it, touches d the circle: Therefore the ſtraight lines AB, BC, CA do each of them touch the circle, and the circle EFG is inſcribed in the triangle ABC. Which was to be done.

c 26. 1.

d 16. 3.

PROP. V. PROB.

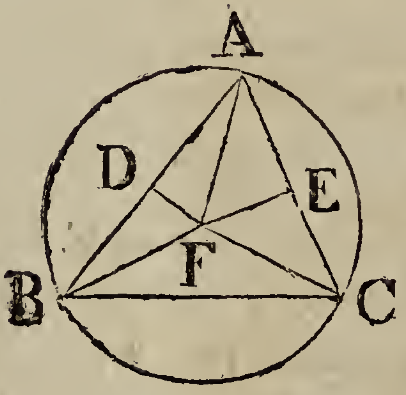
See N.

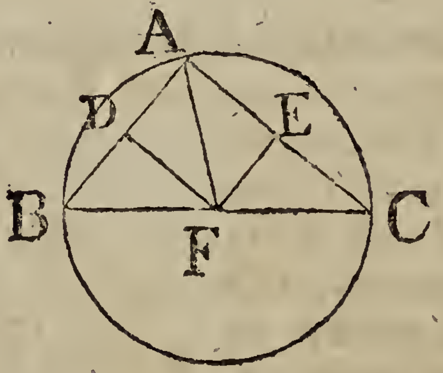
TO deſcribe a circle about a given triangle.

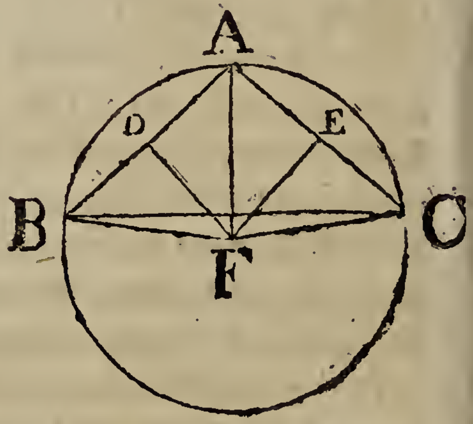
Let the given triangle be ABC; it is required to deſcribe a circle about ABC.

a 10. 1. Biſect a AB, AC in the points D, E, and from theſe points

b 11. 1. draw DF, EF at right angles b to AB, AC; DF, EF produced







meet one another; For, if they do not meet, they are parallel, wherefore AB, AC, which are at right angles to them, are parallel; which is abſurd: Let them meet in F, and join FA; alſo, if the point F be not in BC, join BF, CF: Then, becauſe AD is equal to DB, and DF common, and at right angles to AB, the baſe AF is equal c to the baſe FB: In like manner, it may be ſhown that CF is equal to FA; and therefore BF is equal to FC; and FA, FB, FC are equal to one another; wherefore

c 4. 1.

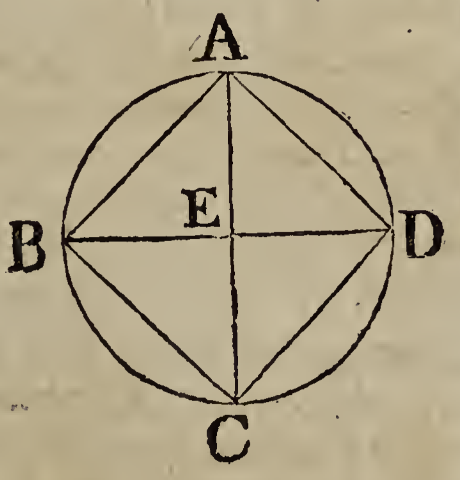
wherefore the circle deſcribed from the centre F, at the diſtance of one of them, ſhall paſs through the extremities of the other two, and be deſcribed about the triangle ABC. Which was to be done.

Cor. And it is manifeſt, that when the centre of the circle falls within the triangle, each of its angles is leſs than a right angle, each of them being in a ſegment greater than a ſemicircle; but, when the centre is in one of the ſides of the triangle, the angle oppoſite to this ſide, being in a ſemicircle, is a right angle; and, if the centre falls without the triangle, the angle oppoſite to the ſide beyond which it is, being in a ſegment leſs than a ſemicircle, is greater than a right angle: Wherefore, if the given triangle be acute angled, the centre of the circle falls within it; if it be a right angled triangle, the centre is in the ſide oppoſite to the right angle; and, if it be an obtuſe angled triangle, the centre falls without the triangle, beyond the ſide oppoſite to the obtuſe angle.

PROP. VI. PROB.

To inſcribe a ſquare in a given circle.

Let ABCD be the given circle; it is required to inſcribe a ſquare in ABCD.



Draw the diameters AC, BD at right angles to one another; and join AB, BC, CD, DA; becauſe BE is equal to ED, for E is the centre, and that EA is common, and at right angles to BD; the baſe BA is equal [a] to the baſe AD; and, for the ſame reaſon, BC, CD are each of them equal to BA or AD; therefore the quadrilateral figure ABCD is equilateral. It is alſo rectangular; for the ſtraight line BD, being the diameter of the circle ABCD, BAD is a ſemicircle; wherefore the angle BAD is a right [b] angle; for the ſame reaſon each of the angles ABC, BCD, CDA is a right angle; therefore the quadrilateral figure ABCD is rectangular, and it has been ſhewn to be equilateral; therefore it is a ſquare; and it is inſcribed in the circle ABCD. Which was to be done.

a 4. 1.

b 31. 3.

PROP.

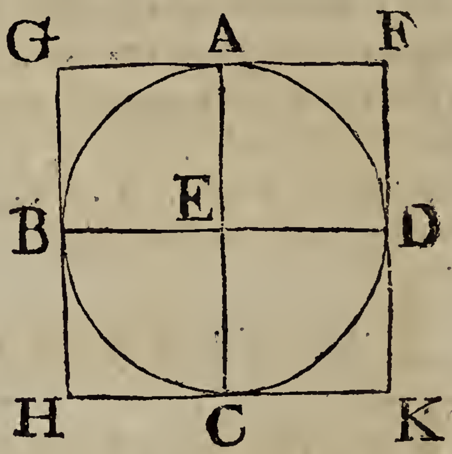
Book IV.

PROP. VII. PROB.

TO deſcribe a ſquare about a given circle.

Let ABCD be the given circle; it is required to deſcribe a ſquare about it.

Draw two diameters AC, BD of the circle ABCD, at right angles to one another, and through the points A, B, C, D
a 17. 3. draw [a] FG, GH, HK, KF touching the circle; and becauſe FG touches the circle ABCD, and EA is drawn from the centre
b 18. 3. E to the point of contact A, the angles at A are right [b] angles; for the ſame reaſon, the angles at the points B, C, D are right angles; and becauſe the angle AEB is a right angle, as likewiſe is EBG, GH
c 28. 1. is parallel [c] to AC; for the ſame reaſon, AC is parallel to FK, and in like manner GF, HK may each of them be demonſtrated to be parallel to BED; therefore the figures GK, GC, AK, FB, BK are parallelograms; and GF
d 34. 1. is therefore equal [d] to HK, and GH to FK; and becauſe AC is equal to BD, and that AC is equal to each of the two GH, FK; and BD to each of the two GF, HK: GH, FK are each of them equal to GF or HK; therefore the quadrilateral figure FGHK is equilateral. It is alſo rectangular; for GBEA being a parallelogram, and AEB a right angle, AGB [d] is likewiſe a right angle: In the ſame manner, it may be ſhown that the angles at H, K, F are right angles; therefore the quadrilateral figure FGHK is rectangular, and it was demonſtrated to be equilateral; therefore it is a ſquare; and it is deſcribed about the circle ABCD. Which was to be done.



PROP. VIII. PROB.

TO inſcribe a circle in a given ſquare.

Let ABCD be the given ſquare; it is required to inſcribe a circle in ABCD.

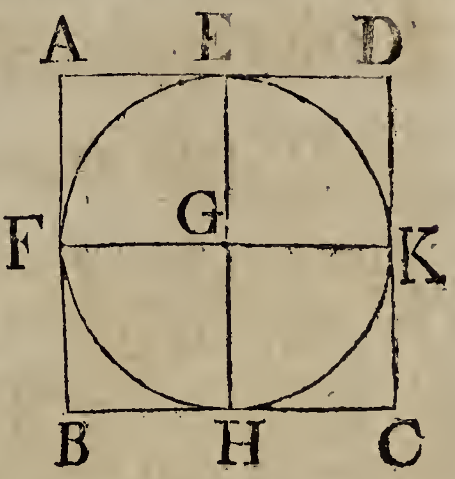
a 10. 1. Biſect [a] each of the ſides AB, AD, in the points F, E, and
b 31. 1. through E draw [b] EH parallel to AB or DC, and through F draw

draw FK parallel to AD or BC; therefore each of the figures AK, KB, AH, HD, AG, GC, BG, GD is a parallelogram, and their oppoſite ſides are equal [c]; and becauſe AD is equal to AB, and that AE is the half of AD, and AF the half of AB, AE is equal to AF; wherefore the ſides oppoſite to theſe are equal, viz. FG to GE; in the ſame manner, it may be demonſtrated that GH, GK are each of them equal to FG or GE; therefore the four ſtraight lines GE, GF, GH, GK, are equal to one another; and the circle deſcribed from the centre G, at the diſtance of one of them, ſhall paſs thro' the extremities of the other three, and touch the ſtraight lines AB, BC, CD, DA; becauſe the angles at the points E, F, H, K are right [d] angles, and that the ſtraight line which is drawn from the extremity of a diameter, at right angles to it, touches the circle [e]; therefore each of the ſtraight lines AB, BC, CD, DA touches the circle, which therefore is inſcribed in the ſquare ABCD. Which was to be done.

c 34. 1.

d 29. 1.

e 16. 3.



PROP. IX. PROB.

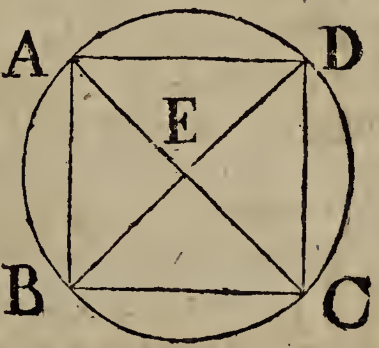
TO deſcribe a circle about a given ſquare.

Let ABCD be the given ſquare; it is required to deſcribe a circle about it.

Join AC, BD cutting one another in E; and becauſe DA is equal to AB, and AC common to the triangles DAC, BAC, the two ſides DA, AC are equal to the two BA, AC, and the baſe DC is equal to the baſe BC; wherefore the angle DAC is equal [a] to the angle BAC, and the angle DAB is biſected by the ſtraight line AC: In the ſame manner, it may be demonſtrated that the angles ABC, BCD, CDA are ſeverally biſected by the ſtraight lines BD, AC; therefore, becauſe the angle DAB is equal to the angle ABC, and that the angle EAB is the half of DAB, and EBA the half of ABC; the angle EAB is equal to the angle EBA; wherefore the ſide EA is equal [b] to the ſide EB: In the ſame manner, it may be demonſtrated

a 8. 1.

b 6. 1.



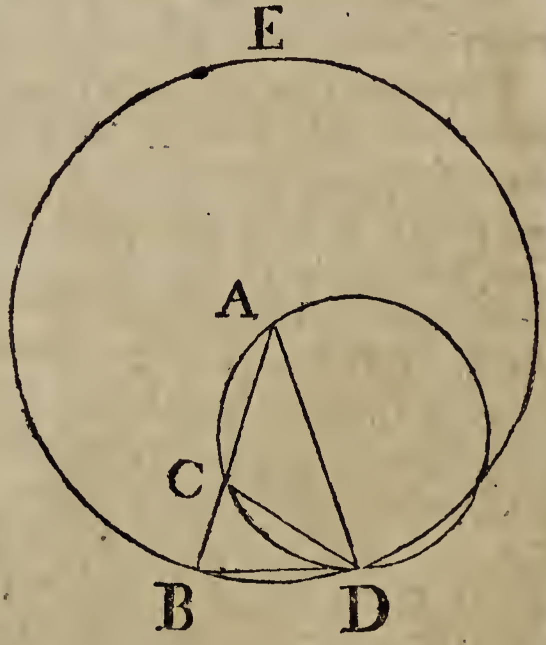
demonſtrated that the ſtraight lines EC, ED are each of them equal to EA or EB; therefore the four ſtraight lines EA, EB, EC, ED are equal to one another; and the circle deſcribed from the centre E, at the diſtance of one of them, ſhall paſs through the extremities of the other three, and be deſcribed about the ſquare ABCD. Which was to be done.

PROP. X. PROB.

TO deſcribe an iſoſceles triangle, having each of the angles at the baſe double of the third angle.

a 11. 2. Take any ſtraight line AB, and divide[a] it in the point C, ſo that the rectangle AB, BC be equal to the ſquare of CA; and from the centre A, at the diſtance AB, deſcribe the circle BDE,
b 1. 4. in which place[b] the ſtraight line BD equal to AC, which is not greater than the diameter of the circle BDE; join DA, DC,
c 5. 4. and about the triangle ADC deſcribe[c] the circle ACD; the triangle ABD is ſuch as is required, that is, each of the angles ABD, ADB is double of the angle BAD.

Becauſe the rectangle AB, BC is equal to the ſquare of AC, and that AC is equal to BD, the rectangle AB, BC is equal to the ſquare of BD; and becauſe from the point B without the circle ACD two ſtraight lines BCA, BD are drawn to the circumference, one of which cuts, and the other meets the circle, and that the rectangle AB, BC contained by the whole of the cutting line, and the part of it without the circle, is equal to the ſquare of BD which meets it;
d 37. 3. the ſtraight line BD touches[d] the circle ACD; and becauſe BD touches the circle, and DC is drawn from the point of contact D, the angle BDC is equal
e 32. 3. [e] to the angle DAC in the alternate ſegment of the circle; to each of theſe add the angle CDA; therefore the whole angle BDA is equal to the two
f 32. 1. angles CDA, DAC; but the exterior angle BCD is equal[f] to the angles CDA, DAC; therefore alſo BDA is equal to BCD; but



but BDA is equal [g] to the angle CBD, becauſe the ſide AD is equal to the ſide AB; therefore CBD, or DBA is equal to BCD; and conſequently the three angles BDA, DBA, BCD, are equal to one another; and becauſe the angle DBC is equal to the angle BCD, the ſide BD is equal [h] to the ſide DC; but BD was made equal to CA; therefore alſo CA is equal to CD, and the angle CDA equal [g] to the angle DAC; therefore the angles CDA, DAC together, are double of the angle DAC: But BCD is equal to the angles CDA, DAC; therefore alſo BCD is double of DAC, and BCD is equal to each of the angles BDA, DBA; each therefore of the angles BDA, DBA is double of the angle DAB; wherefore an iſoſceles triangle ABD is deſcribed, having each of the angles at the baſe double of the third angle. Which was to be done.

[g] 5. 1.

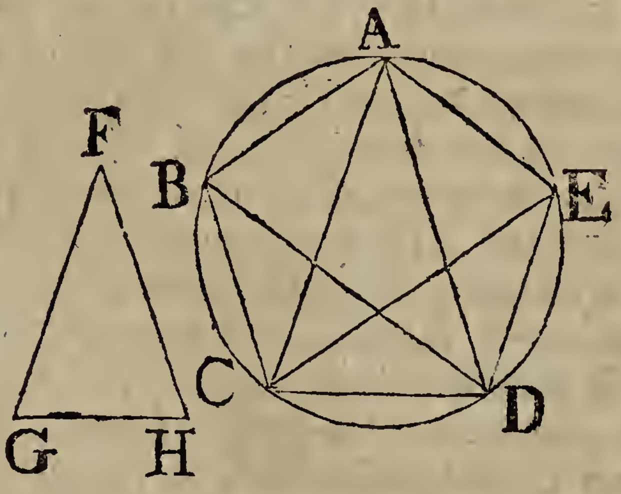
[h] 6. 1.

PROP. XI. PROB.

TO inſcribe an equilateral and equiangular pentagon in a given circle.

Let ABCDE be the given circle; it is required to inſcribe an equilateral and equiangular pentagon in the circle ABCDE.

Deſcribe [a] an iſoſceles triangle FGH, having each of the angles at G, H, double of the angle at F; and in the circle ABCDE inſcribe [b] the triangle ACD equiangular to the triangle FGH, ſo that the angle CAD be equal to the angle at F, and each of the angles ACD, CDA equal to the angle at G or H; wherefore each of the angles ACD, CDA is double of the angle CAD. Biſect [c] the angles ACD, CDA by the ſtraight lines CE, DB; and join AB, BC, DE, EA. ABCDE is the pentagon required.



Becauſe each of the angles ACD, CDA is double of CAD, and are biſected by the ſtraight lines CE, DB, the five angles DAC, ACE, ECD, CDB, BDA are equal to one another; but equal angles ſtand upon equal [d] circumferences; therefore the five circumferences AB, BC, CD, DE, EA are equal to one another:

[a] 10. 4.

[b] 2. 4.

[c] 9. 1.

[d] 26. 3.

another: And equal circumferences are ſubtended by equal [e] ſtraight lines; therefore the five ſtraight lines AB, BC, CD, DE, EA are equal to one another. Wherefore the pentagon ABCDE is equilateral. It is alſo equiangular; becauſe the circumference AB is equal to the circumference DE: If to each be added BCD, the whole ABCD is equal to the whole EDCB: And the angle AED ſtands on the circumference ABCD, and the angle BAE on the circumference EDCB; therefore the angle BAE is equal [f] to the angle AED: For the ſame reaſon, each of the angles ABC, BCD, CDE is equal to the angle BAE, or AED: Therefore the pentagon ABCDE is equiangular; and it has been ſhown that it is equilateral. Wherefore, in the given circle, an equilateral and equiangular pentagon has been inſcribed. Which was to be done.

e 29. 3.

f 27. 3.

PROP. XII. PROB.

TO deſcribe an equilateral and equiangular pentagon about a given circle.

Let ABCDE be the given circle; it is required to deſcribe an equilateral and equiangular pentagon about the circle ABCDE.

Let the angles of a pentagon, inſcribed in the circle, by the laſt propoſition, be in the points A, B, C, D, E, ſo that the circumferences AB, BC, CD, DE, EA are equal [a]; and thro' the points A, B, C, D, E draw GH, HK, KL, LM, MG, touching [b] the circle; take the centre F, and join FB, FK, FC, FL, FD: And becauſe the ſtraight line KL touches the circle ABCDE in the point C, to which FC is drawn from the centre F, FC is perpendicular [c] to KL; therefore each of the angles at C is a right angle: For the ſame reaſon, the angles at the points B, D are right angles: And becauſe FCK is a right angle, the ſquare of FK is equal [d] to the ſquares of FC, CK: For the ſame reaſon, the ſquare of FK is equal to the ſquares of FB, BK: Therefore the ſquares of FC, CK are equal to the ſquares of FB, BK, of which the ſquare of FC is equal to the ſquare of FB; the remaining ſquare of CK is therefore equal to the

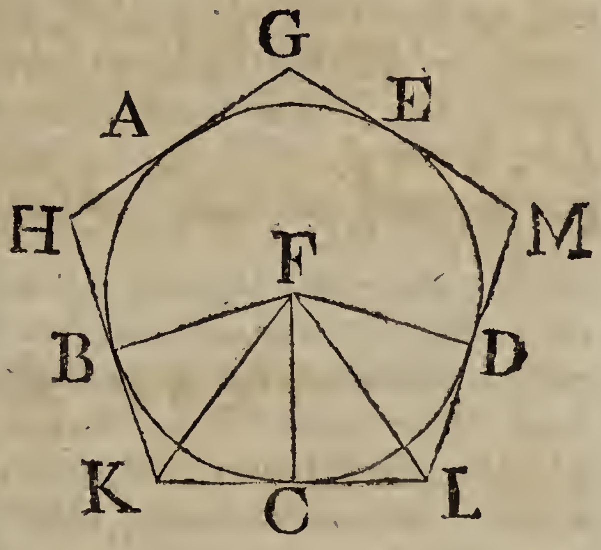
a 11. 4.

b 17. 3.

c 18. 3.

d 47. 1.

the remaining ſquare of BK, and the ſtraight line CK equal to BK: And becauſe FB is equal to FC, and FK common to the triangles BFK, CFK, the two BF, FK are equal to the two CF, FK; and the baſe BK is equal to the baſe KC; therefore the angle BFK is equal [e] to the angle KFC, and the angle BKF to FKC; wherefore the angle BFC is double of the angle KFC, and BKC double of FKC: For the ſame reaſon, the angle CFD is double of the angle CFL, and CLD double of CLF: And becauſe the circumference BC is equal to the circumference CD, the angle BFC is equal [f] to the angle CFD; and BFC is double of the angle KFC, and CFD double of CFL; therefore the angle KFC is equal to the angle CFL; and the right angle FCK is equal to the right angle FCL: Therefore, in the two triangles FKC, FLC, there are two angles of one equal to two angles of the other, each to each, and the ſide FC, which is adjacent to the equal angles in each, is common to both; therefore the other ſides ſhall be equal [g] to the other ſides, and the third angle to the third angle: Therefore the ſtraight line KC is equal to CL, and the angle FKC to the angle FLC: And becauſe KC is equal to CL, KL is double of KC: In the ſame manner, it may be ſhown that HK is double of BK: And becauſe BK is equal to KC, as was demonſtrated, and that KL is double of KC, and HK double of BK, HK ſhall be equal to KL: In like manner, it may be ſhown that GH, GM, ML are each of them equal to HK or KL: Therefore the pentagon GHKLM is equilateral. It is alſo equiangular; for, ſince the angle FKC is equal to the angle FLC, and that the angle HKL is double of the angle FKC, and KLM double of FLC, as was before demonſtrated, the angle HKL is equal to KLM: And in like manner it may be ſhown, that each of the angles KHG, HGM, GML is equal to the angle HKL or KLM: Therefore the five angles GHK, HKL, KLM, LMG, MGH being equal to one another, the pentagon GHKLM is equiangular: And it is equilateral, as was demonſtrated; and it is deſcribed about the circle ABCDE. Which was to be done.



e 8. 1.

f 27. 3.

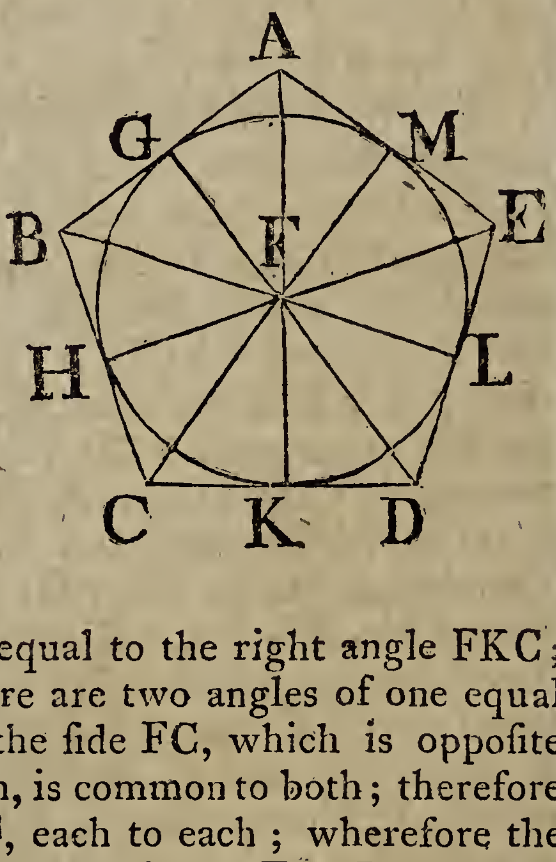
g 26. 1.

PROP. XIII. PROB.

TO inſcribe a circle in a given equilateral and equiangular pentagon.

Let ABCDE be the given equilateral and equiangular pentagon; it is required to inſcribe a circle in the pentagon ABCDE.

a 9. 1. Biſect [a] the angles BCD, CDE by the ſtraight lines CF, DF, and from the point F, in which they meet, draw the ſtraight lines FB, FA, FE: Therefore, ſince BC is equal to CD, and CF common to the triangles BCF, DCF, the two ſides BC, CF are equal to the two DC, CF; and the angle BCF is equal to the angle b 4. 1. DCF; therefore the baſe BF is equal [b] to the baſe FD, and the other angles to the other angles, to which the equal ſides are oppoſite; therefore the angle CBF is equal to the angle CDF: And becauſe the angle CDE is double of CDF, and that CDE is equal to CBA, and CDF to CBF; CBA is alſo double of the angle CBF; therefore the angle ABF is equal to the angle CBF; wherefore the angle ABC is biſected by the ſtraight line BF: In the ſame manner it may be demonſtrated, that the angles BAE, AED, are biſected by the ſtraight lines AF, c 12. 1. FE: From the point F draw [c] FG, FH, FK, FL, FM perpendiculars to the ſtraight lines AB, BC, CD, DE, EA: And becauſe the angle HCF is equal to KCF, and the right angle FHC equal to the right angle FKC; in the triangles FHC, FKC there are two angles of one equal to two angles of the other, and the ſide FC, which is oppoſite to one of the equal angles in each, is common to both; therefore d 26. 1. the other ſides ſhall be equal [d], each to each; wherefore the perpendicular FH is equal to the perpendicular FK: In the ſame manner it may be demonſtrated that FL, FM, FG are each of them equal to FH or FK: Therefore the five ſtraight lines FG, FH, FK, FL, FM are equal to one another: Wherefore the circle deſcribed from the centre F, at the diſtance of one of theſe five, ſhall paſs through the extremities of the other four, and touch



touch the ſtraight lines AB, BC, CD, DE, EA, becauſe the angles at the points G, H, K, L, M are right angles; and that a ſtraight line drawn from the extremity of the diameter of a circle at right angles to it, touches [e] the circle: Therefore each of the ſtraight lines AB, BC, CD, DE, EA touches the circle; wherefore it is inſcribed in the pentagon ABCDE. Which was to be done.

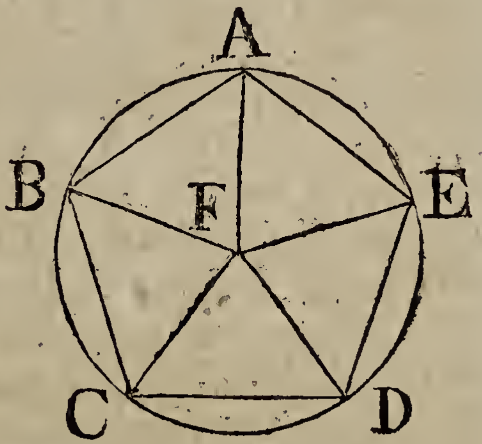
e 16. 3.

PROP. XIV. PROB.

TO deſcribe a circle about a given equilateral and equiangular pentagon.

Let ABCDE be the given equilateral and equiangular pentagon; it is required to deſcribe a circle about it.

Biſect [a] the angles BCD, CDE by the ſtraight lines CF, FD, and from the point F, in which they meet, draw the ſtraight lines FB, FA, FE to the points B, A, E. It may be demonſtrated, in the ſame manner as in the preceding propoſition, that the angles CBA, BAE, AED are biſected by the ſtraight lines FB, FA, FE: And becauſe the angle BCD is equal to the angle CDE, and that FCD is the half of the angle BCD, and CDF the half of CDE; the angle FCD is equal to FDC; wherefore the ſide CF is equal [b] to the ſide FD: In like manner it may be demonſtrated that FB, FA, FE are each of them equal to FC or FD: Therefore the five ſtraight lines FA, FB, FC, FD, FE are equal to one another; and the circle deſcribed from the centre F, at the diſtance of one of them, ſhall paſs through the extremities of the other four, and be deſcribed about the equilateral and equiangular pentagon ABCDE. Which was to be done.



a 9. 1.

b 6. 1.

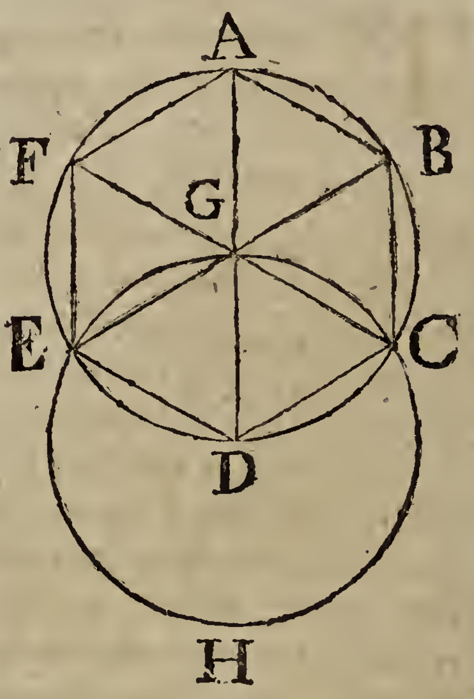
PROP. XV. PROB.

See N. TO inſcribe an equilateral and equiangular hexagon in a given circle.

Let ABCDEF be the given circle; it is required to inſcribe an equilateral and equiangular hexagon in it.

Find the centre G of the circle ABCDEF, and draw the diameter AGD; and from D as a centre, at the diſtance DG, deſcribe the circle EGCH, join EG, CG, and produce them to the points B, F; and join AB, BC, CD, DE, EF, FA: The hexagon ABCDEF is equilateral and equiangular.

Becauſe G is the centre of the circle ABCDEF, GE is equal to GD: And becauſe D is the centre of the circle EGCH, DE is equal to DG; wherefore GE is equal to ED, and the triangle EGD is equilateral; and therefore its three angles EGD, GDE, DEG are equal to one another, becauſe the angles at
a 5. 1. the baſe of an iſoſceles triangle are equal [a]; and the three angles
b 32. 1. of a triangle are equal [b] to two right angles; therefore the angle EGD is the third part of two right angles: In the ſame manner it may be demonſtrated that the angle DGC is alſo the third part of two right angles: And becauſe the ſtraight line GC makes with EB the
c 13. 1. adjacent angles EGC, CGB equal [c] to two right angles; the remaining angle CGB is the third part of two right angles; therefore the angles EGD, DGC, CGB, are equal to one another: And to theſe are equal [d] the
d 15. 1. vertical oppoſite angles BGA, AGF, FGE: Therefore the ſix angles EGD, DGC, CGB, BGA, AGF, FGE are equal to one another: But equal
e 26. 3. angles ſtand upon equal [e] circumferences; therefore the ſix circumferences AB, BC, CD, DE, EF, FA are equal to one another;
f 29. 3. And equal circumferences are ſubtended by equal [f] ſtraight lines; therefore the ſix ſtraight lines are equal to one another, and the hexagon ABCDEF is equilateral. It is alſo equiangular; for, ſince the circumference AF is equal to ED, to each of theſe add the circumference ABCD; therefore the whole circumference FABCD ſhall be equal to the whole EDCBA: And



And the angle FED ſtands upon the circumference FABCD, and the angle AFE upon EDCBA; therefore the angle AFE is equal to FED: In the ſame manner it may be demonſtrated that the other angles of the hexagon ABCDEF are each of them equal to the angle AFE or FED: Therefore the hexagon is equiangular; and it is equilateral, as was ſhown; and it is inſcribed in the given circle ABCDEF. Which was to be done.

Cor. From this it is manifeſt, that the ſide of the hexagon is equal to the ſtraight line from the centre, that is, to the ſemidiameter of the circle.

And if through the points A, B, C, D, E, F there be drawn ſtraight lines touching the circle, an equilateral and equiangular hexagon ſhall be deſcribed about it, which may be demonſtrated from what has been ſaid of the pentagon; and likewiſe a circle may be inſcribed in a given equilateral and equiangular hexagon, and circumſcribed about it, by a method like to that uſed for the pentagon.

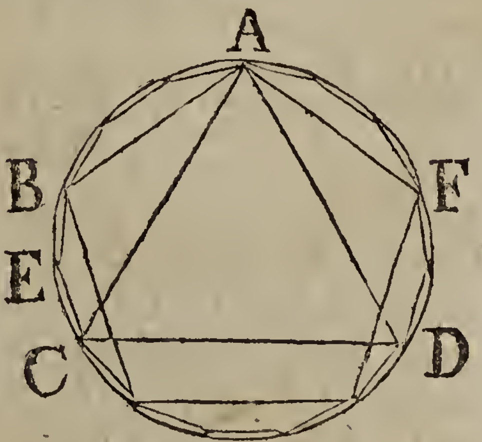
PROP. XVI. PROB.

To inſcribe an equilateral and equiangular quindecagon in a given circle. See N.

Let ABCD be the given circle; it is required to inſcribe an equilateral and equiangular quindecagon in the circle ABCD.

Let AC be the ſide of an equilateral triangle inſcribed[a] in the circle, and AB the ſide of an equilateral and equiangular pentagon inſcribed[b] in the ſame; therefore, of ſuch equal parts as the whole circumference ABCDF contains fifteen, the circumference ABC, being the third part of the whole, contains five; and the circumference AB, which is the fifth part of the whole, contains three; therefore BC their difference containes two of the ſame parts: Biſect[c] BC in E; therefore BE, EC are, each of them, the fifteenth part of the whole circumference ABCD: Therefore, if the ſtraight lines BE, EC be drawn, and ſtraight lines equal to them be placed[d] around in the whole circle, an equilateral and equiangular quindecagon ſhall be inſcribed in it. Which was to be done.

a 2. 4.
b 11. 4.
c 30. 3.
d 1. 4.



Book IV.

And, in the ſame manner as was done in the pentagon, if, through the points of diviſion made by inſcribing the quindecagon, ſtraight lines be drawn touching the circle, an equilateral and equiangular quindecagon ſhall be deſcribed about it: And likewiſe, as in the pentagon, a circle may be inſcribed in a given equilateral and equiangular quindecagon, and circumſcribed about it.

THE

ELEMENTS

OF

EUCLID.

BOOK V.

DEFINITIONS.

I.

A LESS magnitude is ſaid to be a part of a greater magnitude, when the leſs meaſures the greater, that is, ‘when the leſs is contained a certain number of times exactly ‘in the greater.’

II.

A greater magnitude is ſaid to be a multiple of a leſs, when the greater is meaſured by the leſs, that is, ‘when the greater ‘contains the leſs a certain number of times exactly.’

III.

‘Ratio is a mutual relation of two magnitudes of the ſame ‘kind to one another, in reſpect of quantity.’ See N.

IV.

Magnitudes are ſaid to have a ratio to one another, when the leſs can be multiplied ſo as to exceed the other.

V.

The firſt of four magnitudes is ſaid to have the ſame ratio to the the ſecond, which the third has to the fourth, when any equimultiples whatſoever of the firſt and third being taken, and any equimultiples whatſoever of the ſecond and fourth; if the multiple of the firſt be leſs than that of the ſecond, the multiple of the third is alſo leſs than that of the fourth; or, if the multiple of the firſt be equal to that of the ſecond, the multiple of the third is alſo equal to that of the fourth;

or, if the multiple of the firſt be greater than that of the ſecond, the multiple of the third is alſo greater than that of the fourth.

VI.

Magnitudes which have the ſame ratio are called proportionals.

N. B. ‘When four magnitudes are proportionals, it is ‘uſually expreſſed by ſaying, the firſt is to the ſecond, as the ‘third to the fourth.’

VII.

When of the equimultiples of four magnitudes (taken as in the fifth definition) the multiple of the firſt is greater than that of the ſecond, but the multiple of the third is not greater than the multiple of the fourth; then the firſt is ſaid to have to the ſecond a greater ratio than the third magnitude has to the fourth; and, on the contrary, the third is ſaid to have to the fourth a leſs ratio than the firſt has to the ſecond.

VIII.

“Analogy, or proportion, is the ſimilitude of ratios.”

IX.

Proportion conſiſts in three terms at leaſt.

X.

See N. When three magnitudes are proportionals, the firſt is ſaid to have to the third the duplicate ratio of that which it has to the ſecond.

XI.

When four magnitudes are continual proportionals, the firſt is ſaid to have to the fourth the triplicate ratio of that which it has to the ſecond, and ſo on, quadruplicate, &c. increaſing the denomination ſtill by unity, in any number of proportionals.

Definition A, to wit, of compound ratio.

When there are any number of magnitudes of the ſame kind, the firſt is ſaid to have to the laſt of them the ratio compounded of the ratio which the firſt has to the ſecond, and of the ratio which the ſecond has to the third, and of the ratio which the third has to the fourth, and ſo on unto the laſt magnitude.

For example, if A, B, C, D be four magnitudes of the ſame kind, the firſt A is ſaid to have to the laſt D the ratio compounded of the ratio of A to B, and of the ratio of B to C, and of the ratio of C to D; or, the ratio of A to D is ſaid to be compounded of the ratios of A to B, B to C, and C to D:

And if A has to B the ſame ratio which E has to F; and B to C, the ſame ratio that G has to H; and C to D, the ſame that K has to L; then, by this definition, A is ſaid to have to D the ratio compounded of ratios which are the ſame with the ratios of E to F, G to H, and K to L: and the ſame thing is to be underſtood when it is more briefly expreſſed, by ſaying A has to D the ratio compounded of the ratios of E to F, G to H, and K to L.

In like manner, the ſame things being ſuppoſed, if M has to N the ſame ratio which A has to D; then, for ſhortneſs ſake, M is ſaid to have to N, the ratio compounded of the ratios of E to F, G to H, and K to L.

XII.

In proportionals, the antecedent terms are called homologous to one another, as alſo the conſequents to one another.

'Geometers make uſe of the following technical words to ſignify certain ways of changing either the order or magnitude of proportionals, ſo as that they continue ſtill to be proportionals.'

XIII.

Permutando, or alternando, by permutation, or alternately; this word is uſed when there are four proportionals, and it is inferred, that the firſt has the ſame ratio to the third, which the ſecond has to the fourth; or that the firſt is to the third, as the ſecond to the fourth: As is ſhewn in the 16th prop. of this 5th book. See N.

XIV.

Invertendo, by inverſion: When there are four proportionals, and it is inferred, that the ſecond is to the firſt, as the fourth to the third. Prop. B. book 5.

XV.

Componendo, by compoſition; when there are four proportionals, and it is inferred, that the firſt, together with the ſecond, is to the ſecond, as the third, together with the fourth, is to the fourth. 18th prop. book 5.

XVI.

Dividendo, by diviſion; when there are four proportionals, and it is inferred, that the exceſs of the firſt above the ſecond, is to the ſecond, as the exceſs of the third above the fourth, is to the fourth. 17th prop. book 5.

XVII.

Convertendo, by converſion; when there are four proportionals, and it is inferred, that the firſt is to its exceſs above the ſecond, as the third to its exceſs above the fourth. Prop. E. book 5.

XVIII.

Ex æquali (ſc. diſtantia), or ex æquo, from equality of diſtance; when there is any number of magnitudes more than two, and as many others, ſo that they are proportionals when taken two and two of each rank, and it is inferred, that the firſt is to the laſt of the firſt rank of magnitudes, as the firſt is to the laſt of the others: ‘ Of this there are the two following kinds, which ariſe from the different order in which the magnitudes are taken two and two.’

XIX.

Ex æquali, from equality; this term is uſed ſimply by itſelf, when the firſt magnitude is to the ſecond of the firſt rank, as the firſt to the ſecond of the other rank; and as the ſecond is to the third of the firſt rank, ſo is the ſecond to the third of the other; and ſo on in order, and the inference is as mentioned in the preceding definition; whence this is called ordinate proportion. It is demonſtrated in 22d prop. book 5.

XX.

Ex æquali, in proportione perturbata, ſeu inordinata, from equality, in perturbate or diſorderly proportion *; this term is uſed when the firſt magnitude is to the ſecond of the firſt rank, as the laſt but one is to the laſt of the ſecond rank; and as the ſecond is to the third of the firſt rank, ſo is the laſt but two to the laſt but one of the ſecond rank; and as the third is to the fourth of the firſt rank, ſo is the third from the laſt to the laſt but two of the ſecond rank; and ſo on in a croſs order: And the inference is as in the 18th definition. It is demonſtrated in the 23d prop. of book 5.

AXIOMS.

I.

EQUIMULTIPLES of the ſame, or of equal magnitudes, are equal to one another.

II. Thoſe

---

* 4 Prop. lib. 2. Archimedis de ſphæra et cylindro.

II.

Thoſe magnitudes of which the ſame, or equal magnitudes, are equimultiples, are equal to one another.

III.

A multiple of a greater magnitude is greater than the ſame multiple of a leſs.

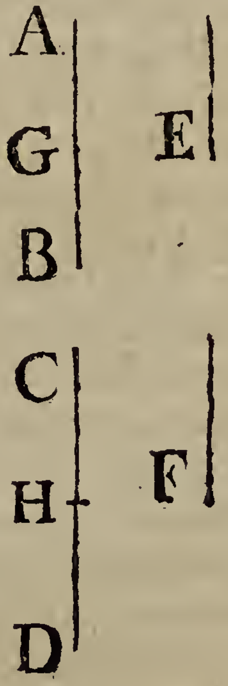
IV.

That magnitude of which a multiple is greater than the ſame multiple of another, is greater than that other magnitude.

PROP. I. THEOR.

IF any number of magnitudes be equimultiples of as many, each of each; what multiple ſoever any one of them is of its part, the ſame multiple ſhall all the firſt magnitudes be of all the other.

Let any number of magnitudes AB, CD be equimultiples of as many others E, F, each of each; whatſoever multiple AB is of E, the ſame multiple ſhall AB and CD together be of E and F together.



Becauſe AB is the ſame multiple of E that CD is of F, as many magnitudes as are in AB equal to E, ſo many are there in CD equal to F. Divide AB into magnitudes equal to E, viz. AG, GB; and CD into CH, HD equal each of them to F: The number therefore of the magnitudes CH, HD ſhall be equal to the number of the others AG, GB: And becauſe AG is equal to E, and CH to F, therefore AG and CH together are equal to [a] E and F together: For the ſame reaſon, becauſe GB is equal to E, and HD to F; GB and HD together are equal to E and F together. Wherefore, as many magnitudes as are in AB equal to E, ſo many are there in AB, CD together equal to E and F together. Therefore, whatſoever multiple AB is of E, the ſame multiple is AB and CD together of E and F together.

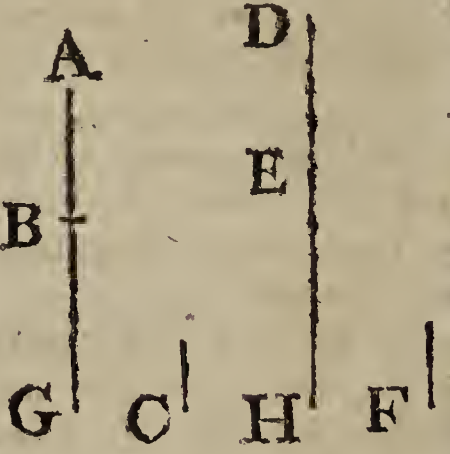
[a] Ax. 2. 5.

Therefore, if any magnitudes, how many ſoever, be equimultiples of as many, each of each, whatſoever multiple any one of them is of its part, the ſame multiple ſhall all the firſt magnitudes be of all the other: ' For the ſame demonſtration

‘ holds in any number of magnitudes, which was here applied ‘ to two.’ Q. E. D.

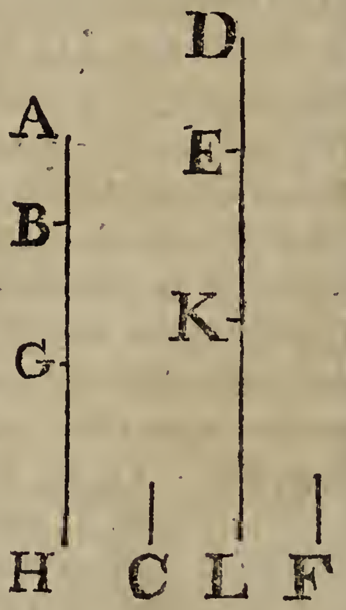
PROP. II. THEOR.

IF the firſt magnitude be the ſame multiple of the ſecond that the third is of the fourth, and the fifth the ſame multiple of the ſecond that the ſixth is of the fourth; then ſhall the firſt together with the fifth be the ſame multiple of the ſecond, that the third together with the ſixth is of the fourth.



Let AB the firſt, be the ſame multiple of C the ſecond, that DE the third is of F the fourth; and BG the fifth, the ſame multiple of C the ſecond, that EH the ſixth is of F the fourth: Then is AG the firſt, together with the fifth, the ſame multiple of C the ſecond, that DH the third, together with the ſixth, is of F the fourth.

Becauſe AB is the ſame multiple of C, that DE is of F; there are as many magnitudes in AB equal to C, as there are in DE equal to F: In like manner, as many as there are in BG equal to C, ſo many are there in EH equal to F: As many, then, as are in the whole AG equal to C, ſo many are there in the whole DH equal to F: therefore AG is the ſame multiple of C, that DH is of F; that is, AG the firſt and fifth together, is the ſame multiple of the ſecond C, that DH the third and ſixth together is of the fourth F. If therefore, the firſt be the ſame multiple, &c. Q. E. D.



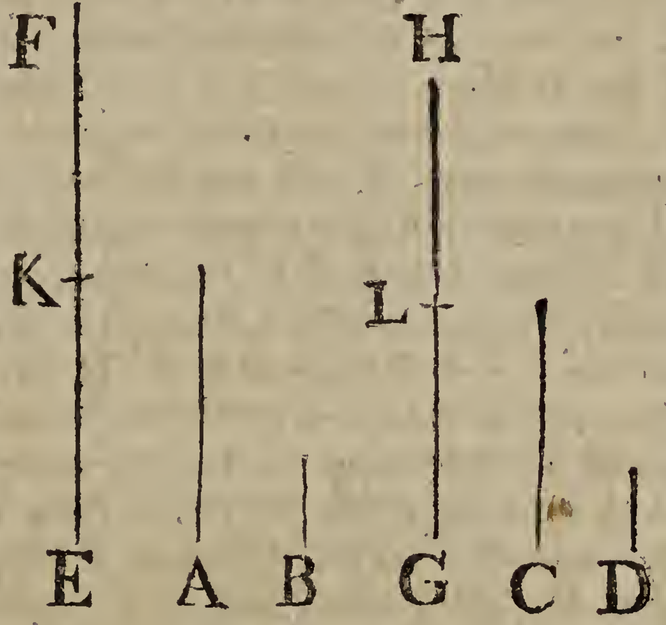
Cor. ‘ From this it is plain, that, if any ‘ number of magnitudes AB. BG, GH, ‘ be multiples of another C; and as many ‘ DE, EK, KL be the ſame multiples of ‘ F, each of each; the whole of the firſt, ‘ viz. AH, is the ſame multiple of C, ‘ that the whole of the laſt, viz. DL, is ‘ of F.’

PROP. III. THEOR.

IF the firſt be the ſame multiple of the ſecond, which the third is of the fourth; and if of the firſt and third there be taken equimultiples, theſe ſhall be equimultiples, the one of the ſecond, and the other of the fourth.

Let A the firſt, be the ſame multiple of B the ſecond, that C the third is of D the fourth; and of A, C let the equimultiples EF, GH be taken: Then EF is the ſame multiple of B, that GH is of D.

Becauſe EF is the ſame multiple of A, that GH is of C, there are as many magnitudes in EF equal to A, as are in GH equal to C: Let EF be divided into the magnitudes EK, KF, each equal to A, and GH into GL, LH, each equal to C: The number therefore of the magnitudes EK, KF, ſhall be equal to the number of the others GL, LH: And becauſe A is the ſame multiple of B, that C is of D, and that EK is equal to A, and GL to C; therefore EK is the ſame multiple of B, that GL is of D: For the ſame reaſon, KF is the ſame multiple of B, that LH is of D; and ſo, if there be more parts in EF, GH equal to A, C: Becauſe, therefore, the firſt EK is the ſame multiple of the ſecond B, which the third GL is of the fourth D, and that the fifth KF is the ſame multiple of the ſecond B, which the ſixth LH is of the fourth D; EF the firſt, together with the fifth, is the ſame multiple [a] of the ſecond B, which GH the third, together with the ſixth, is of the fourth D. If, therefore, the firſt, &c. Q. E. D.



[a] 2. 5.

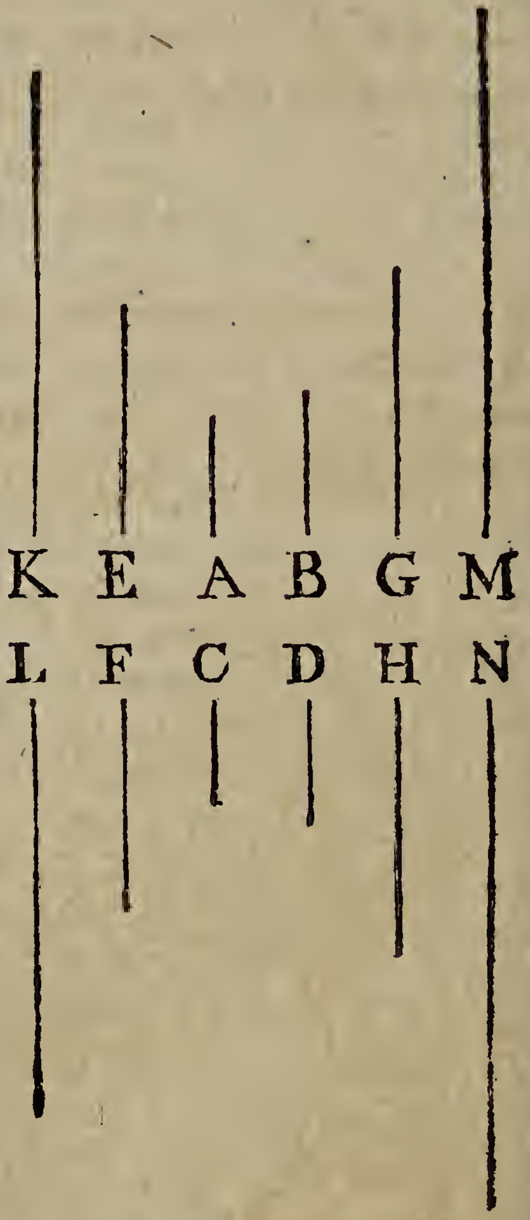
PROP.

PROP. IV. THEOR.

See N.

IF the firſt of four magnitudes has the ſame ratio to the ſecond which the third has to the fourth; then any equimultiples whatever of the firſt and third ſhall have the ſame ratio to any equimultiples of the ſecond and fourth, viz. ‘ the equimultiple of the firſt ſhall have ‘ the ſame ratio to that of the ſecond, which the equi- ‘ multiple of the third has to that of the fourth.’

Let A the firſt, have to B the ſecond, the ſame ratio which the third C has to the fourth D; and of A and C let there be taken any equimultiples whatever E, F; and of B and D any equimultiples whatever G, H: Then E has the ſame ratio to G, which F has to H.



Take of E and F any equimultiples whatever K, L, and of G, H, any equimultiples whatever M, N: Then, becauſe E is the ſame multiple of A, that F is of C; and of E and F have been taken equimultiples K, L; therefore K is the ſame multiple of A, that L is of C[a]: For the ſame reaſon, M is the ſame multiple of B, that N is of D: And becauſe, as A is to B, ſo is C to D[b], and of A and C have been taken certain equimultiples K, L; and of B and D have been taken certain equimultiples M, N; if therefore K be greater than M, L is greater than N; and if equal, equal; if leſs, leſs[c]. And K, L are any equimultiples whatever of E, F; and M, N any whatever of G, H: As therefore E is to G, ſo is [c]F to H. Therefore, if the firſt, &c. Q. E. D.

[a] 3. 5.

[b] Hypoth.

[c] 5. def. 5.

See N.

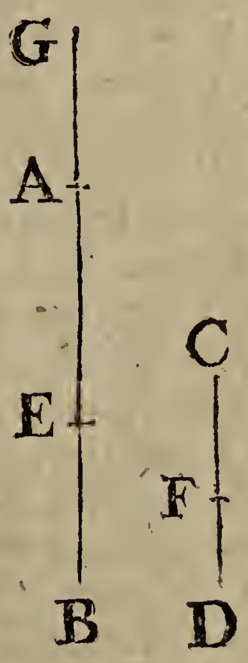
Cor. Likewiſe, if the firſt has the ſame ratio to the ſecond, which the third has to the fourth, then alſo any equimultiples whatever of the firſt and third have the ſame ratio to the ſecond and fourth: And in like manner, the firſt and the third have the ſame ratio to any equimultiples whatever of the ſecond and fourth.

Let A the firſt, have to B the ſecond, the ſame ratio which the third C has to the fourth D, and of A and C let E and F be any equimultiples whatever; then E is to B, as F to D.

Take of E, F any equimultiples whatever K, L, and of B, D any equimultiples whatever G, H; then it may be demonſtrated, as before, that K is the ſame multiple of A, that L is of C: And becauſe A is to B, as C is to D, and of A and C certain equimultiples have been taken, viz. K and L; and of B and D certain equimultiples G, H; therefore, if K be greater than G, L is greater than H; and if equal, equal; if leſs, leſs [c]: c 5. Def. 5. And, K, L are any equimultiples of E, F, and G, H any whatever of B, D; as therefore E, is to B, ſo is F to D: And in the ſame way the other caſe is demonſtrated.

PROP. V. THEOR.

IF one magnitude be the ſame multiple of another, See N. which a magnitude taken from the firſt is of a magnitude taken from the other; the remainder ſhall be the ſame multiple of the remainder, that the whole is of the whole.



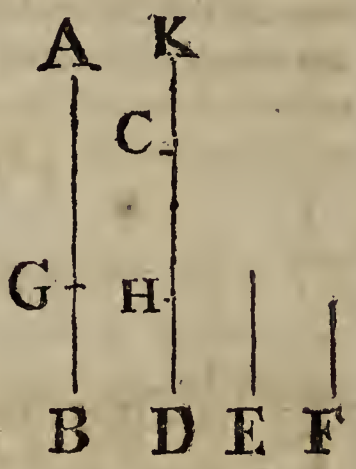
Let the magnitude AB be the ſame multiple of CD, that AE taken from the firſt, is of CF taken from the other; the remainder EB ſhall be the ſame multiple of the remainder FD, that the whole AB is of the whole CD.

Take AG the ſame multiple of FD, that AE is of CF: therefore AE is [a] the ſame multiple of CF, that EG is of CD: a 1. 5. But AE, by the hypotheſis, is the ſame multiple of CF, that AB is of CD: Therefore EG is the ſame multiple of CD that AB is of CD; wherefore EG is equal to AB [b]. b 1. Ax. 5. Take from them the common magnitude AE; the remainder AG is equal to the remainder EB. Wherefore, ſince AE is the ſame multiple of CF, that AG is of FD, and that AG is equal to EB; therefore AE is the ſame multiple of CF, that EB is of FD: But AE is the ſame multiple of CF, that AB is of CD ; therefore EB is the ſame multiple of FD, that AB is of CD. Therefore, if any magnitude, &c. Q. E. D.

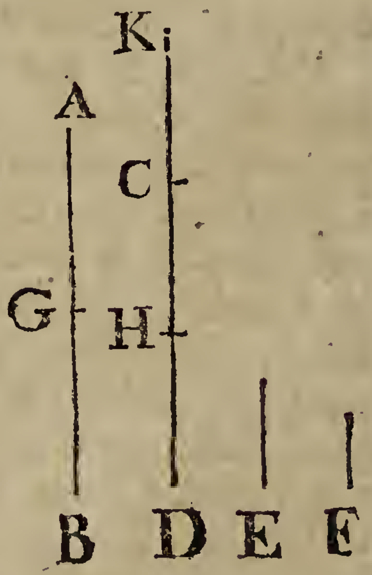
PROP. VI. THEOR.

See N. IF two magnitudes be equimultiples of two others, and if equimultiples of theſe be taken from the firſt two, the remainders are either equal to theſe others, or equimultiples of them.

Let the two magnitudes AB, CD be equimultiples of the two E, F, and AG, CH taken from the firſt two be equimultiples of the ſame E, F ; the remainders GB, HD are either equal to E, F, or equimultiples of them.



Firſt, let GB be equal to E ; HD is equal to F : Make CK equal to F ; and becauſe AG is the ſame multiple of E, that CH is of F, and that GB is equal to E, and CK to F ; therefore AB is the ſame multiple of E, that KH is of F. But AB, by the hypotheſis, is the ſame multiple of E that CD is of F ; therefore KH is the ſame multiple of F, that CD is of F ; a 1. Ax. 5. wherefore KH is equal to CD[a] : Take away the common magnitude CH, then the remainder KC is equal to the remainder HD : But KC is equal to F ; HD therefore is equal to F.



But let GB be a multiple of E ; then HD is the ſame multiple of F : Make CK the ſame multiple of F, that GB is of E : And becauſe AG is the ſame multiple of E, that CH is of F ; and GB the ſame multiple of E, that CK is of F : therefore AB is the ſame multiple of E, b 2. 5. that KH is of F[b] : But AB is the ſame multiple of E, that CD is of F ; therefore KH is the ſame multiple of F, that CD is of it : wherefore KH is equal to CD[a] : Take away CH from both ; therefore the remainder KC is equal to the remainder HD : And becauſe GB is the ſame multiple of E, that KC is of F, and that KC is equal to HD ; therefore HD is the ſame multiple of F, that GB is of E : If therefore two magnitudes, &c. Q. E. D.

PROP. A. THEOR.

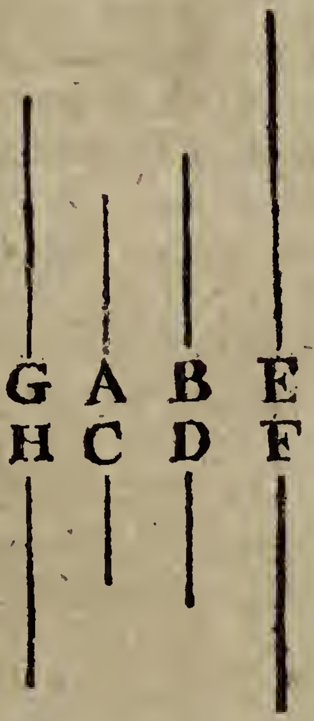
IF the firſt of four magnitudes has to the ſecond, the ſame ratio which the third has to the fourth; then, if the firſt be greater than the ſecond, the third is alſo greater than the fourth; and, if equal, equal; if leſs, leſs. See N.

Take any equimultiples of each of them, as the doubles of each; then, by def. 5th of this book, if the double of the firſt be greater than the double of the ſecond, the double of the third is greater than the double of the fourth; but, if the firſt be greater than the ſecond, the double of the firſt is greater than the double of the ſecond; wherefore alſo the double of the third is greater than the double of the fourth; therefore the third is greater than the fourth: In like manner, if the firſt be equal to the ſecond, or leſs than it, the third can be proved to be equal to the fourth, or leſs than it. Therefore, if the firſt, &c. Q. E. D.

PROP. B. THEOR.

IF four magnitudes are proportionals, they are proportionals alſo when taken inverſely. See N.

If the magnitude A be to B, as C is to D, then alſo inverſely B is to A, as D to C.



Take of B and D any equimultiples whatever E and F; and of A and C any equimultiples whatever G and H. Firſt, Let E be greater than G, then G is leſs than E; and, becauſe A is to B, as C is to D, and of A and C, the firſt and third, G and H are equimultiples; and of B and D, the ſecond and fourth, E and F are equimultiples; and that G is leſs than E, H is alſo [a] leſs than F; that is, F is greater than H; if therefore E be greater than G, F is greater than H: In like manner, if E be equal to G, F may be ſhown to be equal to H; and, if leſs, leſs; and E, F are any equimultiples whatever of B and D, and G, H any whatever of A and C; therefore, as B

[a] 5. def. 5.

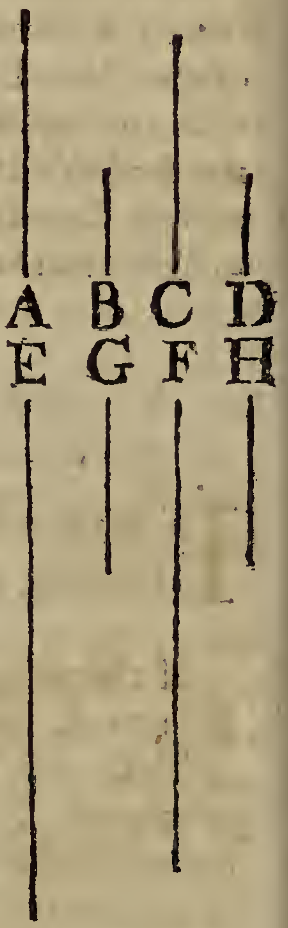
is to A, ſo is D to C. If, then, four magnitudes, &c. Q. E. D.

PROP. C. THEOR.

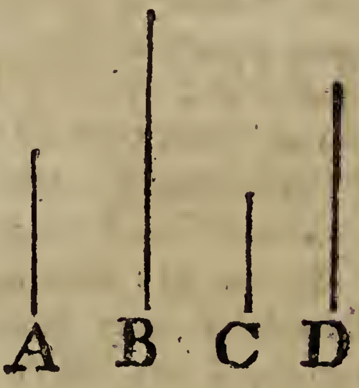
See N. IF the firſt be the ſame multiple of the ſecond, or the ſame part of it, that the third is of the fourth; the firſt is to the ſecond, as the third is to the fourth.

Let the firſt A be the ſame multiple of B the ſecond, that C the third is of the fourth D: A is to B as C is to D.

Take of A and C any equimultiples whatever E and F; and of B and D any equimultiples whatever G and H: Then, becauſe A is the ſame multiple of B that C is of D; and that E is the ſame multiple of A, that F is of C; E is the ſame multiple of B, that
a 3. 5. F is of D[a]; therefore E and F are the ſame multiples of B and D: But G and H are equimultiples of B and D; therefore, if E be a greater multiple of B, than G is, F is a greater multiple of D, than H is of D; that is, if E be greater than G, F is greater than H: In like manner, if E be equal to G, or leſs; F is equal to H, or leſs than it. But E, F are equimultiples, any whatever, of A, C, and G, H any equimultiples whatever of B,
b 5. def. 5. D. Therefore A is to B, as C is to D[b].



Next, Let the firſt A be the ſame part of the ſecond B, that the third C is of the fourth D: A is to B, as C is to D: For B is the ſame multiple of A, that D is of C: wherefore, by the preceding caſe, B is to A, as D is to C; and in-
c B. 5. verſely[c] A is to B, as C is to D. Therefore, if the firſt be the ſame multiple, &c. Q. E. D.



PROP.

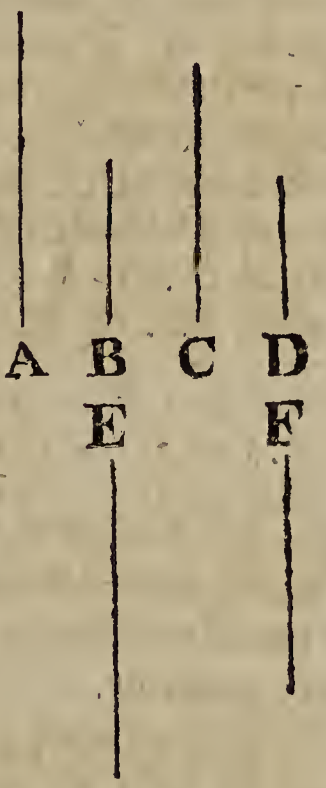
PROP. D. THEOR.

IF the firſt be to the ſecond as the third to the fourth, and if the firſt be a multiple, or part of the ſecond; the third is the ſame multiple, or the ſame part of the fourth. See N.

Let A be to B, as C is to D; and firſt let A be a multiple of B; C is the ſame multiple of D.

Take E equal to A, and whatever multiple A or E is of B, make F the ſame multiple of D: Then, becauſe A is to B, as C is to D; and of B the ſecond, and D the fourth equimultiples have been taken E and F; A is to E, as C to F [a]: But A is equal to E, therefore C is equal to F [b]: And F is the ſame multiple of D, that A is of B. Wherefore C is the ſame multiple of D, that A is of B.

a Cor. 4. 5.
b A. 5.



Next, Let the firſt A be a part of the ſecond B; C the third is the ſame part of the fourth D.

See the figure at the foot of the preceding page.

Becauſe A is to B, as C is to D; then, inverſely, B is [c] to A, as D to C: But A is a part of B, therefore B is a multiple of A; and, by the preceding caſe, D is the ſame multiple of C, that is, C is the ſame part of D, that A is of B: Therefore, if the firſt, &c. Q. E. D.

c B. 5.

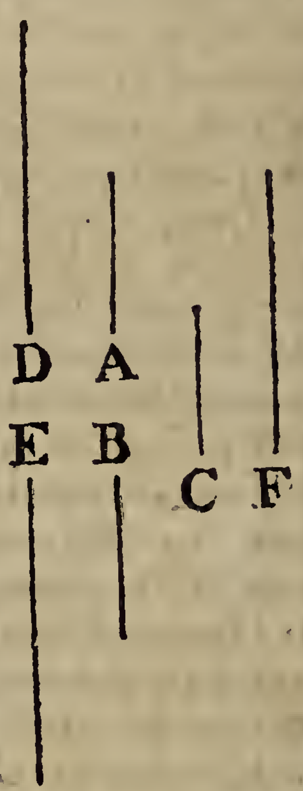
PROP. VII. THEOR.

EQUAL magnitudes have the ſame ratio to the ſame magnitude; and the ſame has the ſame ratio to equal magnitudes.

Let A and B be equal magnitudes, and C any other. A and B have each of them the ſame ratio to C, and C has the ſame ratio to each of the magnitudes A and B.

Take of A and B any equimultiples whatever D and E, and of C any multiple whatever F: Then, becauſe D is the ſame multiple of A, that E is of B, and that A is a 1. Ax. 5. equal to B; D is [a] equal to E: Therefore, if D be greater than F, E is greater than F; and if equal, equal; if leſs, leſs: And D, E are any equimultiples of A, B, and F is any mul- b 5. def. 5. tiple of C. Therefore [b], as A is to C, ſo is B to C.

Likewiſe C has the ſame ratio to A, that it has to B: For, having made the ſame conſtruction, D may in like manner be ſhown equal to E: Therefore, if F be greater than D, it is likewiſe greater than E; and if equal, equal; if leſs, leſs: And F is any multiple whatever of C, and D, E are any equimultiples whatever of A, B. Therefore C is to A, as C is to B [b]. Therefore equal magnitudes, &c. Q. E. D.

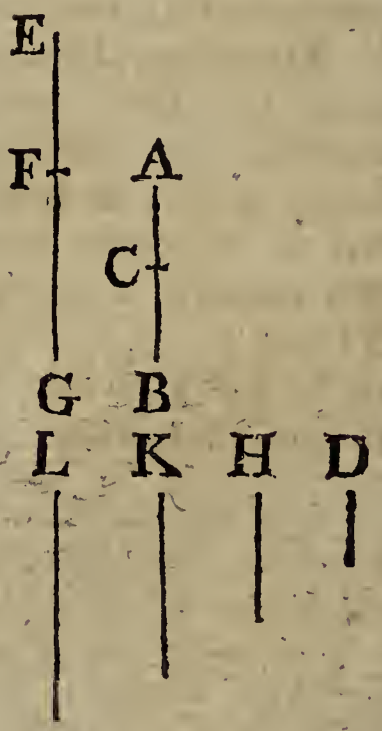


PROP. VIII. THEOR.

See N. OF unequal magnitudes, the greater has a greater ratio to the ſame than the leſs has; and the ſame magnitude has a greater ratio to the leſs, than it has to the greater.

Let AB, BC be unequal magnitudes, of which AB is the greater, and let D be any magnitude whatever: AB has a greater ratio to D than BC to D: And D has a greater ratio to BC than unto AB.

Fig. 1.



If the magnitude which is not the greater of the two AC, CB, be not leſs than D, take EF, FG, the doubles of AC, CB, as in Fig. 1. But, if that which is not the greater of the two AC, CB be leſs than D (as in Fig. 2. and 3.) this magnitude can be multiplied, ſo as to become greater than D, whether it be AC, or CB. Let it be multiplied, until it become greater than D, and let the other be multiplied as often; and let EF be the multiple thus taken of AC, and FG the ſame multiple of CB: Therefore EF and FG are each of them greater than

D: And in every one of the caſes, take H the double of D, K its triple, and ſo on, till the multiple of D be that which firſt becomes greater than FG: Let L be that multiple of D which is firſt greater than FG, and K the multiple of D which is next leſs than L.

Then, becauſe L is the multiple of D, which is the firſt that becomes greater than FG, the next preceding multiple K is not greater than FG; that is, FG is not leſs than K: And ſince EF is the ſame multiple of AC, that FG is of CB; FG is the ſame multiple of CB, that EG is of AB [a]; wherefore EG and FG are equimultiples of AB and CB: And it was ſhown, that FG was not leſs than K, and, by the conſtruction, EF is greater than D; therefore the whole EG is greater than K and D together: But, K together with D, is equal to L; therefore EG is greater than L; but FG is not greater than L; and EG, FG are equimultiples of AB, BC, and L is a multiple of D; therefore [b] AB has to D a greater ratio than BC has to D.

[a] 1. 5.

[b] 7. Def. 5.

Alſo D has to BC a greater ratio than it has to AB: For, Having made the ſame conſtruction, it may be ſhown, in like manner, that L is greater than FG, but that it is not greater than EG: and L is a multiple of D; and FG, EG are equimultiples of CB, AB; therefore D has to CB a greater ratio [b] than it has to AB. Wherefore, of unequal magnitudes, &c. Q. E. D.

Fig. 2.

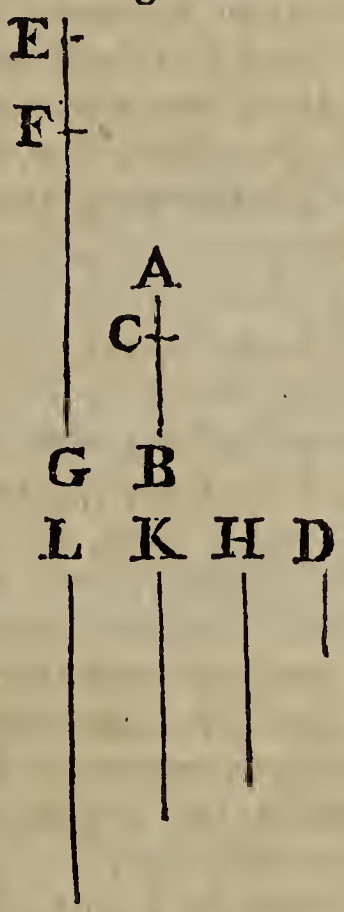
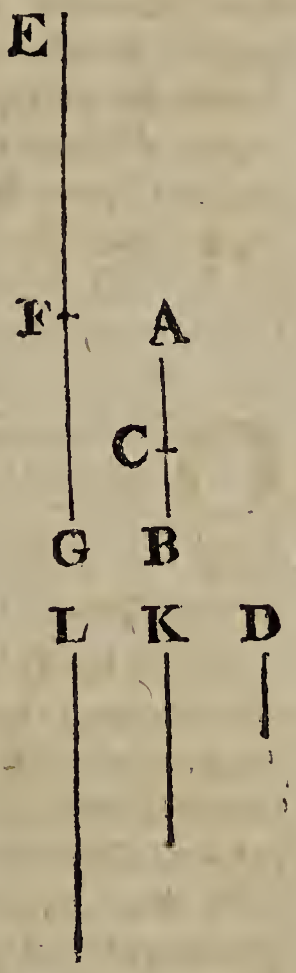


Fig. 3.



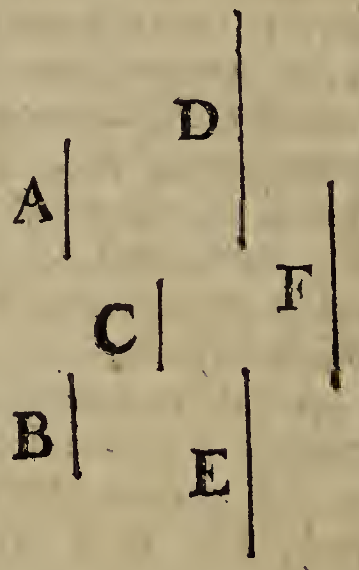
PROP. IX. THEOR.

See N.

MAGNITUDES which have the ſame ratio to the ſame magnitude are equal to one another; and thoſe to which the ſame magnitude has the ſame ratio are equal to one another.

Let A, B have each of them the ſame ratio to C: A is equal to B: For, if they are not equal, one of them is greater than the other; let A be the greater; then, by what was ſhown in the preceding propoſition, there are ſome equimultiples of A and B, and ſome multiple of C ſuch, that the multiple of A is greater than the multiple of C, but the multiple of B is not greater than that of C. Let ſuch multiples be taken, and let D, E, be the equimultiples of A, B, and F the multiple of C, ſo that D may be greater than F, and E not greater than F; But, becauſe A is to C, as B is to C, and of A, B, are taken equimultiples D, E, and of C is taken a multiple F; and that D is greater than F; E ſhall alſo be greater than F [a]; but E is not greater than F, which is impoſſible; A therefore and B are not unequal; that is, they are equal.

[a] 5. Def. 5.



Next, let C have the ſame ratio to each of the magnitudes A and B; A is equal to B: For, if they are not, one of them is greater than the other; let A be the greater; therefore, as was ſhown in Prop. 8th, there is ſome multiple F of C, and ſome equimultiples E and D, of B and A ſuch, that F is greater than E, and not greater than D; but becauſe C is to B, as C is to A, and that F, the multiple of the firſt, is greater than E, the multiple of the ſecond; F the multiple of the third, is greater than D, the multiple of the fourth [a]: But F is not greater than D, which is impoſſible. Therefore, A is equal to B. Wherefore, magnitudes which, &c. Q. E. D.

PROP.

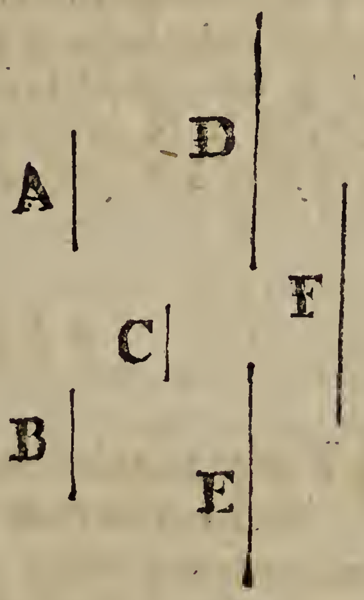
PROP. X. THEOR.

THAT magnitude which has a greater ratio than another has unto the ſame magnitude is the greater of the two: And that magnitude to which the ſame has a greater ratio than it has unto another magnitude is the leſſer of the two. See N.

Let A have to C a greater ratio than B has to C; A is greater than B: For, becauſe A has a greater ratio to C, than B has to C, there are [a] ſome equimultiples of A and B, and ſome multiple of C ſuch, that the multiple of A is greater than the multiple of C, but the multiple of B is not greater than it: Let them be taken, and let D, E be equimultiples of A, B, and F a multiple of C ſuch, that D is greater than F, but E is not greater than F: Therefore D is greater than E: And, becauſe D and E are equimultiples of A and B, and D is greater than E; therefore A is [b] greater than B.

[a] 7. def. 5.

[b] 4. Ax. 5.



Next, Let C have a greater ratio to B than it has to A; B is leſs than A: For [a] there is ſome multiple F of C, and ſome equimultiples E and D of B and A ſuch, that F is greater than E, but is not greater than D: E therefore is leſs than D; and becauſe E and D are equimultiples of B and A, therefore B is [b] leſs than A. That magnitude, therefore, &c. Q. E. D.

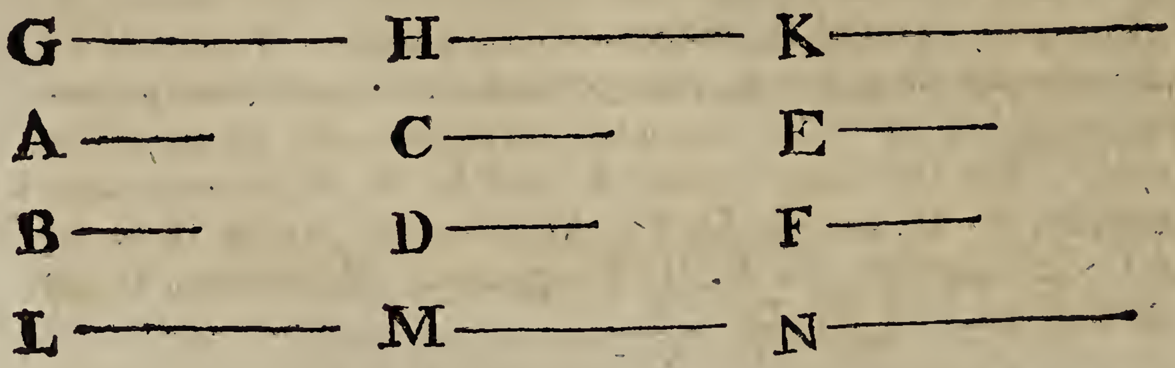
PROP. XI. THEOR.

RATIOS that are the ſame to the ſame ratio, are the ſame to one another.

Let A be to B as C is to D; and as C to D, ſo let E be to F; A is to B, as E to F.

Take of A, C, E, any equimultiples whatever G, H, K; and of B, D, F, any equimultiples whatever L, M, N. Therefore, ſince A is to B, as C to D, and G, H are taken equimultiples of

A, C, and L, M of B, D; if G be greater than L, H is greater than M; and if equal, equal; and if leſs, leſs [a]. Again, be- a 5. Def. 5. cauſe C is to D, as E is to F, and H, K are taken equimultiples of C, E; and M, N, of D, F; if H be greater than M, K is greater than N; and if equal, equal; and if leſs, leſs: But, if



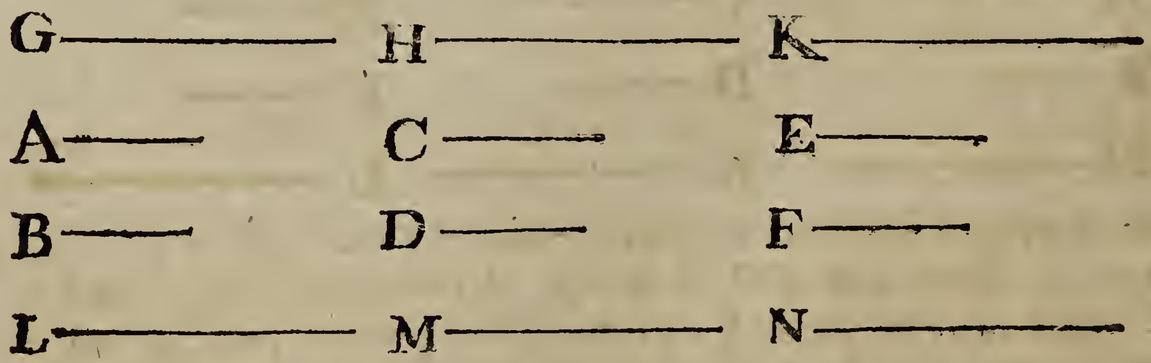
G be greater than L, it has been ſhewn that H is greater than M; and if equal, equal; and if leſs, leſs; therefore, if G be greater than L, K is greater than N; and if equal, equal; and if leſs, leſs: And G, K are any equimultiples whatever of A, E; and L, N any whatever of B, F: Therefore, as A is to B, ſo is E to F [a]. Wherefore, ratios that, &c. Q. E. D.

PROP. XII. THEOR.

IF any number of magnitudes be proportionals, as one of the antecedents is to its conſequent, ſo ſhall all the antecedents taken together be to all the conſequents.

Let any number of magnitudes A, B, C, D, E, F, be proportionals; that is, as A is to B, ſo C to D, and E to F: As A is to B, ſo ſhall A, C, E together be to B, D, F together.

Take of A, C, E any equimultiples whatever G, H, K;



and of B, D, F any equimultiples whatever L, M, N: Then, becauſe A is to B, as C is to D, and as E to F; and that G, H,

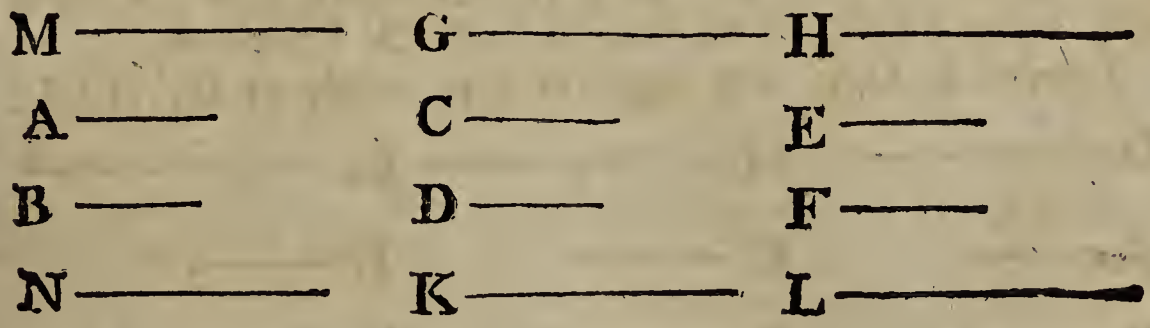
K are equimultiples of A, C, E, and L, M, N equimultiples of B, D, F; ſf G be greater than L, H is greater than M, and K greater than N; and if equal, equal; and if leſs, leſs [a]. Where- a 5. def. 5.
fore, if G be greater than L, then G, H, K together are greater than L, M, N together; and if equal, equal; and if leſs, leſs. And G, and G, H, K together are any equimultiples of A, and A, C, E together; becauſe, if there be any number of magnitudes equimultiples of as many, each of each, whatever multiple one of them is of its part, the ſame multiple is the whole of the whole [b]: For the ſame reaſon L, and L, M, N are any equi- b 1. 5.
multiples of B, and B, D, F: As therefore A is to B, ſo are A, C, E together to B, D, F together. Wherefore, if any number, &c. Q. E. D.

PROP. XIII. THEOR.

IF the firſt has to the ſecond the ſame ratio which the See N.
third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the ſixth; the firſt ſhall alſo have to the ſecond a greater ratio than the fifth has to the ſixth.

Let A the firſt, have the ſame ratio to B the ſecond, which C the third, has to D the fourth, but C the third, to D the fourth, a greater ratio than E the fifth, to F the ſixth: Alſo the firſt A ſhall have to the ſecond B a greater ratio than the fifth E to the ſixth F.

Becauſe C has a greater ratio to D, than E to F, there are ſome equimultiples of C and E, and ſome of D and F ſuch, that the multiple of C is greater than the multiple of D, but



the multiple of E is not greater than the multiple of F [a]: Let a 7. def. 5.
ſuch be taken, and of C, E let G, H be equimultiples, and K, L equimultiples of D, F, ſo that G be greater than K, but H not greater than L; and whatever multiple G is of C, take M the ſame multiple of A; and whatever multiple K is of D, take N the ſame multiple of B: Then, becauſe A is to B, as C to D,

D, and of A and C, M and G are equimultiples: And of B and D, N and K are equimultiples; if M be greater than N, G is
b 5. def. 5. greater than K; and if equal, equal; and if leſs, leſs[b]; but G is greater than K, therefore M is greater than N: But H is not greater than L; and M, H are equimultiples of A, E; and N, L equimultiples of B, F: Therefore A has a greater ratio to
c 7. def. 5. B, than E has to F[c]. Wherefore, if the firſt, &c. Q. E. D.

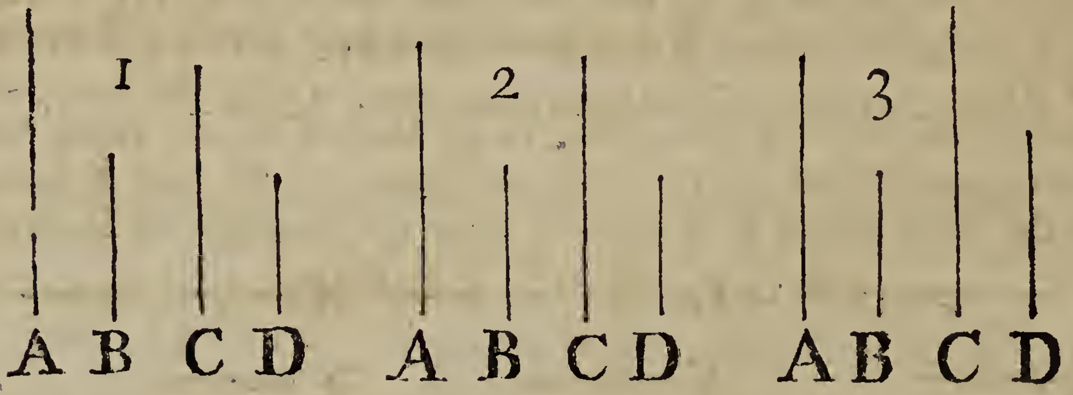
Cor. And if the firſt have a greater ratio to the ſecond, than the third has to the fourth, but the third the ſame ratio to the fourth, which the fifth has to the ſixth; it may be demonſtrated, in like manner, that the firſt has a greater ratio to the ſecond, than the fifth has to the ſixth.

PROP. XIV. THEOR.

See N. IF the firſt has to the ſecond, the ſame ratio which the third has to the fourth; then, if the firſt be greater than the third, the ſecond ſhall be greater than the fourth; and if equal, equal; and if leſs, leſs.

Let the firſt A, have to the ſecond B, the ſame ratio which the third C, has to the fourth D; if A be greater than C, B is greater than D.

Becauſe A is greater than C, and B is any other magnitude,
a 8. 5. A has to B a greater ratio than C to B[a]: But, as A is to B, ſo is C to D; therefore alſo C has to D a greater ratio than C
b 13. 5. has to B[b]. But of two magnitudes, that to which the ſame
c 10. 5. has the greater ratio is the leſſer[c]: Wherefore D is leſs than B; that is, B is greater than D.



Secondly, if A be equal to C, B is equal to D: For A is to
d 9. 5. B, as C, that is A, to D; B therefore is equal to D[d].

Thirdly, if A be leſs than C, B ſhall be leſs than D: For C is greater than A, and becauſe C is to D, as A is to B, D is greater than B, by the firſt caſe; wherefore B is leſs than D. Therefore, if the firſt, &c. Q. E. D.

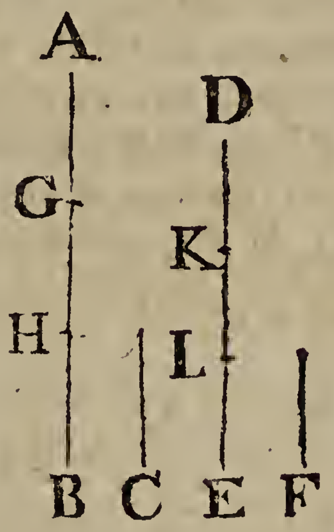
PROP.

PROP. XV. THEOR.

MAGNITUDES have the ſame ratio to one another which their equimultiples have.

Let AB be the ſame multiple of C, that DE is of F: C is to F, as AB to DE.

Becauſe AB is the ſame multiple of C, that DE is of F; there are as many magnitudes in AB equal to C, as there are in DE equal to F: Let AB be divided into magnitudes, each equal to C, viz. AG, GH, HB; and DE into magnitudes, each equal to F, viz. DK, KL, LE: Then the number of the firſt AG, GH, HB, ſhall be equal to the number of the laſt DK, KL, LE: And becauſe AG, GH, HB are all equal, and that DK, KL, LE are alſo equal to one another; therefore AG is to DK, as GH to KL, and as HB to LE[a]: And as one of the antecedents to its conſequent, ſo are all the antecedents together to all the conſequents together[b]; wherefore, as AG is to DK, ſo is AB to DE: But AG is equal to C, and DK to F: Therefore, as C is to F, ſo is AB to DE. Therefore magnitudes, &c. Q. E. D.



[a] 7. 5.

[b] 12. 5.

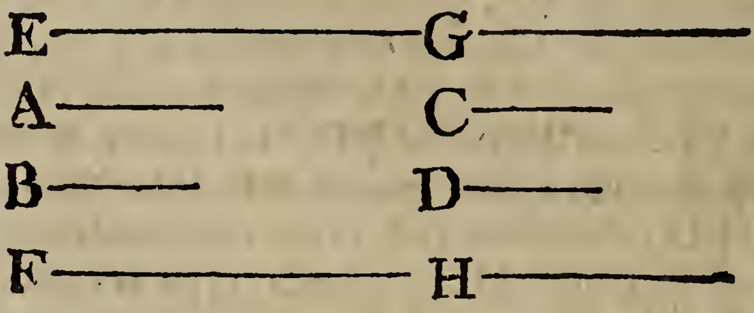
PROP. XVI. THEOR.

IF four magnitudes of the ſame kind be proportionals they ſhall alſo be proportionals when taken alternately.

Let the four magnitudes A, B, C, D be proportionals, viz. as A to B, ſo C to D: They ſhall alſo be proportionals when taken alternately; that is, A is to C, as B to D.

Take of A and B any equimultiples whatever E and F; and of C and D take any equimultiples whatever G and H: and

becauſe E is the ſame multiple of A, that F is of B, and that
magnitudes have the ſame ratio to one another which their
a 15. 5. equimultiples have [a]; therefore A is to B, as E is to F: But as
A is to B, ſo is C to
D: Wherefore as C
b 11. 5. is to D, ſo [b] is E to
F: Again, becauſe
G, H are equimultiples
of C, D, as C
is to D, ſo is G to
H [a]; but as C is to
D, ſo is E to F. Wherefore, as E is to F, ſo is G to H [b]. But,
when four magnitudes are proportionals, if the firſt be greater
than the third, the ſecond ſhall be greater than the fourth; and
c 14. 5. if equal, equal; if leſs, leſs [c]. Wherefore, if E be greater than
G, F likewiſe is greater than H; and if equal, equal; if leſs,
leſs: And E, F are any equimultiples whatever of A, B; and
G, H any whatever of C, D. Therefore A is to C, as B to
d 5. def. 5. D [d]. If then four magnitudes, &c. Q. E. D.



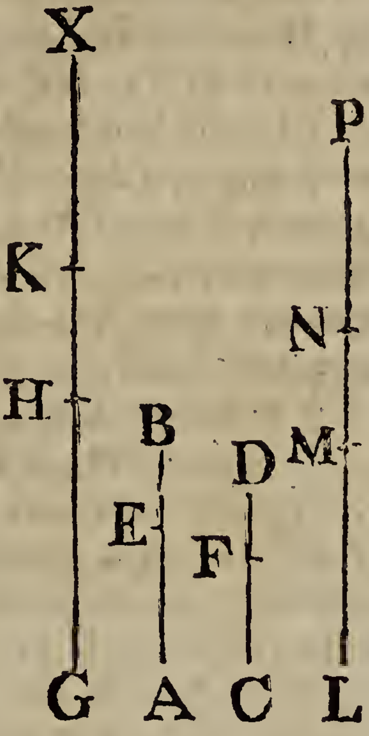
PROP. XVII. THEOR.

See N. IF magnitudes, taken jointly, be proportionals, they ſhall alſo be proportionals when taken ſeparately; that is, if two magnitudes together have to one of them the ſame ratio which two others have to one of theſe, the remaining one of the firſt two ſhall have to the other the ſame ratio which the remaining one of the laſt two has to the other of theſe.

Let AB, BE, CD, DF be the magnitudes taken jointly which are proportionals; that is, as AB to BE, ſo is CD to DF; they ſhall alſo be proportionals taken ſeparately, viz. as AE to EB, ſo CF to FD.

Take of AE, EB, CF, FD any equimultiples whatever GH,
HK, LM, MN; and again, of EB, FD, take any equimultiples
whatever KX, NP: And becauſe GH is the ſame multiple of
a 1. 5. AE, that HK is of EB, wherefore GH is the ſame multiple [a] of
AE, that GK is of AB: But GH is the ſame multiple of AE,
that LM is of CF; wherefore GK is the ſame multiple of AB,

that LM is of CF. Again, becauſe LM is the ſame multiple of CF, that MN is of FD ; therefore LM is the ſame multiple [a] of CF, that LN is of CD ; But LM was ſhown to be the ſame a 1. 5. multiple of CF, that GK is of AB ; GK therefore is the ſame multiple of AB, that LN is of CD ; that is, GK, LN are equimultiples of AB, CD. Next, becauſe HK is the ſame multiple of EB, that MN is of FD ; and that KX is alſo the ſame multiple of EB, that NP is of FD ; therefore HX is the ſame multiple [b] of EB, that MP is of FD. b 2. And becauſe AB is to BE, as CD is to DF, and that of AB and CD, GK and LN are equimultiples, and of EB and FD, HX and MP are equimultiples ; if GK be greater than HX, then LN is greater than MP ; and if equal, equal ; and if leſs, leſs [c] : c 5. def. 5. But if GH be greater than KX, by adding the common part HK to both, GK is greater than HX ; wherefore alſo LN is greater than MP ; and by taking away MN from both, LM is greater than NP : Therefore, if GH be greater than KX, LM is greater than NP. In like manner it may be demonſtrated, that if GH be equal to KX, LM likewiſe is equal to NP ; and if leſs, leſs : And GH, LM are any equimultiples whatever of AE, CF, and KX, NP are any whatever of EB, FD. Therefore [c], as AE is to EB, ſo is CF to FD. If then magnitudes, &c. Q. E. D.

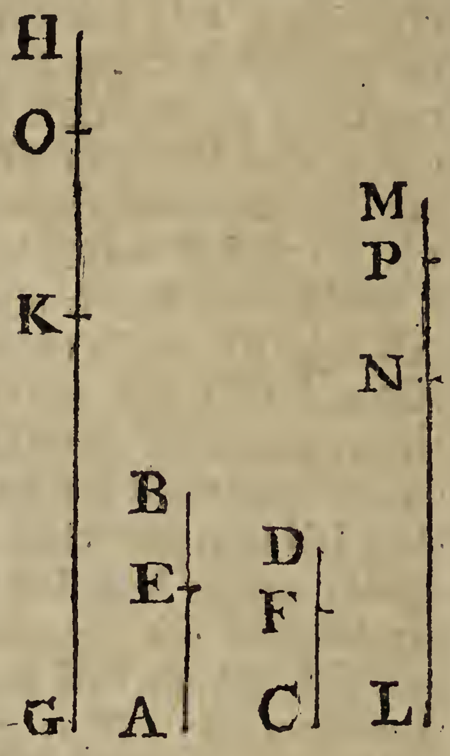


PROP. XVIII. THEOR.

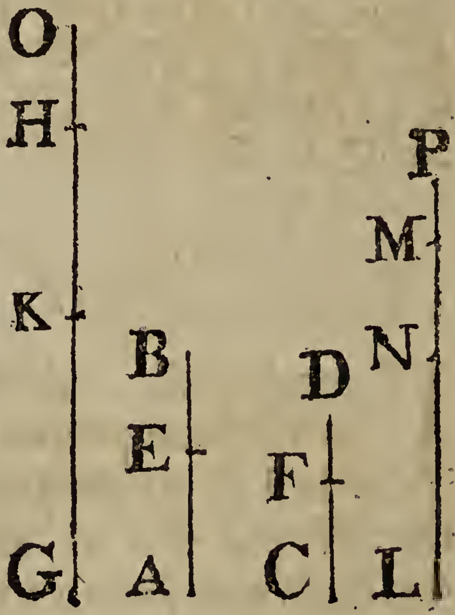
IF magnitudes, taken ſeparately, be proportionals, they ſhall alſo be proportionals when taken jointly, that is, if the firſt be to the ſecond, as the third to the fourth, the firſt and ſecond together ſhall be to the ſecond, as the third and fourth together to the fourth. See N.

Let AE, EB, CF, FD be proportionals ; that is, as AE to EB, ſo is CF to FD ; they ſhall alſo be proportionals when taken jointly ; that is, as AB to BE, ſo CD to DF.

Take of AB, BE, CD, DF any equimultiples whatever GH, HK, LM, MN ; and again, of BE, DF, take any whatever equimultiples KO, NP : And becauſe KO, NP are equimultiples of BE, DF; and that KH, NM are equimultiples likewiſe of BE, DF, if KO, the multiple of BE, be greater than KH, which is a multiple of the ſame BE, NP, likewiſe the multiple of DF, ſhall be greater than MN, the multiple of the ſame DF; and if KO be equal to KH, NP ſhall be equal to NM; and if leſs, leſs.



Firſt, Let KO not be greater than KH, therefore NP is not greater than NM: And becauſe GH, HK are equimultiples of AB, BE, and that AB is greater than BE, therefore GH is a 3. Ax. 5. greater [a] than HK; but KO is not greater than KH, wherefore GH is greater than KO. In like manner it may be ſhewn, that LM is greater than NP. Therefore, if KO be not greater than KH, then GH, the multiple of AB, is always greater than KO, the multiple of BE; and likewiſe LM, the multiple of CD, greater than NP, the multiple of DF.

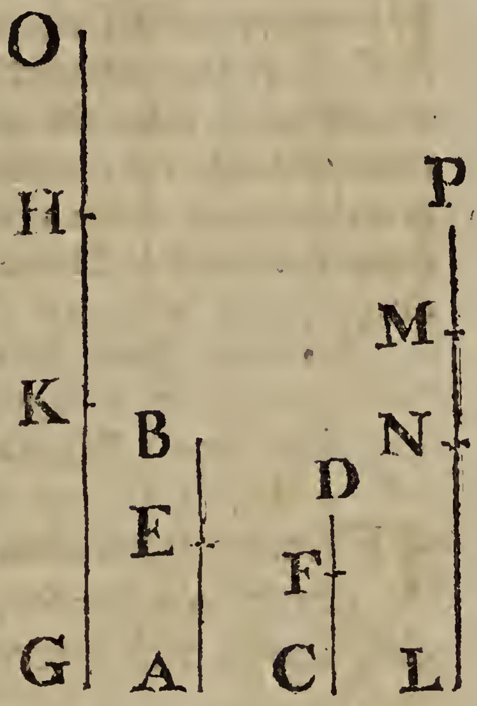


Next, Let KO be greater than KH: therefore, as has been ſhown, NP is greater than NM: And becauſe the whole GH is the ſame multiple of the whole AB, that HK is of BE, the remainder GK is the ſame multiple of b 5. 5. the remainder AE that GH is of AB [b]: which is the ſame that LM is of CD. In like manner, becauſe LM is the ſame multiple of CD, that MN is of DF, the remainder LN is the ſame multiple of the remainder CF, that the whole LM is of the whole CD [b]: But it was ſhown that LM is the ſame multiple of CD, that GK is of AE; therefore GK is the ſame multiple of AE, that LN is of CF; that is, GK, LN are equimultiples of AE, CF: And becauſe KO, NP are equimultiples of BE, DF, if from KO, NP there be taken KH, NM, which are likewiſe equimultiples of BE, DF, the remainders HO, MP are either equal to BE, c 6. 5. DF, or equimultiples of them [c]. Firſt, Let HO, MP, be equal to BE, DF; and becauſe AE is to EB, as CF to FD, and that GK, LN are equimultiples of AE, CF; GK ſhall be to EB, as LN to FD [d]: But HO is equal to EB, and MP to FD; wherefore GK is to HO, as LN to MP. If therefore GK be greater than HO, LN is greater than MP; and if equal, equal; and if leſs [e], leſs.

d Cor. 4. 5.
e A. 5.

But let HO, MP be equimultiples of EB, FD; and becauſe AE is to EB, as CF to FD, and that of AE, CF are taken equimultiples GK, LN; and of EB, FD, the equimultiples HO, MP; if GK be greater than HO, LN is greater than MP; and if equal, equal; and if leſs, leſs [f]; which was likewiſe ſhown in the preceding caſe. If therefore GH be greater than KO, taking KH from both, GK is greater than HO; wherefore alſo LN is greater than MP; and conſequently, adding NM to both, LM is greater than NP: Therefore, if GH be greater than KO, LM is greater than NP. In like manner it may be ſhown, that if GH be equal to KO, LM is equal to NP; and if leſs, leſs. And in the caſe in which KO is not greater than KH, it has been ſhown that GH is always greater than KO, and likewiſe LM than NP: But GH, LM are any equimultiples of AB, CD, and KO, NP are any whatever of BE, DF; therefore [f], as AB is to BE, ſo is CD to DF. If then magnitudes, &c. Q. E. D.

f 5. Def. 5.



PROP. XIX. THEOR.

IF a whole magnitude be to a whole, as a magnitude taken from the firſt, is to a magnitude taken from the other; the remainder ſhall be to the remainder, as the whole to the whole.

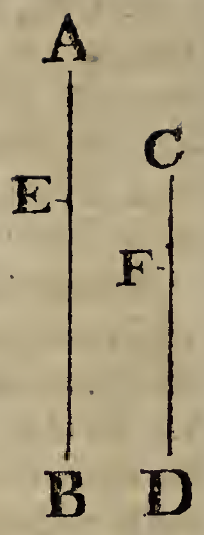
See N.

Let the whole AB, be to the whole CD, as AE, a magnitude taken from AB, to CF, a magnitude taken from CD; the remainder EB ſhall be to the remainder FD, as the whole AB to the whole CD.

Becauſe AB is to CD, as AE to CF; likewiſe, alternately [a],

a 16. 5.

BA is to AE, as DC to CF: And becauſe, if magnitudes, taken jointly, be proportionals, they are
b 17. 5. alſo proportionals [b] when taken ſeparately; therefore, as BE is to DF, ſo is EA to FC; and alternately, as BE is to EA, ſo is DF to FC: But, as AE to CF, ſo by the hypotheſis, is AB to CD; therefore alſo BE, the remainder, ſhall be to the remainder DF, as the whole AB to the whole CD: Wherefore, if the whole, &c. Q. E. D.



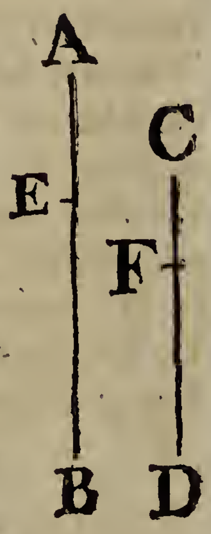
Cor. If the whole be to the whole, as a magnitude taken from the firſt, is to a magnitude taken from the other; the remainder likewiſe is to the remainder, as the magnitude taken from the firſt to that taken from the other: The demonſtration is contained in the preceding.

PROP. E. THEOR.

IF four magnitudes be proportionals, they are alſo proportionals by converſion, that is, the firſt is to its exceſs above the ſecond, as the third to its exceſs above the fourth.

Let AB be to BE, as CD to DF; then BA is to AE, as DC to CF.

Becauſe AB is to BE, as CD to DF, by diviſion
a 17. 5. [a], AE is to EB, as CF to FD; and by inver-
b B. 5. ſion [b], BE is to EA, as DF to FC. Wherefore, by
c 18. 5. compoſition [c], BA is to AE, as DC is to CF: If, therefore, four, &c. Q. E. D.



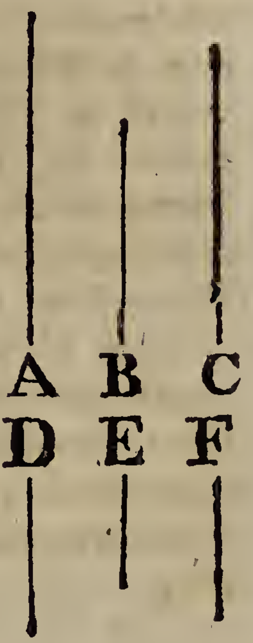
PROP. XX. THEOR.

See N. IF there be three magnitudes, and other three, which, taken two and two, have the ſame ratio; if the firſt be greater than the third, the fourth ſhall be greater than the ſixth; and if equal, equal; and if leſs, leſs.

Let

Book V.

Let A, B, C be three magnitudes, and D, E, F other three, which, taken two and two, have the ſame ratio, viz. as A is to B, ſo is D to E; and as B to C, ſo is E to F. If A be greater than C, D ſhall be greater than F; and, if equal, equal; and if leſs, leſs.

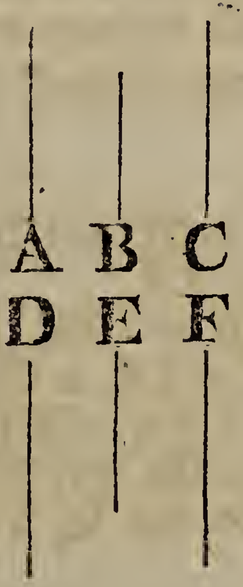


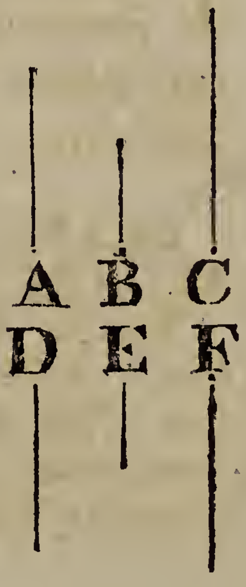
Becauſe A is greater than C, and B is any other magnitude, and that the greater has to the ſame magnitude a greater ratio than the leſs has to it [a]; therefore A has to B a greater ratio than C has to B: But as D is to E, ſo is A to B; therefore [b] D has to E a greater ratio than C to B; and becauſe B is to C, as E to F, by inverſion, C is to B, as F is to E; and D was ſhown to have to E a greater ratio than C to B; therefore D has to E a greater ratio than F to E [c]: But the magnitude which has a greater ratio than another to the ſame magnitude, is the greater of the two [d]: D is therefore greater than F.

a 8. 5.
b 13. 5.
c Cor. 13. 5.
d 10. 5.

Secondly, Let A be equal to C; D ſhall be equal to F: Becauſe A and C are equal to one another, A is to B, as C is to B [e]: But A is to B, as D to E; and C is to B, as F to E; wherefore D is to E, as F to E [f]; and therefore D is equal to F [g].

e 7. 5.
f 11. 5.
g 9. 5.





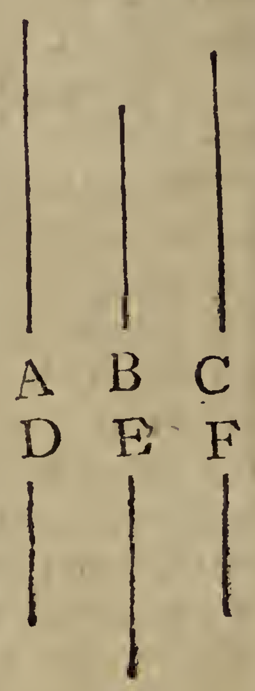
Next, Let A be leſs than C; D ſhall be leſs than F: For C is greater than A, and, as was ſhown in the firſt caſe, C is to B, as F to E, and in like manner B is to A, as E to D; therefore F is greater than D, by the firſt caſe; and therefore D is leſs than F. Therefore, if there be three, &c. Q. E. D.

PROP. XXI. THEOR.

IF there be three magnitudes, and other three, which have the ſame ratio taken two and two, but in a croſs order; if the firſt magnitude be greater than the third, the fourth ſhall be greater than the ſixth; and if equal, equal; and if leſs, leſs.

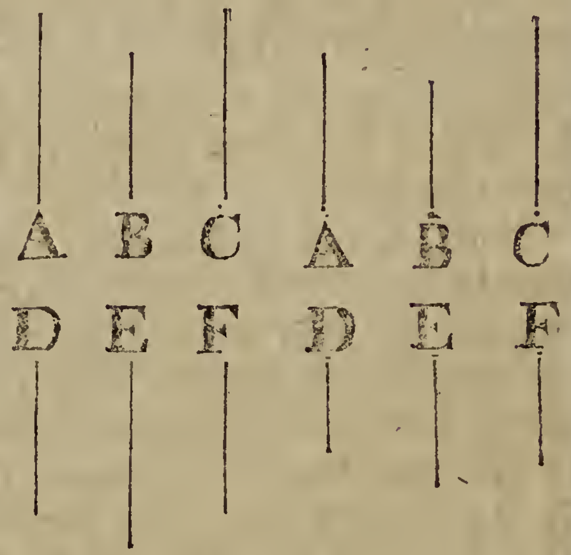
See N.

Let A, B, C be three magnitudes, and D, E, F other three, which have the ſame ratio, taken two and two, but in a croſs order, viz. as A is to B, ſo is E to F, and as B is to C, ſo is D to E. If A be greater than C, D ſhall be greater than F; and if equal, equal; and if leſs, leſs.



Becauſe A is greater than C, and B is any
a 8. 5. other magnitude, A has to B a greater ratio [a]
than C has to B: But as E to F, ſo is A to B:
b 13. 5. therefore [b] E has to F a greater ratio than C to
B: And becauſe B is to C, as D to E, by inverſion, C is to B, as E to D: And E was ſhown
to have to F a greater ratio than C to B; there-
c Cor. 13. 5. fore E has to F a greater ratio than E to D [c];
but the magnitude to which the ſame has a
greater ratio than it has to another, is the leſſer
d 10. 5. of the two [d]: F therefore is leſs than D; that
is, D is greater than F.

Secondly, Let A be equal to C; D ſhall be equal to F. Be-
e 7. 5. cauſe A and C are equal, A is [e] to B, as C is to B: But A is
to B, as E to F; and C is to B,
as E to D; wherefore E is to F
f 11. 5. as E to D [f]; and therefore D is
g 9. 5. equal to F [g].



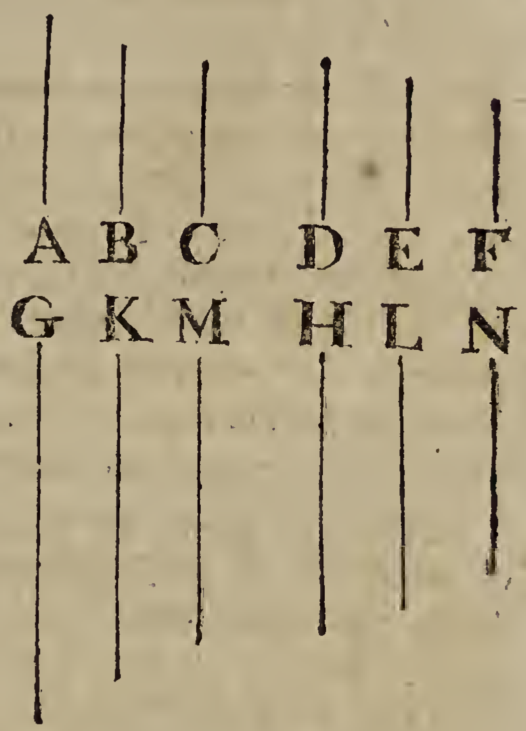
Next, Let A be leſs than C; D ſhall be leſs than F. For C is greater than A, and, as was ſhown, C is to B, as E to D, and in like manner B is to A, as F to E; therefore F is greater than D, by caſe firſt; and therefore D is leſs than F. Therefore, if there be three, &c. Q. E. D.

PROP. XXII. THEOR.

See N. If there be any number of magnitudes, and as many others, which, taken two and two in order, have the ſame ratio; the firſt ſhall have to the laſt of the firſt magnitudes the ſame ratio which the firſt of the others has to the laſt. N. B. *This is uſually cited by the words "ex æquali," or "ex æquo."*

Firſt, Let there be three magnitudes A, B, C, and as many others D, E, F, which taken two and two, have the ſame ratio, that is, ſuch that A is to B as D to E; and as B is to C, ſo is E to F; A ſhall be to C, as D to F.

Take of A and D any equimultiples whatever G and H; and of B and E any equimultiples whatever K and L; and of C and F any whatever M and N: Then, becauſe A is to B, as D to E, and that G, H are equimultiples of A, D, and K, L equimultiples of B, E; as G is to K, ſo is [a] H to L: a 4. 5.
For the ſame reaſon, K is to M, as L to N: and becauſe there are three magnitudes G, K, M, and other three H, L, N, which, two and two, have the ſame ratio; if G be greater than M, H is greater than N; and if equal, equal; and if leſs, leſs [b]; and G, H are b 20. 5.
any equimultiples whatever of A, D, and M, N are any equimultiples whatever of C, F: Therefore [c], as A is to C, ſo is D c 5. Def. 5.
to F.



Next, Let there be four magnitudes, A, B, C, D, and other four, E, F, G, H, which two and two have the ſame ratio, viz. as A is to B, ſo is E to F; and as B to C, ſo F to G; and as C to D, ſo G to H: A ſhall be to D, as E to H.

A. B. C. D.
E. F. G. H.

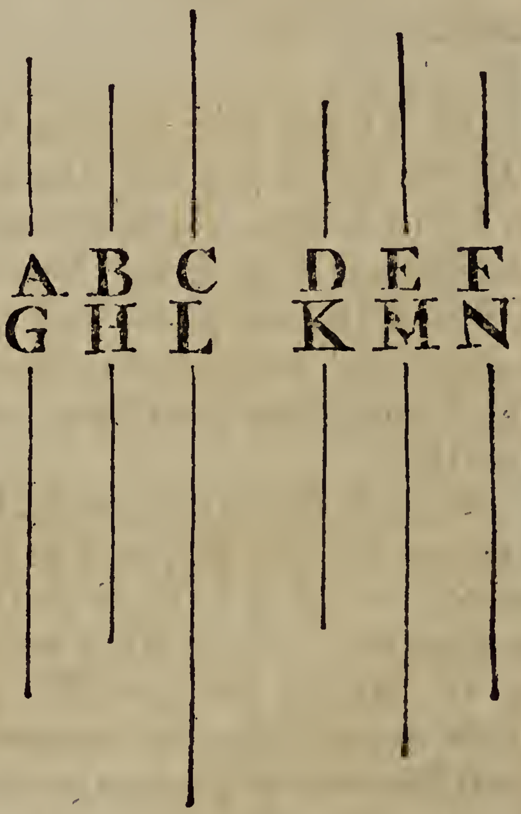
Becauſe A, B, C are three magnitudes, and E, F, G other three, which, taken two and two, have the ſame ratio; by the foregoing caſe, A is to C, as E to G: But C is to D, as G is to H; wherefore again, by the firſt caſe, A is to D, as E to H; and ſo on, whatever be the number of magnitudes. Therefore, if there be any number, &c. Q. E. D.

PROP. XXIII. THEOR.

Sec N. IF there be any number of magnitudes, and as many others, which, taken two and two, in a croſs order, have the ſame ratio; the firſt ſhall have to the laſt of the firſt magnitudes the ſame ratio which the firſt of the others has to the laſt. N. B. *This is uſually cited by the words, " ex æquali in propoportione perturbata;" or " ex " æquo perturbato.*

Firſt, Let there be three magnitudes A, B, C, and other three D, E, F, which, taken two and two, in a croſs order, have the ſame ratio, that is, ſuch that A is to B, as E to F; and as B is to C, ſo is D to E: A is to C, as D to F.

Take of A, B, D any equimultiples whatever G, H, K; and of C, E, F any equimultiples whatever L, M, N: And becauſe G, H are equimultiples of A, B, and that magnitudes have the ſame ratio which their equimultiples have [a]; as A is to B, ſo is G to H: And for the ſame reaſon, as E is to F, ſo is M to N: But as A is to B, ſo is E to F; as therefore G is to H, ſo is M to N [b]. And becauſe as B is to C, ſo is D to E, and that H, K are equimultiples of B, D, and L, M of C, E; as H is to L, ſo is [c] K to M: And it has been ſhown that G is to H, as M to N: Then, becauſe there are three magnitudes G, H, L, and other three K, M, N which have the ſame ratio taken two and two in a croſs order; if G be greater than L, K is greater than N; and if equal, equal; and if leſs, leſs [d]; and G, K are any equimultiples whatever of A, D; and L, N any whatever of C, F; as, therefore, A is to C, ſo is D to F.



Next,

[a] 15. 5.

[b] 11. 5.

[c] 4. 5.

[d] 21. 5.

Next, Let there be four magnitudes, A, B, C, D, and other four E, F, G, H, which taken two and two in a croſs order, have the ſame ratio. viz. A to B, as G to H; B to C, as F to G; and C to D, as E to F: A is to D, as E to H.

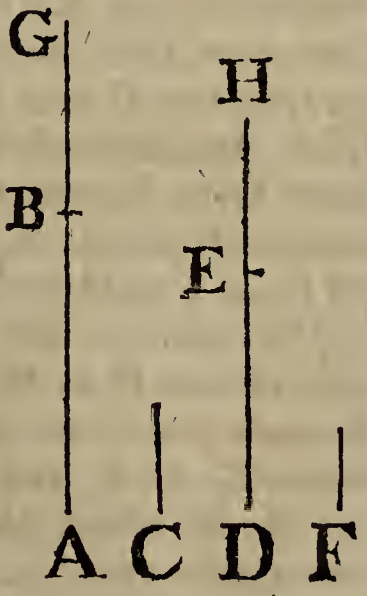
A. B. C. D.
E. F. G. H.

Becauſe A, B, C, are three magnitudes, and F, G, H other three, which, taken two and two in a croſs order, have the ſame ratio; by the firſt caſe, A is to C, as F to H: But C is to D, as E is to F; wherefore again, by the firſt caſe, A is to D, as E to H: And ſo on whatever be the number of magnitudes. Therefore, if there be any number, &c. Q. E. D.

PROP. XXIV. THEOR.

IF the firſt has to the ſecond the ſame ratio which the third has to the fourth; and the fifth to the ſecond, the ſame ratio which the ſixth has to the fourth; the firſt and fifth together ſhall have to the ſecond, the ſame ratio which the third and ſixth together have to the fourth. See N.

Let AB the firſt, have to C the ſecond, the ſame ratio which DE the third, has to F the fourth; and let BG the fifth, have to C the ſecond, the ſame ratio which EH the ſixth, has to F the fourth: AG, the firſt and fifth together, ſhall have to C the ſecond, the ſame ratio which DH, the third and ſixth together, has to F the fourth.



Becauſe BG is to C, as EH to F; by inverſion, C is to BG, as F to EH: And becauſe, as AB is to C, ſo is DE to F; and as C to BG, ſo F to EH; ex æquali [a], AB is to BG, as DE to EH: And becauſe theſe magnitudes are proportionals, they ſhall likewiſe be proportionals when taken jointly [b]; as therefore AG is to GB, ſo is DH to HE; but as GB to C, ſo is HE to F. Therefore, ex æquali [a], as AG is to C, ſo is DH to F. Wherefore, if the firſt, &c. Q. E. D.

a 22. 5.

b 18. 5.

Cor. 1. If the ſame hypotheſis be made as in the propoſition, the exceſs of the firſt and fifth ſhall be to the ſecond, as the exceſs of the third and ſixth to the fourth: The demonſtration of this is the ſame with that of the propoſition, if diviſion be uſed inſtead of compoſition.

Cor. 2. The propoſition holds true of two ranks of magnitudes, whatever be their number, of which each of the firſt rank has to the ſecond magnitude the ſame ratio that the correſponding one of the ſecond rank has to a fourth magnitude; as is manifeſt.

PROP. XXV. THEOR.

If four magnitudes of the ſame kind are proportionals, the greateſt and leaſt of them together are greater than the other two together.

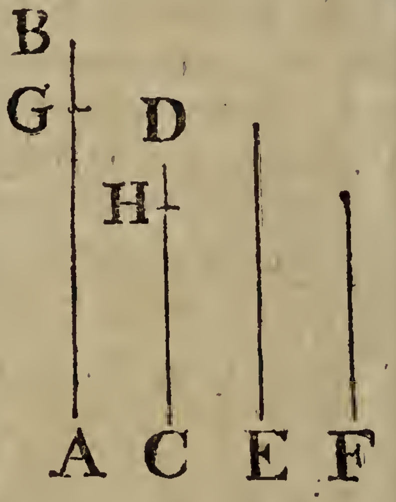
Let the four magnitudes AB, CD, E, F be proportionals, viz. AB to CD, as E to F; and let AB be the greateſt of them, and conſequently F the leaſt [a]. AB, together with F, are greater than CD, together with E.

a A. & 14. 5.

Take AG equal to E, and CH equal to F: Then becauſe as AB is to CD, ſo is E to F, and that AG is equal to E, and CH equal to F; AB is to CD, as AG to CH. And becauſe AB the whole, is to the whole CD, as AG is to CH, likewiſe the remainder GB ſhall be to the remainder HD, as the whole AB is to the whole [b] CD: But AB is greater than CD, therefore [c] GB is greater than HD: And becauſe AG is equal to E, and CH to F; AG and F together are equal to CH and E together. If therefore to the unequal magnitudes GB, HD, of which GB is the greater, there be added equal magnitudes, viz. to GB the two AG and F, and CH and E to HD; AB and F together are greater than CD and E. Therefore, if four magnitudes, &c. Q. E. D.

b 19. 5.

c A. 5.



PROP. F. THEOR.

See N.

Ratios which are compounded of the ſame ratios, are the ſame with one another.

Let A be to B, as D to E; and B to C, as E to F: The ratio which is compounded of the ratios of A to B, and B to C, which, by the definition of compound ratio, is the ratio of A to C, is the ſame with the ratio of D to F, which, by the ſame definition, is compounded of the ratios of D to E, and E to F.

A. B. C.
D. E. F.

Becauſe there are three magnitudes A, B, C, and three others D, E, F, which, taken two and two in order, have the ſame ratio; ex æquali, A is to C, as D to F [a].

Next, Let A be to B, as E to F, and B to C, as D to E; therefore, *ex æquali in proportione perturbata* [b], A is to C, as D to F; that is, the ratio of A to C, which is compounded of the ratios of A to B, and B to C, is the ſame with the ratio of D to F, which is compounded of the ratios of D to E, and E to F: And in like manner the propoſition may be demonſtrated whatever be the number of ratios in either caſe.

A. B. C.
D. E. F.

[a] 22. 5.

[b] 23. 5.

PROP. G. THEOR.

See N.

IF ſeveral ratios be the ſame with ſeveral ratios, each to each; the ratio which is compounded of ratios which are the ſame with the firſt ratios, each to each, is the ſame with the ratio compounded of ratios which are the ſame with the other ratios, each to each.

Let A be to B, as E to F; and C to D, as G to H: And let A be to B, as K to L; and C to D, as L to M: Then the ratio of K to M, by the definition of compound ratio, is compounded of the ratios of K to L, and L to M, which are the ſame with the ratios of A to B, and C to D: And as E to F, ſo let N be to O; and as G to H, ſo let O be to P; then the ratio of N to P is compounded of the ratios of N to O, and O to P, which are the ſame with the ratios of E to F, and G to H: And it is to be ſhewn that the ratio of K to M, is the ſame with the ratio of N to P, or that K is to M, as N to P.

A. B. C. D. K. L. M.
E. F. G. H. N. O. P.

Becauſe K is to L, as (A to B, that is, as E to F, that is, as) N to O; and as L to M, ſo is (C to D, and ſo is G to H,

and

and ſo is) O to P: Ex æquali [a] K is to M, as N to P. Therefore, if ſeveral ratios, &c. Q. E. D.

a 22. 5.

PROP. H. THEOR.

See N.

IF a ratio compounded of ſeveral ratios be the ſame with a ratio compounded of any other ratios, and if one of the firſt ratios, or a ratio compounded of any of the firſt, be the ſame with one of the laſt ratios, or with the ratio compounded of any of the laſt; then the ratio compounded of the remaining ratios of the firſt, or the remaining ratio of the firſt, if but one remain, is the ſame with the ratio compounded of thoſe remaining of the laſt, or with the remaining ratio of the laſt.

Let the firſt ratios be thoſe of A to B, B to C, C to D, D to E, and E to F; and let the other ratios be thoſe of G to H, H to K, K to L, and L to M: Alſo, let the ratio of A to F, which is compounded of [a] the firſt ratios be the ſame with the ratio of G to M, which is compounded of the other ratios: And beſides, let the ratio of A to D, which is compounded of the ratios of A to B, B to C, C to D, be the ſame with the ratio of G to K, which is compounded of the ratios of G to H, and H to K: Then the ratio compounded of the remaining firſt ratios, to wit, of the ratios of D to E and E to F, which compounded ratio is the ratio of D to F, is the ſame with the ratio of K to M, which is compounded of the remaining ratios of K to L, and L to M of the other ratios.

a Definition of compounded ratio.

A. B. C. D. E. F.
G. H. K. L. M.

Becauſe, by the hypotheſis, A is to D, as G to K, by inverſion [b], D is to A, as K to G; and as A is to F, ſo is G to M; therefore [c], ex æquali, D is to F, as K to M. If therefore a ratio which is, &c. Q. E. D.

b B. 5.
c 22. 5.

PROP.

PROP. K. THEOR.

IF there be any number of ratios, and any number of other ratios ſuch, that the ratio compounded of ratios which are the ſame with the firſt ratios, each to each, is the ſame with the ratio compounded of ratios which are the ſame, each to each, with the laſt ratios; and if one of the firſt ratios, or the ratio which is compounded of ratios which are the ſame with ſeveral of the firſt ratios, each to each, be the ſame with one of the laſt ratios, or with the ratio compounded of ratios which are the ſame, each to each, with ſeveral of the laſt ratios: Then the ratio compounded of ratios which are the ſame with the remaining ratios of the firſt, each to each, or the remaining ratio of the firſt, if but one remain; is the ſame with the ratio compounded of ratios which are the ſame with thoſe remaining of the laſt, each to each, or with the remaining ratio of the laſt. See N.

Let the ratios of A to B, C to D, E to F be the firſt ratios; and the ratios of G to H, K to L, M to N, O to P, Q to R, be the other ratios: And let A be to B, as S to T; and C to D, as T to V, and E, to F, as V to X: Therefore, by the definition of compound ratio, the ratio of S to X is compounded of the ratios of S to T, T to V, and V to X, which are the ſame with the ratios of A to B, C to D, E to F, each to each: Alſo, as G to H, ſo let Y be to Z; and K to L, as Z to a; M to N, as a to b, O to P, as b to c; and Q to R, as c to d: Therefore, by the ſame definition, the ratio of Y to d is compounded of the ratios of Y to Z, Z to a, a to b, b to c, and c to

h, k, l.
A, B; C, D; E, F. S, T, V, X.
G, H; K, L; M, N; O, P; Q, R. Y, Z, a, b, c, d.
e, f, g. m, n, o, p.

c to d, which are the ſame, each to each, with the ratios of G to H, K to L, M to N, O to P, and Q to R: Therefore, by the hypotheſis, S is to X, as Y to d: Alſo, let the ratio of A to B, that is, the ratio of S to T, which is one of the firſt ratios, be the ſame with the ratio of e to g, which is compounded of the ratios of e to f, and f to g, which, by the hypotheſis, are the ſame with the ratios of G to H, and K to L, two of the other ratios; and let the ratio of h to l be that which is compounded of the ratios of h to k, and k to l, which are the ſame with the remaining firſt ratios, viz. of C to D, and E to F; alſo, let the ratio of m to p, be that which is compounded of the ratios of m to n, n to o, and o to p, which are the ſame, each to each, with the remaining other ratios, viz. of M to N, O to P, and Q to R: Then the ratio of h to l is the ſame with the ratio of m to p, or h is to l, as m to p.

h, k, l.
A, B; C, D; E, F. S, T, V, X.
G, H; K, L; M, N; O, P; Q, R. Y, Z, a, b, c, d.
e, f, g. m, n, o, p.

Becauſe e is to f, as (G to H, that is, as) Y to Z; and f is to g, as (K to L, that is, as) Z to a; therefore, ex æquali, e is to g, as Y to a: And by the hypotheſis, A is to B, that is, S to T, as e to g; wherefore S is to T, as Y to a; and, by inverſion, T is to S, as a to Y; and S is to X, as Y to d; therefore, ex æquali, T is to X, as a to d: Alſo, becauſe h is to k as (C to D, that is, as) T to V; and k is to l, as (E to F, that is, as) V to X; therefore, ex æquali, h is to l, as T to X: In like manner, it may be demonſtrated, that m is to p, as a to d:
a 11. 5. And it has been ſhown, that T is to X, as a to d; therefore [a] h is to l, as m to p. Q. E. D.

The propoſitions G and K are uſually, for the ſake of brevity, expreſſed in the ſame terms with propoſitions F and H: And therefore it was proper to ſhow the true meaning of them when they are ſo expreſſed; eſpecially ſince they are very frequently made uſe of by geometers.

THE

ELEMENTS

OF

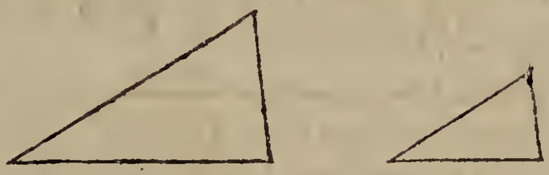
EUCLID.

BOOK VI.

DEFINITIONS.

I.

SIMILAR rectilineal figures are thoſe which have their ſeveral angles equal, each to each, and the ſides about the equal angles proportionals.



II.

See N.

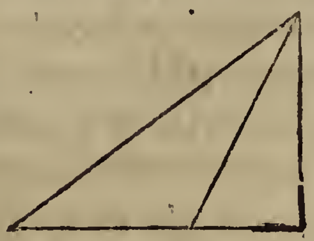
" Reciprocal figures, viz. triangles and parallelograms, are " ſuch as have their ſides about two of their angles propor- " tionals in ſuch manner, that a ſide of the firſt figure is to " a ſide of the other, as the remaining ſide of this other is to " the remaining ſide of the firſt."

III.

A ſtraight line is ſaid to be cut in extreme and mean ratio, when the whole is to the greater ſegment, as the greater ſegment is to the leſs.

IV.

The altitude of any figure is the ſtraight line drawn from its vertex perpendicular to the baſe.



PROP.

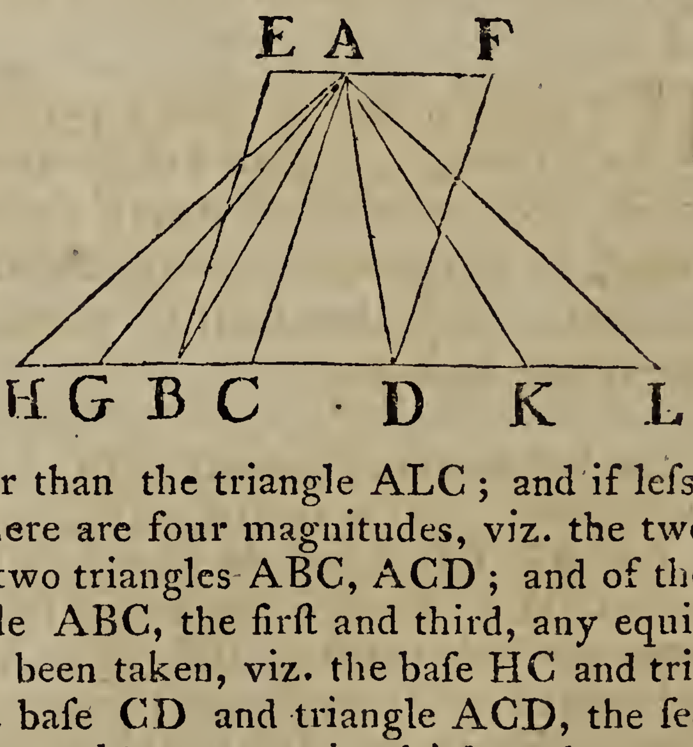
PROP. I. THEOR.

See N.

TRiangles and parallelograms of the ſame altitude are one to another as their baſes.

Let the triangles ABC, ACD, and the parallelograms EC, CF have the ſame altitude, viz. the perpendicular drawn from the point A to BD: Then, as the baſe BC is to the baſe CD, ſo is the triangle ABC to the triangle ACD, and the parallelogram EC to the parallelogram CF.

Produce BD both ways to the points H, L, and take any number of ſtraight lines BG, GH, each equal to the baſe BC; and DK, KL, any number of them, each equal to the baſe CD; and join AG, AH, AK, AL: Then, becauſe CB, BG, GH are all equal, the triangles AHG, AGB, ABC are all equal [a]: Therefore, whatever multiple the baſe HC is of the baſe BC, the ſame multiple is the triangle AHC of the triangle ABC: For the ſame reaſon, whatever multiple the baſe LC is of the baſe CD, the ſame multiple is the triangle ALC of the triangle ADC: And if the baſe HC be equal to the baſe CL, the triangle AHC is alſo equal to the triangle ALC [a]; and if the baſe HC be greater than the baſe CL, likewiſe the triangle AHC is greater than the triangle ALC; and if leſs, leſs: Therefore, ſince there are four magnitudes, viz. the two baſes BC, CD, and the two triangles ABC, ACD; and of the baſe BC and the triangle ABC, the firſt and third, any equimultiples whatever have been taken, viz. the baſe HC and triangle AHC; and of the baſe CD and triangle ACD, the ſecond and fourth, have been taken any equimultiples whatever, viz. the baſe CL and triangle ALC; and that it has been ſhown, that, if the baſe HC be greater than the baſe CL, the triangle AHC is greater than the triangle ALC; and if equal, equal; and if leſs, leſs: Therefore [b], as the baſe BC is to the baſe CD, ſo is the triangle ABC to the triangle ACD.



[a] 38. 1.

[b] 5. def. 5.

And becauſe the parallelogram CE is double of the triangle

ABC [c], and the parallelogram CF double of the triangle ACD, and that magnitudes have the ſame ratio which their equimultiples have [d]; as the triangle ABC is to the triangle ACD, ſo is the parallelogram EC to the parallelogram CF: And becauſe it has been ſhown, that, as the baſe BC is to the baſe CD, ſo is the triangle ABC to the triangle ACD; and as the triangle ABC is to the triangle ACD, ſo is the parallelogram EC to the parallelogram CF; therefore, as the baſe BC is to the baſe CD, ſo is [e] the parallelogram EC to the parallelogram CF. Wherefore triangles, &c. Q. E. D.

c 41. 1.
d 15. 5.
e 11. 5.

Cor. From this it is plain, that triangles and parallelograms that have equal altitudes, are one to another as their baſes.

Let the figures be placed ſo as to have their baſes in the ſame ſtraight line; and having drawn perpendiculars from the vertices of the triangles to the baſes, the ſtraight line which joins the vertices is parallel to that in which their baſes are [f], becauſe the perpendiculars are both equal and parallel to one another. Then, if the ſame conſtruction be made as in the propoſition, the demonſtration will be the ſame.

f 33. 1.

PROP. II. THEOR.

IF a ſtraight line be drawn parallel to one of the ſides of a triangle, it ſhall cut the other ſides, or thoſe produced, proportionally: And if the ſides, or the ſides produced, be cut proportionally, the ſtraight line which joins the points of ſection ſhall be parallel to the remaining ſide of the triangle.

See N.

Let DE be drawn parallel to BC, one of the ſides of the triangle ABC: BD is to DA, as CE to EA.

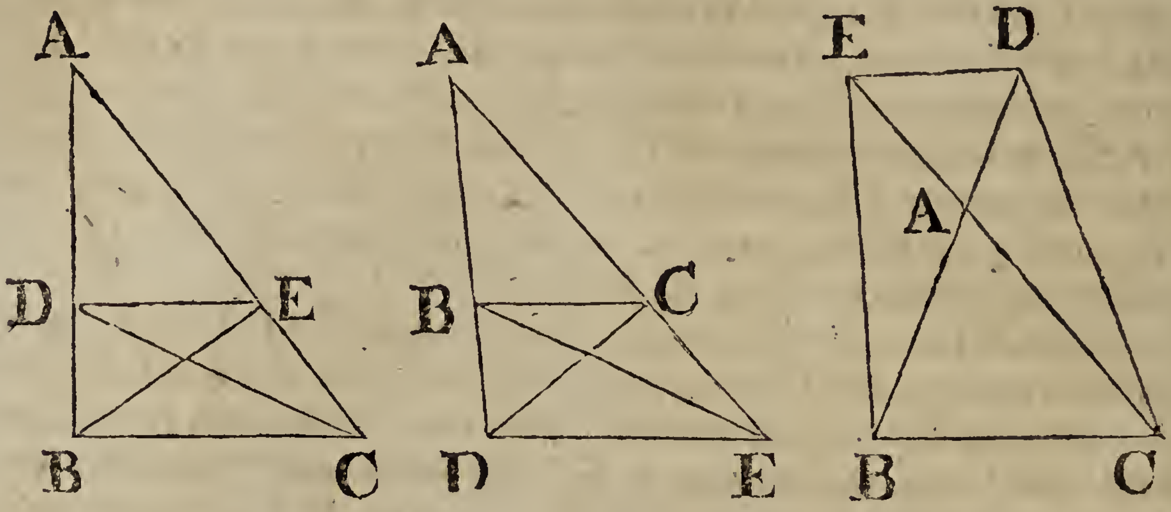
Join BE, CD; then the triangle BDE is equal to the triangle CDE [a], becauſe they are on the ſame baſe DE, and between the ſame parallels DE, BC: ADE is another triangle, and equal magnitudes have to the ſame, the ſame ratio [b]; therefore, as the triangle BDE to the triangle ADE, ſo is the triangle CDE to the triangle ADE; but as the triangle BDE to the triangle ADE, ſo is [c] BD to DA, becauſe having the ſame altitude, viz. the perpendicular drawn from the point E to AB, they are to one another as their baſes; and for the ſame reaſon, as

a 37. 1.
b 7. 5.
c 1. 6.

as the triangle CDE to the triangle ADE, ſo is CE to EA. Therefore, as BD to DA, ſo is CE to EA[d].

d 11. 5.

Next, Let the ſides AB, AC of the triangle ABC, or theſe



produced, be cut proportionally in the points D, E, that is, ſo that BD be to DA, as CE to EA, and join DE; DE is parallel to BC.

The ſame conſtruction being made, Becauſe as BD to DA, ſo is CE to EA; and as BD to DA, ſo is the triangle BDE to the triangle ADE[e]; and as CE to EA, ſo is the triangle CDE to the triangle ADE; therefore the triangle BDE is to the triangle ADE, as the triangle CDE to the triangle ADE; that is, the triangles BDE, CDE have the ſame ratio to the triangle ADE; and therefore[f] the triangle BDE is equal to the triangle CDE: and they are on the ſame baſe DE; but equal triangles on the ſame baſe are between the ſame parallels[g]; therefore DE is parallel to BC. Wherefore, if a ſtraight line, &c. Q. E. D.

e 1. 6.

f 9. 5.

g 39. 1.

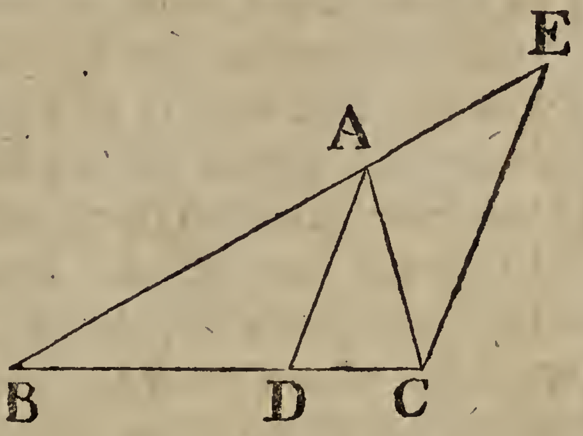
PROP. III. THEOR.

See N.

IF the angle of a triangle be divided into two equal angles, by a ſtraight line which alſo cuts the baſe; the ſegments of the baſe ſhall have the ſame ratio which the other ſides of the triangle have to one another: And if the ſegments of the baſe have the ſame ratio which the other ſides of the triangle have to one another, the ſtraight line drawn from the vertex to the point of ſection, divides the vertical angle into two equal angles.

Let the angle BAC of any triangle ABC be divided into two equal angles by the ſtraight line AD: BD is to DC, as BA to AC.

Through the point C draw CE parallel [a] to DA, and let BA produced meet CE in E. Becauſe the ſtraight line AC meets the parallels AD, EC, the angle ACE is equal to the alternate angle CAD [b]: But CAD, by the hypotheſis, is equal to the angle BAD; wherefore BAD is equal to the angle ACE. Again, becauſe the ſtraight line BAE meets the parallels AD, EC, the outward angle BAD is equal to the inward and oppoſite angle AEC: But the angle ACE has been proved equal to the angle BAD; therefore alſo ACE is equal to the angle AEC, and conſequently the ſide AE is equal to the ſide [c] AC: And becauſe AD is drawn parallel to one of the ſides of the triangle BCE, viz. to EC, BD is to DC, as BA to AE [d]; but AE is equal to AC; therefore, as BD DC, ſo is BA to AC [e].



a 31. 1.
b 29. 1.
c 6. 1.
d 2. 6.
e 7. 5.

Let now BD be to DC, as BA to AC, and join AD; the angle BAC is divided into two equal angles by the ſtraight line AD.

The ſame conſtruction being made; becauſe, as BD to DC, ſo is BA to AC; and as BD to DC, ſo is BA to AE [d], becauſe AD is parallel to EC; therefore BA is to AC, as BA to AE [f]: Conſequently AC is equal to AE [g], and the angle AEC is therefore equal to the angle ACE [h]: But the angle AEC is equal to the outward and oppoſite angle BAD; and the angle ACE is equal to the alternate angle CAD [b]: Wherefore alſo the angle BAD is equal to the angle CAD: Therefore the angle BAC is cut into two equal angles by the ſtraight line AD. Therefore, if the angle, &c. Q. E. D.

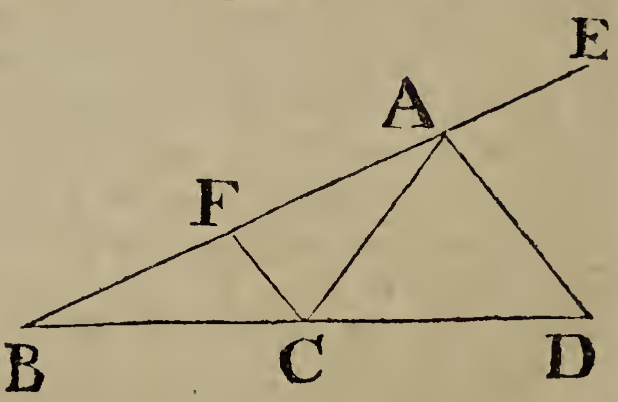
f 11. 5.
g 9. 5.
h 5. 1.

PROP.

PROP. A. THEOR.

IF the outward angle of a triangle made by producing one of its ſides, be divided into two equal angles, by a ſtraight line which alſo cuts the baſe produced; the ſegments betweeen the dividing line and the extremities of the baſe have the ſame ratio which the other ſides of the triangle have to one another: And if the ſegments of the baſe produced, have the ſame ratio which the other ſides of the triangle have, the ſtraight line drawn from the vertex to the point of ſection divides the outward angle of the triangle into two equal angles.

Let the outward angle CAE of any triangle ABC be divided into two equal angles by the ſtraight line AD which meets the baſe produced in D: BD is to DC, as BA to AC.



a 31. 1. Through C draw CF parallel to AD [a]; and becauſe the ſtraight
line AC meets the parallels AD, FC, the angle ACF is equal
b 29. 1. to the alternate angle CAD [b]: But CAD is equal to the angle
c Hyp. DAE [c]; therefore alſo DAE is equal to the angle ACF. Again, becauſe the ſtraight line FAE meets the parallels AD, FC, the outward angle DAE is equal to the inward and oppoſite angle CFA: But the angle ACF has been proved equal to the angle DAE; therefore alſo the angle ACF is equal to the angle CFA, and conſequently the ſide AF is equal to the ſide
d 6. 1. AC [d]: And becauſe AD is parallel to FC, a ſide of the triangle
e 2. 6. BCF, BD is to DC, as BA to AF [e]; but AF is equal to AC; as therefore BD is to DC, ſo is BA to AC.

Let now BD be to DC, as BA to AC, and join AD; the angle CAD is equal to the angle DAE.

The ſame conſtruction being made, becauſe BD is to DC,
f 11. 5. as BA to AC; and that BD is alſo to DC, as BA to AF [f];
g 9. 5. therefore BA is to AC, as BA to AF [g]; wherefore AC is equal
h 5. 1. to AF [h], and the angle AFC equal [h] to the angle ACF: But
the

the angle AFC is equal to the outward angle EAD, and the angle ACF to the alternate angle CAD; therefore alſo EAD is equal to the angle CAD. Wherefore, if the outward, &c. Q. E. D.

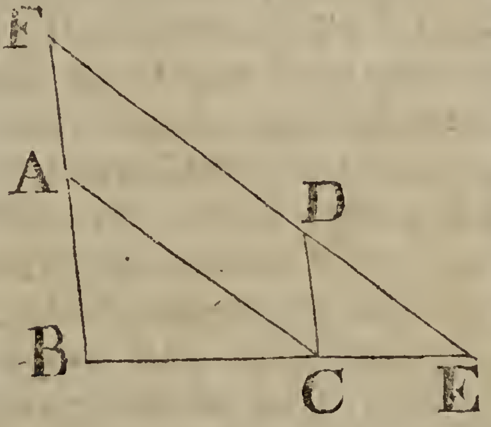
PROP. IV. THEOR.

THE ſides about the equal angles of equiangular triangles are proportionals; and thoſe which are oppoſite to the equal angles are homologous ſides, that is, are the antecedents or conſequents of the ratios.

Let ABC, DCE be equiangular triangles, having the angle ABC equal to the angle DCE, and the angle ACB to the angle DEC, and conſequently [a] the angle BAC equal to the angle CDE. The ſides about the equal angles of the triangles ABC, DCE are proportionals; and thoſe are the homologous ſides which are oppoſite to the equal angles.

a 32. 1.

Let the triangle DCE be placed, ſo that its ſide CE may be contiguous to BC, and in the ſame ſtraight line with it: And becauſe the angles ABC, ACB are together leſs than two right angles [b], ABC, and DEC, which is equal to ACB, are alſo leſs than two right angles; wherefore BA, ED produced ſhall meet [c]; let them be produced and meet in the point F: And becauſe the angle ABC is equal to the angle DCE, BF is parallel [d] to CD. Again, becauſe the angle ACB is equal to the angle DEC, AC is parallel to FE [d]: Therefore FACD is a parallelogram; and conſequently AF is equal to CD, and AC to FD [e]: And becauſe AC is parallel to FE, one of the ſides of the triangle FBE, BA is to AF, as BC to CE [f]: But AF is equal to CD; therefore [g], as BA to CD, ſo is BC to CE; and alternately, as AB to BC, ſo is DC to CE [b]: Again, becauſe CD is parallel to BF, as BC to CE, ſo is FD to DE [f]; but FD is equal to AC; therefore, as BC to CE, ſo is AC to DE: And alternately, as BC to CA, ſo CE to DE: Therefore, becauſe it has been proved that AB is to BC, as DC to CE, and as BC to CA, ſo CE to ED, ex æquali [h], BA is to AC as CD to DE. Therefore the ſides, &c. Q. E. D.



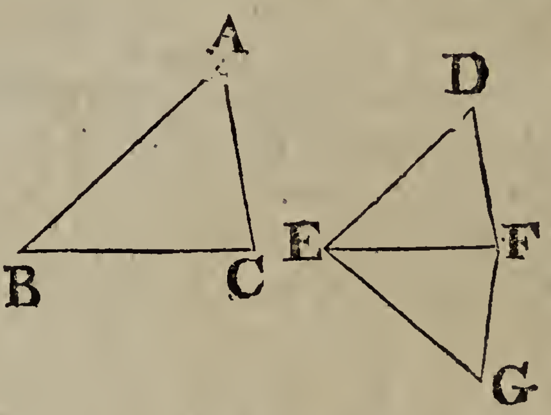
b 17. 1.
c 12. Ax. 1.
d 28. 1.
e 34. 1.
f 2. 6.
g 7. 5.
h 22. 5.

PROP. V. THEOR.

IF the ſides of two triangles, about each of their angles, be proportionals, the triangles ſhall be equiangular, and have their equal angles oppoſite to the homologous ſides.

Let the triangles ABC, DEF have their ſides proportionals, ſo that AB is to BC, as DE to EF; and BC to CA, as EF to FD; and conſequently, ex æquali, BA to AC, as ED to DF; the triangle ABC is equiangular to the triangle DEF, and their equal angles are oppoſite to the homologous ſides, viz. the angle ABC equal to the angle DEF, and BCA to EFD, and alſo BAC to EDF.

a 23. 1. At the points E, F, in the ſtraight line EF, make [a] the angle
FEG equal to the angle ABC, and the angle EFG equal to
BCA; wherefore the remain-
ing angle BAC is equal to the
b 32. 1. remaining angle EGF [b], and
the triangle ABC is there-
fore equiangular to the tri-
angle GEF; and conſequent-
ly they have their ſides op-
poſite to the equal angles pro-
c 4. 6. portionals [c]. Wherefore, as
AB to BC, ſo is GE to EF;
d 11. 5. but as AB to BC, ſo is DE to EF; therefore as DE to EF, ſo [d]
GE to EF: Therefore DE and GE have the ſame ratio to EF,
e 9. 5. and conſequently are equal [e]: For the ſame reaſon, DF is equal
to FG: And becauſe, in the triangles DEF, GEF, DE is equal
to EG, and EF common, the two ſides DE, EF are equal to
the two GE, EF, and the baſe DF is equal to the baſe GF;
f 8. 1. therefore the angle DEF is equal [f] to the angle GEF, and the
other angles to the other angles which are ſubtended by the e-
g 4. 1. qual ſides [g]. Wherefore the angle DFE is equal to the angle
GFE, and EDF to EGF: And becauſe the angle DEF is equal
to the angle GEF, and GEF to the angle ABC; therefore the
angle ABC is equal to the angle DEF: For the ſame reaſon,
the angle ACB is equal to the angle DFE, and the angle at A
to the angle at D. Therefore the triangle ABC is equiangular
to the triangle DEF. Wherefore, if the ſides, &c. Q. E. D.

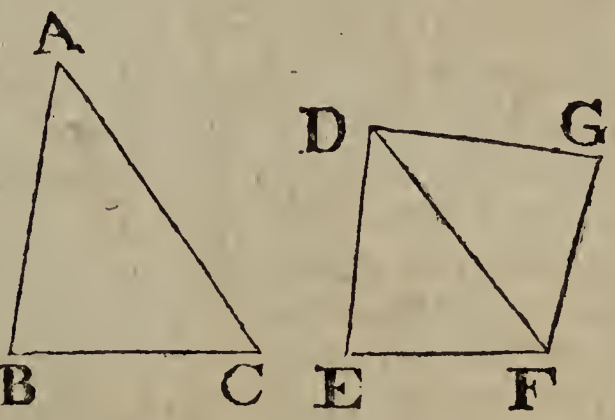


PROP. VI. THEOR.

IF two triangles have one angle of the one equal to one angle of the other, and the ſides about the equal angles proportionals, the triangles ſhall be equiangular, and ſhall have thoſe angles equal which are oppoſite to the homologous ſides.

Let the triangles ABC, DEF have the angle BAC in the one equal to the angle EDF in the other, and the ſides about thoſe angles proportionals; that is, BA to AC, as ED to DF; the triangles ABC, DEF are equiangular, and have the angle ABC equal to the angle DEF, and ACB to DFE.

At the points D, F, in the ſtraight line DF, make [a] the angle FDG equal to either of the angles BAC, EDF; and the angle DFG equal to the angle ACB: Wherefore the remaining angle at B is equal to the remaining one at G [b], and conſequently the triangle ABC is equiangular to the triangle DGF; and therefore as BA to AC, ſo is [c] GD to DF: But, by the hypotheſis, as BA to AC, ſo is ED to DF; as therefore ED to DF, ſo is [d] GD to DF; wherefore ED is equal [e] to DG; and DF is common to the two triangles EDF, GDF: Therefore the two ſides ED, DF are equal to the two ſides GD, DF; and the angle EDF is equal to the angle GDF; wherefore the baſe EF is equal to the baſe FG [f], and the triangle EDF to the triangle GDF, and the remaining angles to the remaining angles, each to each, which are ſubtended by the equal ſides: Therefore the angle DFG is equal to the angle DFE, and the angle at G to the angle at E: But the angle DFG is equal to the angle ACB; therefore the angle ACB is equal to the angle DFE: And the angle BAC is equal to the angle EDF [g]; wherefore alſo the remaining angle at B is equal to the remaining angle at E. Therefore the triangle ABC is equiangular to the triangle DEF. Wherefore, if two triangles, &c. Q. E. D.



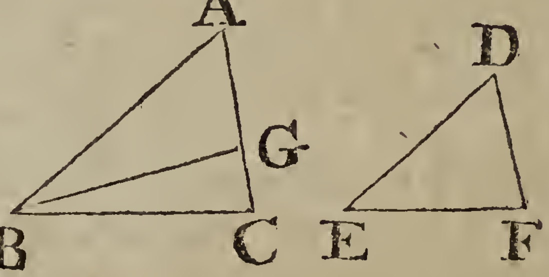
a 23. 1.
b 32. 1.
c 4. 6.
d 11. 5.
e 9. 5.
f 4. 1.
g Hyp.

PROP. VII. THEOR.

See N.

IF two triangles have one angle of the one equal to one angle of the other, and the ſides about two other angles proportionals, then, if each of the remaining angles be either leſs, or not leſs, than a right angle; or if one of them be a right angle: The triangles ſhall be equiangular, and have thoſe angles equal about which the ſides are proportionals.

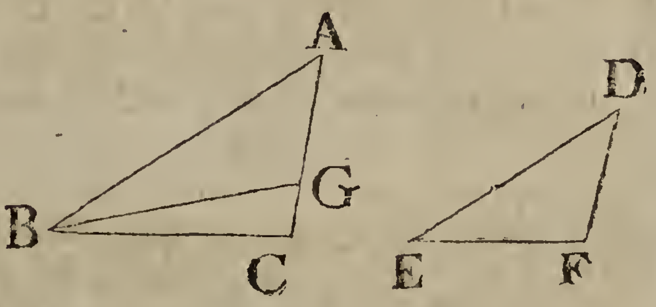
Let the two triangles ABC, DEF have one angle in the one equal to one angle in the other, viz. the angle BAC to the angle EDF, and the ſides about two other angles ABC, DEF proportionals, ſo that AB is to BC, as DE to EF; and, in the firſt caſe, let each of the remaining angles at C, F be leſs than a right angle. The triangle ABC is equiangular to the triangle DEF, viz. the angle ABC is equal to the angle DEF, and the remaining angle at C to the remaining angle at F.



For, if the angles ABC, DEF be not equal, one of them is
greater than the other: let ABC be the greater, and at the
point B, in the ſtraight line
AB, make the angle ABG
a 23. 1. equal to the angle [a] DEF:
And becauſe the angle at A
is equal to the angle at D,
and the angle ABG to the
angle DEF; the remaining
b 32. 1. angle AGB is equal [b] to the
remaining angle DFE: Therefore the triangle ABG is equi-
c 4. 6. angular to the triangle DEF; wherefore [c] as AB is to BG, ſo is
DE to EF; but as DE to EF, ſo, by hypotheſis, is AB to BC;
d 11. 5. therefore as AB to BC, ſo is AB to BG [d]; and becauſe AB
e 9. 5. has the ſame ratio to each of the lines BC, BG; BC is equal [e]
to BG, and therefore the angle BGC is equal to the angle
f 5. 1. BCG [f]: But the angle BCG is, by hypotheſis, leſs than a right
angle; therefore alſo the angle BGC is leſs than a right angle,
g 13. 1. and the adjacent angle AGB muſt be greater than a right angle [g].
But it was proved that the angle AGB is equal to the angle at F;
therefore the angle at F is greater than a right angle: But, by the
hypotheſis, it is leſs than a right angle; which is abſurd. Therefore

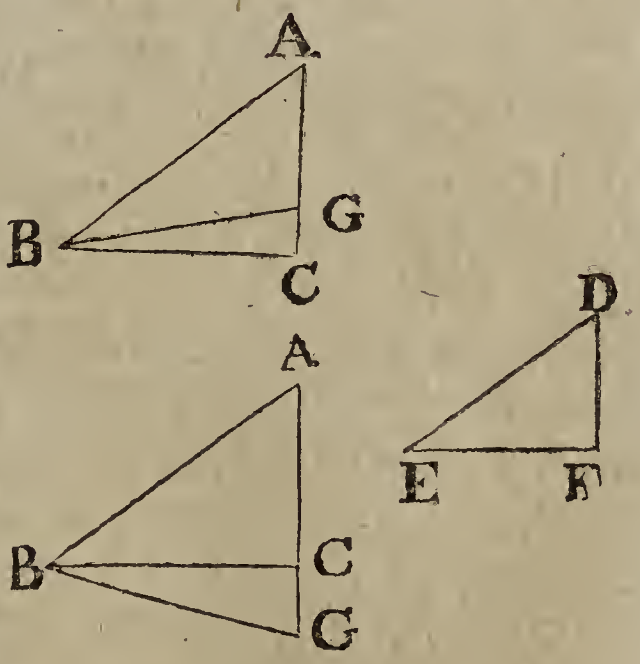
fore the angles ABC, DEF are not unequal, that is, they are equal: And the angle at A is equal to the angle at D; wherefore the remaining angle at C is equal to the remaining angle at F: Therefore the triangle ABC is equiangular to the triangle DEF.

Next, Let each of the angles at C, F be not leſs than a right angle: The triangle ABC is alſo in this caſe equiangular to the triangle DEF.



The ſame conſtruction being made, it may be proved in like manner that BC is equal to BG, and the angle at C equal to the angle BGC: But the angle at C is not leſs than a right angle; therefore the angle BGC is not leſs than a right angle: Wherefore two angles of the triangle BGC are together not leſs than two right angles, which is impoſſible[h]; and therefore the triangle ABC may be proved to be equiangular to the triangle DEF, as in the firſt caſe. h 17. 1.

Laſtly, Let one of the angles at C, F, viz. the angle at C, be a right angle; in this caſe likewiſe the triangle ABC is equiangular to the triangle DEF.



For, if they be not equiangular, make, at the point B of the ſtraight line AB, the angle ABG equal to the angle DEF; then it may be proved, as in the firſt caſe, that BG is equal to BC: But the angle BCG is a right angle, therefore[i] the angle BGC is alſo a right angle; whence two of the angles of the triangle BGC are together not leſs than two right angles, which is impoſſible[h]: Therefore the triangle ABC is equiangular to the triangle DEF. Wherefore, if two triangles, &c. Q. E. D. i 5. 1.

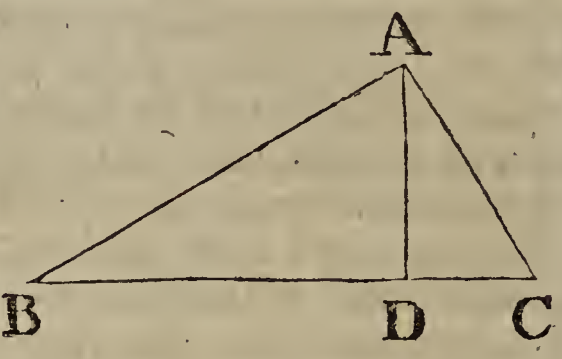
PROP. VIII. THEOR.

See N.

IN a right angled triangle, if a perpendicular be drawn from the right angle to the baſe; the triangles on each ſide of it are ſimilar to the whole triangle, and to one another.

Let ABC be a right angled triangle, having the right angle BAC; and from the point A let AD be drawn perpendicular to the baſe BC: The triangles ABD, ADC are ſimilar to the whole triangle ABC, and to one another.

Becauſe the angle BAC is equal to the angle ADB, each of them being a right angle, and that the angle at B is common to the two triangles ABC, ABD; the remaining angle ACB is equal to the remaining angle BAD [a]: Therefore the triangle ABC is equiangular to the triangle ABD, and the ſides about their equal angles are proportionals [b]; wherefore the triangles are ſimilar [c]: In the like manner it may be demonſtrated, that the triangle ADC is equiangular and ſimilar to the triangle ABC: And the triangles ABD, ADC, being both equiangular and ſimilar to ABC, are equiangular and ſimilar to each other. Therefore, in a right angled, &c. Q. E. D.



a 32. 1.
b 4. 6.
c 1. def. 6.

Cor. From this it is manifeſt, that the perpendicular drawn from the right angle of a right angled triangle to the baſe, is a mean proportional between the ſegments of the baſe: And alſo, that each of the ſides is a mean proportional between the baſe, and its ſegment adjacent to that ſide: Becauſe in the triangles BDA, ADC, BD is to DA, as DA to DC [b]; and in the triangles ABC, DBA, BC is to BA, as BA to BD [b]; and in the triangles ABC, ACD, BC is to CA, as CA to CD [b].

PROP.

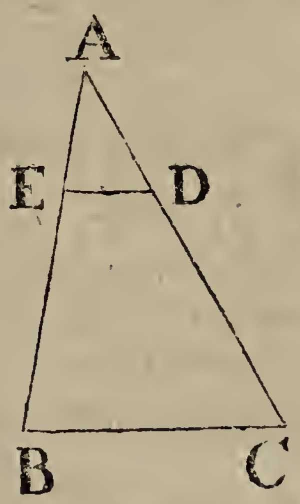
PROP. IX. PROB.

FROM a given ſtraight line to cut off any part required. See N.

Let AB be the given ſtraight line; it is required to cut off any part from it.

From the point A draw a ſtraight line AC making any angle with AB; and in AC take any point D, and take AC the ſame multiple of AD, that AB is of the part which is to be cut off from it; join BC, and draw DE parallel to it: Then AE is the part required to be cut off.

Becauſe ED is parallel to one of the ſides of the triangle ABC, viz. to BC, as CD is to DA, ſo is [a] BE to EA; and, by compoſition [b], CA is to AD, as BA to AE: But CA is a multiple of AD; therefore [c] BA is the ſame multiple of AE: Whatever part therefore AD is of AC, AE is the ſame part of AB: Wherefore, from the ſtraight line AB the part required is cut off. Which was to be done.



a 2. 6.
b 18. 5.
c D. 5.

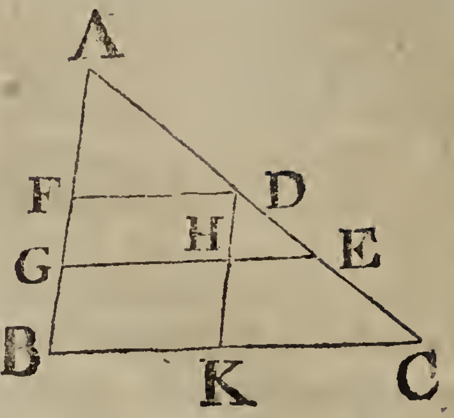
PROP. X. PROB.

TO divide a given ſtraight line ſimilarly to a given divided ſtraight line, that is, into parts that ſhall have the ſame ratios to one another which the parts of the divided given ſtraight line have.

Let AB be the ſtraight line given to be divided, and AC the divided line; it is required to divide AB ſimilarly to AC.

Let AC be divided in the points D, E; and let AB, AC be placed ſo as to contain any angle, and join BC, and through the points D, E, draw [a] DF, EG parallels to it; and through D draw DHK parallel to AB: Therefore each of the figures FH, HB, is a parallelogram; wherefore DH is equal [b] to FG, and HK

a 31. 1.
b 34. 1.

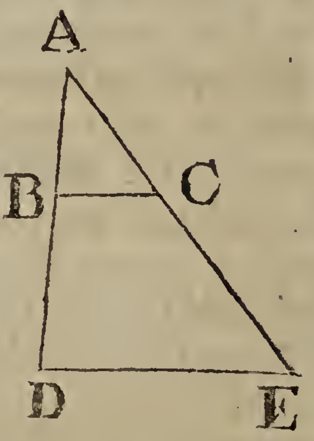


HK to GB: And becauſe HE is parallel to KC, one of the ſides of the triangle DKC, as CE to ED, ſo is c KH to HD: But KH is equal to BG, and HD to GF; therefore, as CE to ED, ſo is BG to GF: Again, becauſe FD is parallel to EG, one of the ſides of the triangle AGE, as ED to DA, ſo is GF to FA: But it has been proved that CE is to ED, as BG to GF; and as ED to DA, ſo GF to FA: Therefore the given ſtraight line AB is divided ſimilarly to AC. Which was to be done. c 2. 6.

PROP. XI. PROB.

TO find a third proportional to two given ſtraight lines.

Let AB, AC be the two given ſtraight lines, and let them be placed ſo as to contain any angle; it is required to find a third proportional to AB, AC.



Produce AB, AC to the points D, E; and make BD equal to AC; and having joined BC, through D, draw DE parallel to it [a]. a 31. 1.

Becauſe BC is parallel to DE, a ſide of the triangle ADE, AB is [b] to BD, as AC to CE: But BD is equal to AC; as therefore AB to AC, ſo is AC to CE. Wherefore to the two given ſtraight lines AB, AC a third proportional CE is found. Which was to be done. b 2. 6.

PROP. XII. PROB.

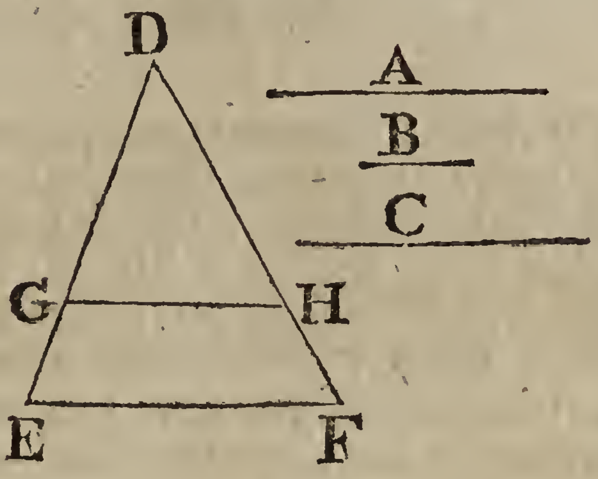
TO find a fourth proportional to three given ſtraight lines.

Let A, B, C be the three given ſtraight lines; it is required to find a fourth proportional to A, B, C.

Take two ſtraight lines DE, DF, containing any angle EDF; and upon theſe make DG equal to A, GE equal to B, and DH equal to C; and having joined GH, draw EF parallel [a] to it through the point E: And becauſe GH is parallel to EF, one of the ſides of the triangle DEF, DG is to GE, as DH to HF [b]; but DG is equal to A, GE to B, and DH to C; therefore, as A is to B, ſo is C to HF. Wherefore to the three given ſtraight lines, A, B, C a fourth proportional HF is found. Which was to be done.

a 31. 1.

b 2. 6.



PROP. XIII. PROB.

TO find a mean proportional between two given ſtraight lines.

Let AB, BC be the two given ſtraight lines; it is required to find a mean proportional between them.

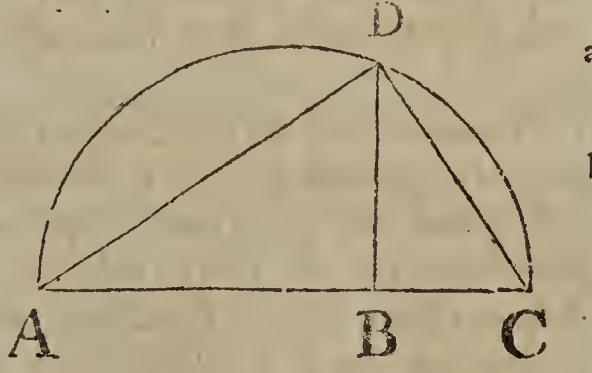
Place AB, BC in a ſtraight line, and upon AC deſcribe the ſemicircle ADC, and from the point B draw [a] BD at right angles to AC, and join AD, DC.

a 11. 1.

Becauſe the angle ADC in a ſemicircle is a right angle [b], and becauſe in the right angled triangle ADC, DB is drawn from the right angle perpendicular to the baſe, DB is a mean proportional between AB, BC the ſegments of the baſe [c]: Therefore between the two given ſtraight lines AB, BC, a mean proportional DB is found. Which was to be done.

b 31. 3.

c Cor. 8. 6.



PROP.

PROP. XIV. THEOR.

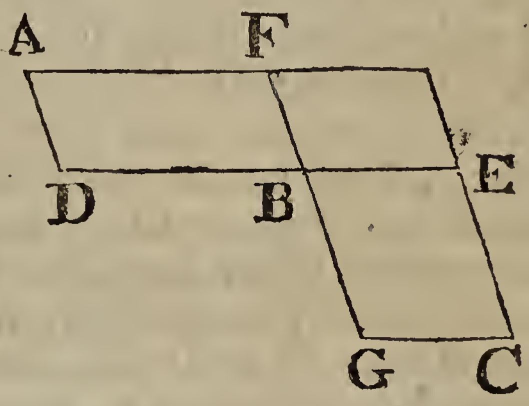
EQUAL parallelograms which have one angle of the one equal to one angle of the other, have their ſides about the equal angles reciprocally proportional: And parallelograms that have one angle of the one equal to one angle of the other, and their ſides about the equal angles reciprocally proportional, are equal to one another.

Let AB, BC be equal parallelograms, which have the angles at B equal, and let the ſides DB, BE be placed in the ſame ſtraight line; wherefore alſo FB, BG are in one ſtraight line [a]; The ſides of the parallelograms AB, BC about the equal angles, are reciprocally proportional; that is, DB is to BE, as GB to BF.

a 14. 1.

Complete the parallelogram FE; and becauſe the parallelogram AB is equal to BC, and that FE is another parallelogram, AB is to FE, as BC to FE [b]: But as AB to FE, ſo is the baſe DB to BE [c]; and, as BC to FE, ſo is the baſe GB to BF; therefore, as DB to BE, ſo is GB to BF [d]. Wherefore, the ſides of the parallelograms AB, BC about their equal angles are reciprocally proportional.

b 7. 5.
c 1. 6.
d 11. 5.



But, let the ſides about the equal angles be reciprocally proportional, viz. as DB to BE, ſo GB to BF; the parallelogram AB is equal to the parallelogram BC.

Becauſe, as DB to BE, ſo is GB to BF; and as DB to BE, ſo is the parallelogram AB to the parallelogram FE; and as GB to BF, ſo is the parallelogram BC to the parallelogram FE; therefore as AB to FE, ſo BC to FE [d]: Wherefore the parallelogram AB is equal [e] to the parallelogram BC. Therefore equal parallelograms, &c. Q. E. D.

e 9. 5.

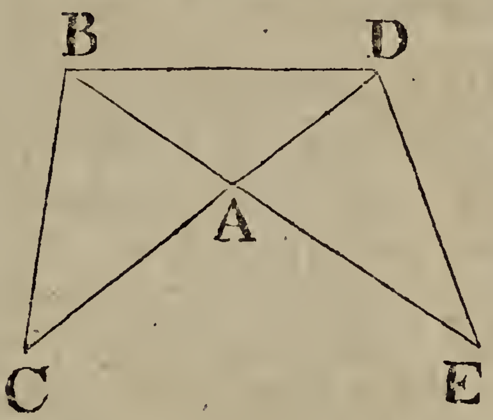
PROP.

PROP. XV. THEOR.

EQUAL triangles which have one angle of the one equal to one angle of the other, have their ſides about the equal angles reciprocally proportional: And triangles which have one angle in the one equal to one angle in the other, and their ſides about the equal angles reciprocally proportional, are equal to one another.

Let ABC, ADE be equal triangles, which have the angle BAC equal to the angle DAE; the ſides about the equal angles of the triangles are reciprocally proportional; that is, CA is to AD, as EA to AB.

Let the triangles be placed ſo that their ſides CA, AD be in one ſtraight line; wherefore alſo EA and AB are in one ſtraight line [a]; and join BD. Becauſe the triangle ABC is equal a 14. 1. to the triangle ADE, and that ABD is another triangle; therefore as the triangle CAB is to the triangle BAD, ſo is triangle EAD to triangle DAB [b]: b 7. 5. But as triangle CAB to triangle BAD, ſo is the baſe CA to AD [c]; c 1. 6. and as triangle EAD to triangle DAB, ſo is the baſe EA to AB [c]; as therefore CA to AD, ſo is EA to AB [d]; d 11. 5. wherefore the ſides of the triangles ABC, ADE about the equal angles are reciprocally proportional.



But let the ſides of the triangles ABC, ADE about the equal angles be reciprocally proportional, viz. CA to AD, as EA to AB; the triangle ABC is equal to the triangle ADE.

Having joined BD as before; becauſe, as CA to AD, ſo is EA to AB; and as CA to AD, ſo is triangle ABC to triangle BAD [c]; and as EA to AB, ſo is triangle EAD to triangle BAD [c]; therefore [d] as triangle BAC to triangle BAD, ſo is triangle EAD to triangle BAD; that is, the trangles BAC, EAD have the ſame ratio to the triangle BAD: Wherefore the triangle ABC is equal [e] to the triangle ADE. Therefore equal e 9. 5. triangles, &c. Q. E. D.

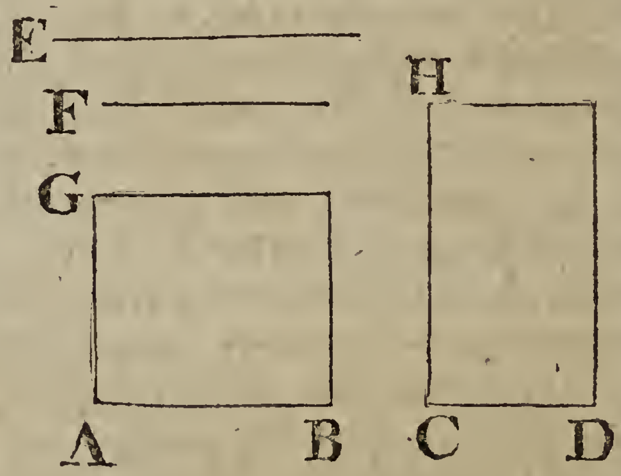
PROP.

PROP. XVI. THEOR.

IF four ſtraight lines be proportionals, the rectangle contained by the extremes is equal to the rectangle contained by the means: And if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four ſtraight lines are proportionals.

Let the four ſtraight lines, AB, CD, E, F be proportionals, viz. as AB to CD ſo E to F; the rectangle contained by AB, F is equal to the rectangle contained by CD, E.

a 11. 1. From the points A, C draw [a] AG, CH at right angles to
AB, CD; and make AG equal to F, and CH equal to E, and
complete the parallelograms BG, DH: Becauſe, as AB to CD,
ſo is E to F; and that E is equal to CH, and F to AG; AB
b 7. 5. is [b] to CD, as CH to AG: Therefore the ſides of the parallelograms BG, DH about the equal angles are reciprocally pro-
portional; but parallelograms which have their ſides about e-
c 14. 6. qual angles reciprocally proportional, are equal to one another [c];
therefore the parallelogram BG is equal to the parallelogram DH: And the parallelogram BG is contained by the ſtraight lines AB, F; becauſe AG is equal to F; and the parallelogram DH is contained by CD and E; becauſe CH is equal to E: Therefore the rectangle contained by the ſtraight lines AB, F is equal to that which is contained by CD and E.



And if the rectangle contained by the ſtraight lines AB, F be equal to that which is contained by CD, E; theſe four lines are proportionals, viz. AB is to CD, as E to F.

The ſame conſtruction being made, becauſe the rectangle contained by the ſtraight lines AB, F is equal to that which is contained by CD, E, and that the rectangle l G is contained by AB, F, becauſe AG is equal to F; and the rectangle DH by CD, E, becauſe CH is equal to E; therefore the parallelogram BG is equal to the parallelogram DH; and they are equiangular:

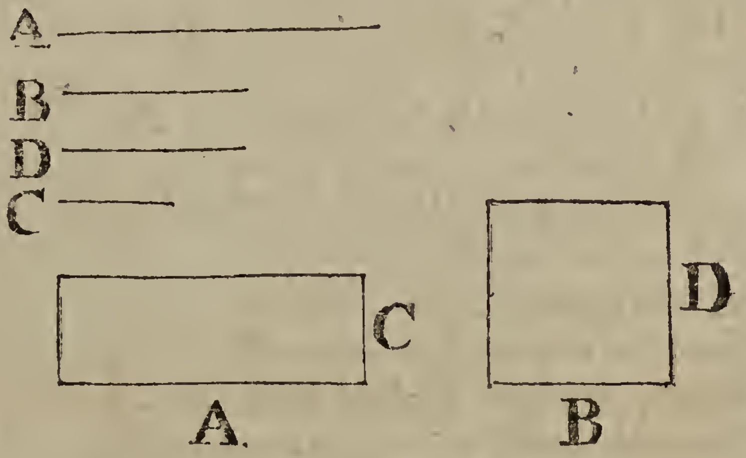
lar: But the ſides about the equal angles of equal parallelograms are reciprocally proportional [c]: Wherefore, as AB to CD, ſo is CH to AG; and CH is equal to E, and AG to F: As [c 14. 6.] therefore AB is to CD, ſo E to F. Wherefore, if four, &c. Q. E. D.

PROP. XVII. THEOR.

IF three ſtraight lines be proportionals, the rectangle contained by the extremes is equal to the ſquare of the mean: And if the rectangle contained by the extremes be equal to the ſquare of the mean, the three ſtraight lines are proportionals.

Let the three ſtraight lines A, B, C be proportionals, viz. as A to B, ſo B to C; the rectangle contained by A, C is equal to the ſquare of B.

Take D equal to B; and becauſe as A to B, ſo B to C, and that B is equal to D; A is [a] to B, as D to C: But if four [a 7. 5.] ſtraight lines be proportionals, the rectangle contained by the extremes is equal to that which is contained by the means [b]: Therefore the rectangle contained by A, C is equal [b 16. 6.] to that contained by B, D: But the rectangle contained by B, D is the ſquare of B; becauſe B is equal to D: Therefore the rectangle contained by A, C is equal to the ſquare of B.



And if the rectangle contained by A, C be equal to the ſquare of B; A is to B, as B to C.

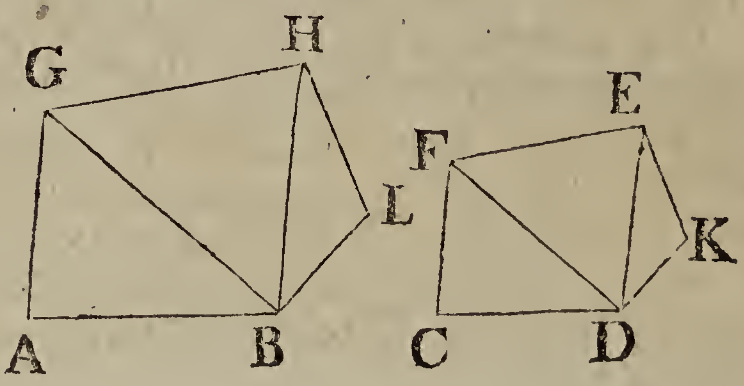
The ſame conſtruction being made, becauſe the rectangle contained by A, C is equal to the ſquare of B, and the ſquare of B is equal to the rectangle contained by B, D, becauſe B is equal to D; therefore the rectangle contained by A, C is equal to that contained by B, D: But if the rectangle contained by the extremes be equal to that contained by the means, the four ſtraight lines are proportionals [b]: Therefore A is to B, as D to

C; but B is equal to D; wherefore as A to B, ſo B to C: Therefore, if three ſtraight lines, &c. Q. E. D.

PROP. XVIII. PROB.

See N. UPON a given ſtraight line to deſcribe a rectilineal figure ſimilar, and ſimilarly ſituated to a given rectilineal figure.

Let AB be the given ſtraight line, and CDEF the given rectilineal figure of four ſides; it is required upon the given ſtraight line AB to deſcribe a rectilineal figure ſimilar, and ſimilarly ſituated to CDEF.



Join DF, and at the points A, B in the ſtraight line AB, a 23. 1. make [a] the angle BAG equal to the angle at C, and the angle ABG equal to the angle CDF; therefore the remaining angle b 32. 1. CFD is equal to the remaining angle AGB [b]: Wherefore the triangle FCD is equiangular to the triangle GAB: Again, at the points G, B in the ſtraight line GB make [a] the angle BGH equal to the angle DFE, and the angle GBH equal to FDE; therefore the remaining angle FED is equal to the remaining angle GHB, and the triangle FDE equiangular to the triangle GBH: Then, becauſe the angle AGB is equal to the angle CFD, and BGH to DFE, the whole angle AGH is equal to the whole CFE: For the ſame reaſon, the angle ABH is equal to the angle CDE; alſo the angle at A is equal to the angle at C, and the angle GHB to FED: Therefore the rectilineal figure ABHG is equiangular to CDEF: But likewiſe theſe figures have their ſides about the equal angles proportionals: Becauſe the triangles GAB, c 4. 6. FCD being equiangular, BA is [c] to AG, as DC to CF; and becauſe AG is to GB, as CF to FD; and as GB to GH, ſo, by reaſon of the equiangular triangles BGH, DFE, is FD to d 22. 5. FE; therefore, ex æquali [d], AG is to GH, as CF to FE: In the ſame manner it may be proved that AB is to BH, as CD to DE: And GH is to HB, as FE to ED [c]. Wherefore, becauſe the rectilineal figures ABHG, CDEF are equiangular, and have their ſides about the equal angles proportionals, they are ſimilar to one another [e]. [e] 1. Def. 6.

Next, let it be required to deſcribe upon a given ſtraight line AB, a rectilineal figure ſimilar, and ſimilarly ſituated to the rectilineal figure CDKEF.

Join DE, and upon the given ſtraight line AB deſcribe the rectilineal figure ABHG ſimilar, and ſimilarly ſituated to the quadrilateral figure CDEF, by the former caſe; and at the points B, H in the ſtraight line BH, make the angle HBL equal to the angle EDK, and the angle BHL equal to the angle DEK; therefore the remaining angle at K is equal to the remaining angle at L: And becauſe the figures ABHG, CDEF are ſimilar, the angle GHB is equal to the angle FED, and BHL is equal to DEK; wherefore the whole angle GHL is equal to the whole angle FEK: For the ſame reaſon the angle ABL is equal to the angle CDK: Therefore the five ſided figures AGHLB, CFEKD are equiangular; and becauſe the figures AGHB, CFED are ſimilar, GH is to HB, as FE to ED; and as HB to HL, ſo is ED to EK [c]; therefore, ex æquali [d], GH is to HL, as FE to EK: For the ſame reaſon, AB is to BL as CD to DK: And BL is to LH, as [c] DK to KE, becauſe the triangles BLH, DKE are equiangular: Therefore, becauſe the five ſided figures AGHLB, CFEKD are equiangular, and have their ſides about the equal angles proportionals, they are ſimilar to one another: And in the ſame manner a rectilineal figure of ſix or more ſides may be deſcribed upon a given ſtraight line ſimilar to one given, and ſo on. Which was to be done. [c] 4. 6. [d] 22. 5.

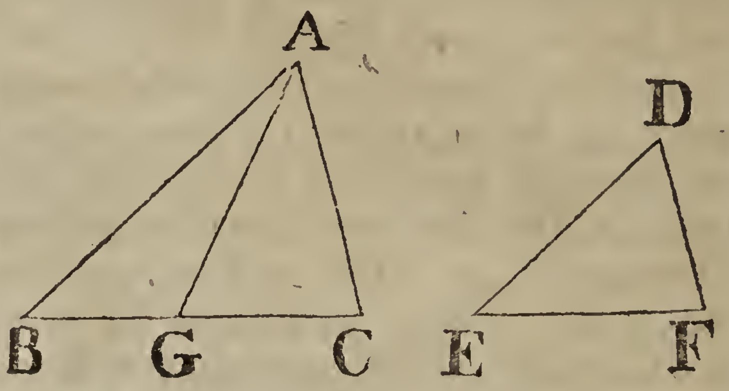
PROP. XIX. THEOR.

SIMILAR triangles are to one another in the duplicate ratio of their homologous ſides.

Let ABC, DEF be ſimilar triangles, having the angle B equal to the angle E, and let AB be to BC, as DE to EF, ſo that the ſide BC is homologous to EF [a]: the triangle ABC has to the triangle DEF, the duplicate ratio of that which BC has to EF. [a] 12. Def. 5.

Take BG a third proportional to BC, EF [b], ſo that BC is to EF, as EF to BG, and join GA: Then, becauſe as AB to BC, ſo DE to EF; alternately [c], AB is to DE, as BC to EF: But as [b] 11. 6. [c] 16. 5.

as BC to EF, ſo is EF to BG; therefore [d] as AB to DE, ſo is
EF to BG: Wherefore the ſides of the triangles ABG, DEF
d 11. 5. which are about the equal angles, are reciprocally proportional:
But triangles which have the ſides about two equal angles reciprocally proportional
are equal to one another
e 15. 6. ther [e]: Therefore the triangle ABG is equal
to the triangle DEF: And becauſe as BC is to EF, ſo EF to BG; and that if three
ſtraight lines be proportionals, the firſt is
f 10. Def. 5. ſaid [f] to have to the third the duplicate ratio of that which it has
to the ſecond; BC therefore has to BG the duplicate ratio of
g 1. 6. that which BC has to EF: But as BC to BG, ſo is [g] the triangle
ABC to the triangle ABG. Therefore the triangle ABC has to the triangle ABG the duplicate ratio of that which BC has to EF: But the triangle ABG is equal to the triangle DEF; wherefore alſo the triangle ABC has to the triangle DEF the duplicate ratio of that which BC has to EF. Therefore ſimilar triangles, &c. Q. E. D.



Cor. From this it is manifeſt, that if three ſtraight lines be proportionals, as the firſt is to the third, ſo is any triangle upon the firſt to a ſimilar, and ſimilarly deſcribed triangle upon the ſecond.

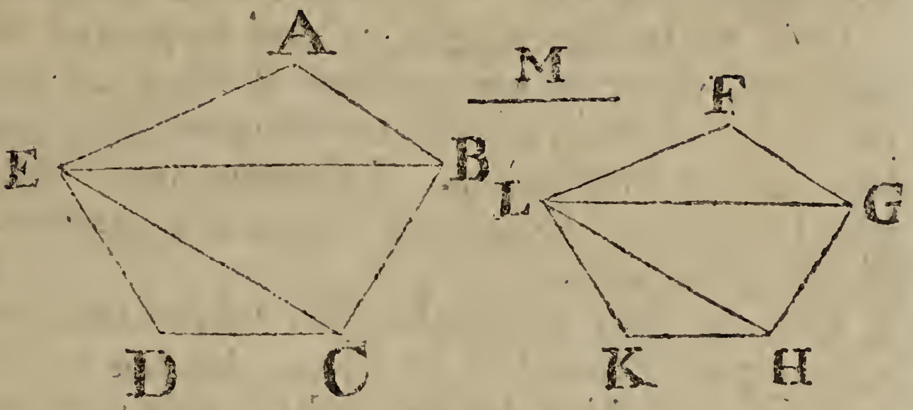
PROP. XX. THEOR.

SIMILAR polygons may be divided into the ſame number of ſimilar triangles, having the ſame ratio to one another that the polygons have; and the polygons have to one another the duplicate ratio of that which their homologous ſides have.

Let ABCDE, FGHKL be ſimilar polygons, and let AB be the homologous ſide to FG: The polygons ABCDE, FGHKL may be divided into the ſame number of ſimilar triangles, whereof each to each has the ſame ratio which the polygons have; and the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which the ſide AB has to the ſide FG.

Join BE, EC, GL, LH: And becauſe the polygon ABCDE is ſimilar to the polygon FGHKL, the angle BAE is equal to the angle GFL [a], and BA is to AE, as GF to FL [a]: Wherefore, becauſe the triangles ABE, FGL have an angle in one equal to an angle in the other, and their ſides about theſe equal angles proportionals, the triangle ABE is equiangular [b], and therefore ſimilar to the triangle FGL [c]; wherefore the angle ABE is equal to the angle FGL: And, becauſe the polygons are ſimilar, the whole angle ABC is equal [a] to the whole angle FGH; therefore the remaining angle EBC is equal to the remaining angle LGH: And becauſe the triangles ABE, FGL are ſimilar, EB is to BA, as LG to GF [a]; and alſo, becauſe the polygons are ſimilar, AB is to BC, as FG to GH [a]; therefore, ex æquali [d], EB is to BC, as LG to GH; that is, the ſides about the equal angles EBC, LGH are proportionals; therefore [d] the triangle EBC is equiangular to the triangle LGH, and ſimilar to it [c]. For the ſame reaſon, the triangle ECD likewiſe is ſimilar to the triangle LHK: Therefore the ſimilar polygons ABCDE, FGHKL are divided into the ſame number of ſimilar triangles.

a 1. def. 6.
b 6. 6.
c 4. 6.
d 22. 5.



Alſo theſe triangles have, each to each, the ſame ratio which the polygons have to one another, the antecedents being ABE, EBC, ECD, and the conſequents FGL, LGH, LHK: And the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which the ſide AB has to the homologous ſide FG.

Becauſe the triangle ABE is ſimilar to the triangle FGL, ABE has to FGL the duplicate ratio [e] of that which the ſide BE has to the ſide GL: For the ſame reaſon, the triangle BEC has to GLH the duplicate ratio of that which BE has to GL: Therefore, as the triangle ABE to the triangle FGL, ſo [f] is the triangle BEC to the triangle GLH. Again, becauſe the triangle EBC is ſimilar to the triangle LGH, EBC has to LGH the duplicate ratio of that which the ſide EC has to the ſide LH: For the ſame reaſon, the triangle ECD has to the triangle

e 19. 6.
f 11. 5.

LHK, the duplicate ratio of that which EC has to LH: As therefore the triangle EBC to the triangle LGH, ſo is [f] the triangle ECD to the triangle LHK: But it has been proved that the triangle EBC is likewiſe to the triangle LGH, as the triangle ABE to the triangle FGL. Therefore, as the triangle ABE is to the triangle FGL, ſo is triangle EBC to triangle LGH, and triangle ECD to triangle LHK: And therefore, as one of the antecedents to one of the conſequents, ſo are all the antecedents to all the conſequents [g]. Wherefore, as the triangle ABE to the triangle FGL, ſo is the polygon ABCDE to the polygon FGHKL: But the triangle ABE has to the triangle FGL, the duplicate ratio of that which the ſide AB has to the homologous ſide FG. Therefore alſo the polygon ABCDE has to the polygon FGHKL the duplicate ratio of that which AB has to the homologous ſide FG. Wherefore ſimilar polygons, &c. Q. E. D.

f 11. 5.

g 12. 5.

Cor. 1. In like manner, it may be proved, that ſimilar four ſided figures, or of any number of ſides, are one to another in the duplicate ratio of their homologous ſides, and it has already been proved in triangles. Therefore, univerſally, ſimilar rectilineal figures are to one another in the duplicate ratio of their homologous ſides.

Cor. 2. And if to AB, FG, two of the homologous ſides, a third proportional M be taken, AB has [h] to M the duplicate ratio of that which AB has to FG: But the four ſided figure or polygon upon AB has to the four ſided figure or polygon upon FG likewiſe the duplicate ratio of that which AB has to FG: Therefore, as AB is to M, ſo is the figure upon AB to the figure upon FG, which was alſo proved in triangles [i]. Therefore, univerſally, it is manifeſt, that if three ſtraight lines be proportionals, as the firſt is to the third, ſo is any rectilineal figure upon the firſt, to a ſimilar and ſimilarly deſcribed rectilineal figure upon the ſecond.

h 10. def. 5.

i Cor. 19. 6.

PROP.

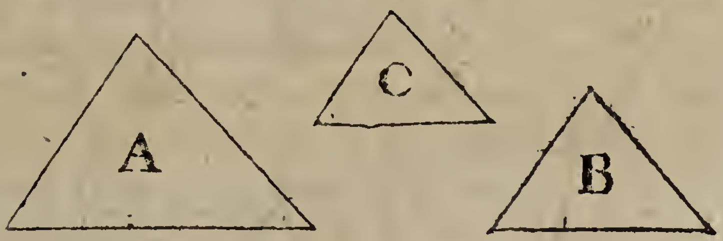
PROP. XXI. THEOR.

Rectilineal figures which are ſimilar to the ſame rectilineal figure, are alſo ſimilar to one another.

Let each of the rectilineal figures A, B be ſimilar to the rectilineal figure C: The figure A is ſimilar to the figure B.

Becauſe A is ſimilar to C, they are equiangular, and alſo have their ſides about the equal angles proportionals [a]. Again, becauſe B is ſimilar to C, they are equiangular, and have their ſides about the equal angles proportionals [a]: Therefore the figures A, B are each of them equiangular to C, and have the ſides about the equal angles of each of them and of C proportionals. Wherefore the rectilineal figures A and B are equiangular [b], and have their ſides about the equal angles proportionals [c]. Therefore A is ſimilar [a] to B. Q. E. D.

a 1. def. 6.
b 1. Ax. 1.
c 11. 5.



PROP. XXII. THEOR.

If four ſtraight lines be proportionals, the ſimilar rectilineal figures ſimilarly deſcribed upon them ſhall alſo be proportionals; and if the ſimilar rectilineal figures ſimilarly deſcribed upon four ſtraight lines be proportionals, thoſe ſtraight lines ſhall be proportionals.

Let the four ſtraight lines AB, CD, EF, GH be proportionals, viz. AB to CD, as EF to GH, and upon AB, CD let the ſimilar rectilineal figures KAB, LCD be ſimilarly deſcribed; and upon EF, GH the ſimilar rectilineal figures MF, NH in like manner: The rectilineal figure KAB is to LCD, as MF to NH.

To AB, CD take a third proportional [a] X; and to EF, GH a third proportional O: And becauſe AB is to CD, as EF to GH, and that CD is [b] to X, as GH to O; wherefore, ex æquali [c], as AB to X, ſo EF to O: But as AB to X, ſo is [d] the rectilineal

a 11. 6.
b 11. 5.
c 22. 5.
d 2. Cor. 20. 6.

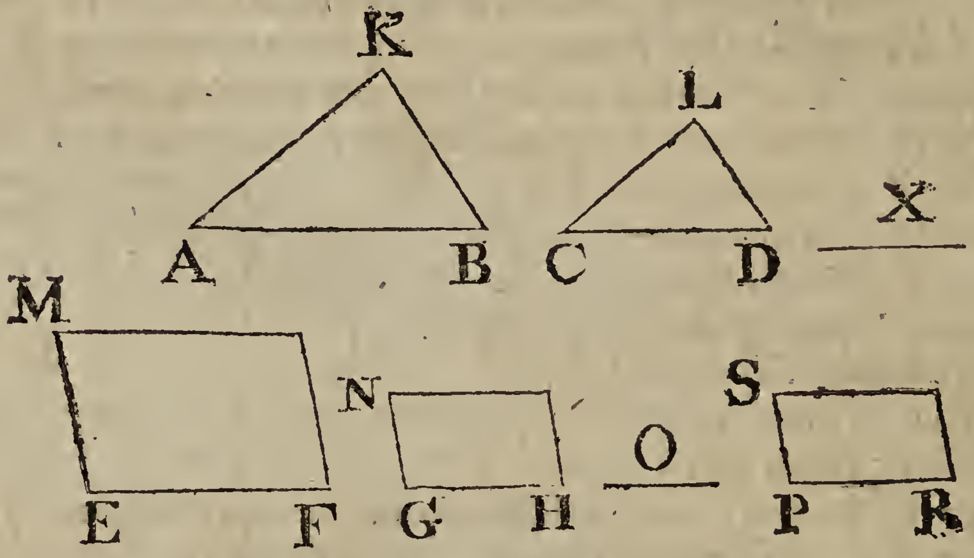
rectilineal KAB to the rectilineal LCD, and as EF to O, ſo is [d] the rectilineal MF to the rectilineal NH: Therefore, as KAB to LCD, ſo [b] is MF to NH.

d 2. Cor. 20. 6.
b 11. 5.

And if the rectilineal KAB be to LCD, as MF to NH; the ſtraight line AB is to CD, as EF to GH.

e 12. 6.
f 18. 6.

Make [e] as AB to CD, ſo EF to PR, and upon PR deſcribe [f] the rectilineal figure SR ſimiliar and ſimilarly ſituated to either



of the figures MF, NH: Then, becauſe as AB to CD, ſo is EF to PR, and that upon AB, CD are deſcribed the ſimilar and ſimilarly ſituated rectilineals KAB, LCD, and upon EF, PR, in like manner, the ſimilar rectilineals MF, SR; KAB is to LCD, as MF to SR; but, by the hypotheſis, KAB is to LCD, as MF to NH; and therefore the rectilineal MF having the ſame ratio to each of the two NH, SR, theſe are equal [g] to one another: They are alſo ſimilar, and ſimilarly ſituated; therefore GH is equal to PR: And becauſe as AB to CD, ſo is EF to PR, and that PR is equal to GH; AB is to CD, as EF to GH. If therefore four ſtraight lines, &c. Q. E. D.

g 9. 5.

PROP. XXIII. THEOR.

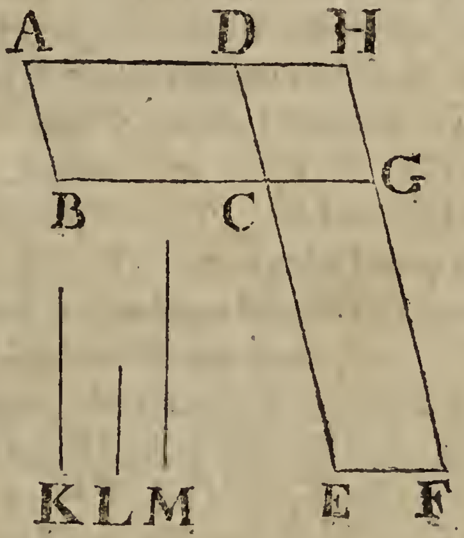
See N.

EQUIANGULAR parallelograms have to one another the ratio which is compounded of the ratios of their ſides.

Let AC, CF be equiangular parallelograms, having the angle BCD equal to the angle ECG: The ratio of the parallelogram AC to the parallelogram CF, is the ſame with the ratio which is compounded of the ratios of their ſides.

Let BC, CG be placed in a ſtraight line; therefore DC and CE are alſo in a ſtraight line [a]; and complete the parallelogram DG; and, taking any ſtraight line K, make [b] as BC to CG, ſo K to L; and as DC to CE, ſo make [b] L to M: Therefore the ratios of K to L, and L to M, are the ſame with the ratios of the ſides, viz. of BC to CG, and DC to CE. But the ratio of K to M is that which is ſaid to be compounded [c] of the ratios of K to L, and L to M: Wherefore alſo K has to M the ratio compounded of the ratios of the ſides: And becauſe as BC to CG, ſo is the parallelogram AC to the parallelogram CH [d]; but as BC to CG, ſo is K to L; therefore K is [e] to L, as the parallelogram AC to the parallelogram CH: Again, becauſe as DC to CE, ſo is the parallelogram CH to the parallelogram CF; but as DC to CE, ſo is L to M; wherefore L is [e] to M, as the parallelogram CH to the parallelogram CF: Therefore, ſince it has been proved, that as K to L, ſo is the parallelogram AC to the parallelogram CH; and as L to M, ſo the parallelogram CH to the parallelogram CF; ex æquali [f], K is to M, as the parallelogram AC to the parallelogram CF: But K has to M the ratio which is compounded of the ratios of the ſides; therefore alſo the parallelogram AC has to the parallelogram CF the ratio which is compounded of the ratios of the ſides. Wherefore equiangular parallelograms, &c. Q. E. D.

a 14. 1.
b 12. 6.
c A. def. 5.
d 1. 6.
e 11. 5.
f 22. 5.



PROP. XXIV. THEOR.

THE parallelograms about the diameter of any parallelogram, are ſimilar to the whole, and to one another.

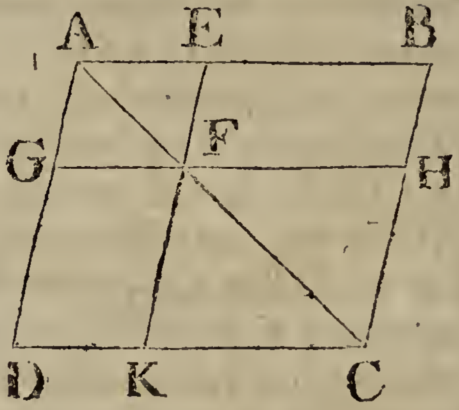
See N.

Let ABCD be a parallelogram, of which the diameter is AC; and EG, HK the parallelograms about the diameter: The parallelograms EG, HK are ſimilar both to the whole parallelogram ABCD, and to one another.

Becauſe DC, GF are parallels, the angle ADC is equal [a] to the angle AGF: For the ſame reaſon, becauſe BC, EF are pa-

a 29.

rallels, the angle ABC is equal to the angle AEF: And each
of the angles BCD, EFG is equal to the oppoſite angle DAB [b],
b 34. 1. and therefore are equal to one another, wherefore the parallelograms
ABCD, AEFG are equiangular: And becauſe the
angle ABC is equal to the angle AEF, and the angle BAC
common to the two triangles BAC, EAF, they are equiangular
c 4. 6. to one another; therefore [c] as AB
to BC, ſo is AE to EF: And becauſe
the oppoſite ſides of parallelograms
d 7. 5. are equal to one another [b], AB is [c] to
AD, as AE to AG; and DC to CB
as GF to FE; and alſo CD to DA
as FG to GA: Therefore the ſides of
the parallelograms ABCD, AEFG about
the equal angles are proportionals;
and they are therefore ſimilar to
e 1. def. 6. one another [e]: For the ſame reaſon, the parallelogram ABCD
is ſimilar to the parallelogram FHCK. Wherefore each of the
parallelograms GE, KH is ſimilar to DB: But rectilineal figures
which are ſimilar to the ſame rectilineal figure, are alſo
f 21. 6. ſimilar to one another [f]; therefore the parallelogram GE is ſimilar
to KH. Wherefore the parallelograms, &c. Q. E. D.



PROP. XXV. PROB.

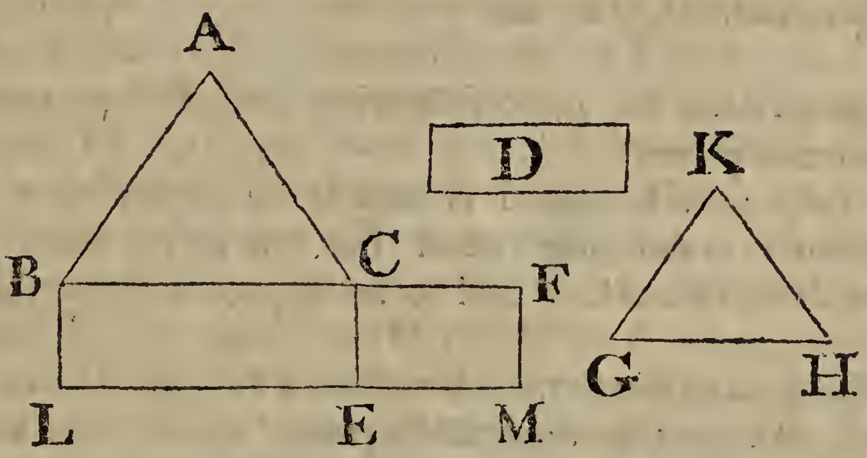
See N. TO deſcribe a rectilineal figure which ſhall be ſimilar to one, and equal to another given rectilineal figure.

Let ABC be the given rectilineal figure, to which the figure
to be deſcribed is required to be ſimilar, and D that to which
it muſt be equal. It is required to deſcribe a rectilineal figure
ſimilar to ABC, and equal to D.

a Cor. 45. 1. Upon the ſtraight line BC deſcribe [a] the parallelogram BE
equal to the figure ABC; alſo upon CE deſcribe [a] the parallelogram
CM equal to D, and having the angle FCE equal
to the angle CBL: Therefore BC and CF are in a ſtraight
b { 29. 1. 14. 1. line [b], as alſo LE and EM: Between BC and CF find [c] a mean
c 13. 6. proportional GH, and upon GH deſcribe [d] the rectilineal figure
d 18. 6. KGH ſimilar and ſimilarly ſituated to the figure ABC:
And becauſe BC is to GH as GH to CF, and if three ſtraight
e 2. Cor. 20. 6. lines be proportionals, as the firſt is to the third, ſo is [e] the

figure upon the firſt to the ſimilar and ſimilarly deſcribed figure upon the ſecond; therefore as BC to CF, ſo is the rectilineal figure ABC to KGH: But as BC to CF, ſo is [f] the parallelogram BE to the parallelogram EF: Therefore as the rectilineal figure ABC is to KGH, ſo is the parallelogram BE to the parallelogram EF [g]: And the rectilineal figure ABC is equal

f 1. 6.
g 11. 5.



to the parallelogram BE; therefore the rectilineal figure KGH is equal [h] to the parallelogram EF: But EF is equal to the figure D; wherefore alſo KGH is equal to D; and it is ſimilar to ABC. Therefore the rectilineal figure KGH has been deſcribed ſimilar to the figure ABC, and equal to D. Which was to be done.

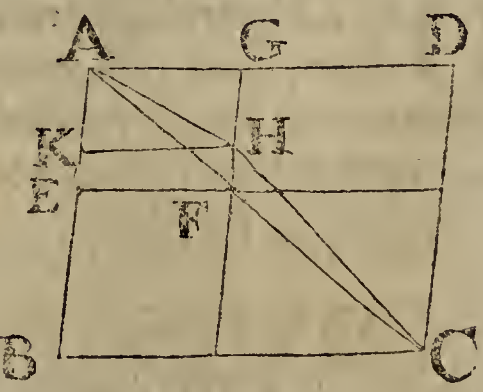
h 14. 5.

PROP. XXVI. THEOR.

IF two ſimilar parallelograms have a common angle, and be ſimilarly ſituated; they are about the ſame diameter.

Let the parallelograms ABCD, AEFG be ſimilar and ſimilarly ſituated, and have the angle DAB common. ABCD and AEFG are about the ſame diameter.

For, if not, let, if poſſible the parallelogram BD have its diameter AHC in a different ſtraight line from AF the diameter of the parallelogram EG, and let GF meet AHC in H; and through H draw HK parallel to AD or BC: Therefore the parallelograms ABCD, AKHG being about the ſame diameter, they are ſimilar to one another [a]: Wherefore as DA to AB, ſo is [b] GA to AK: But becauſe ABCD and AEFG are ſimilar parallelograms,



a 24. 6.
b 1. def. 6.

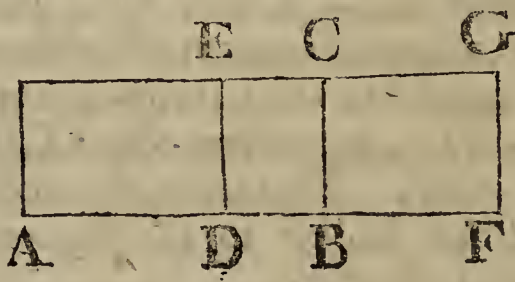
c 11. 5.
d 9. 5.

as DA is to AB, ſo is GA to AE; therefore [c] as GA to AE, ſo GA to AK; wherefore GA has the ſame ratio to each of the ſtraight lines AE, AK; and conſequently AK is equal [d] to AE, the leſs to the greater, which is impoſſible: Therefore ABCD and AKHG are not about the ſame diameter; wherefore ABCD and AEFG muſt be about the ſame diameter. Therefore, if two ſimilar, &c. Q. E. D.

'To underſtand the three following propoſitions more eaſily 'it is to be obſerved,

'1. That a parallelogram is ſaid to be applied to a ſtraight 'line, when it is deſcribed upon it as one of its ſides. Ex. gr. 'the parallelogram AC is ſaid to be applied to the ſtraight line 'AB.

'2. But a parallelogram AE is ſaid to be applied to a ſtraight 'line AB, deficient by a parallelogram, when AD the baſe of 'AE is leſs than AB, and therefore AE is leſs than the parallelogram AC deſcribed upon 'AB in the ſame angle, and between the ſame parallels, by 'the parallelogram DC; and 'DC is therefore called the defect of AE.



'3. And a parallelogram AG is ſaid to be applied to a ſtraight 'line AB, exceeding by a parallelogram, when AF the baſe of 'AG is greater than AB, and therefore AG exceeds AC the 'parallelogram deſcribed upon AB in the ſame angle, and be'tween the ſame parallels, by the parallelogram BG.'

PROP. XXVII. THEOR.

See N.

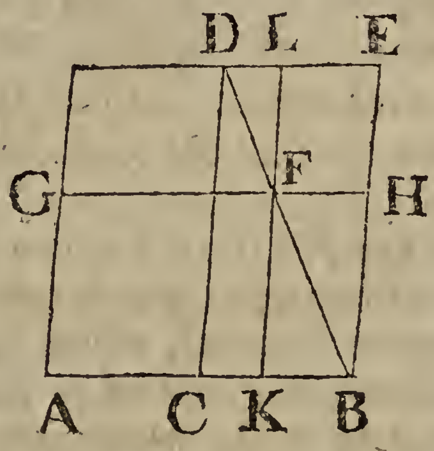
OF all parallelograms applied to the ſame ſtraight line, and deficient by parallelograms, ſimilar and ſimilarly ſituated to that which is deſcribed upon the half of the line; that which is applied to the half, and is ſimilar to its defect, is the greateſt.

Let AB be a ſtraight line divided into two equal parts in C, and let the parallelogram AD be applied to the half AC, which is therefore deficient from the parallelogram upon the whole line AB by the parallelogram CE upon the other half CB: Of all the parallelograms applied to any other parts of

AB, and deficient by parallelograms that are ſimilar, and ſimilarly ſituated to CE, AD is the greateſt.

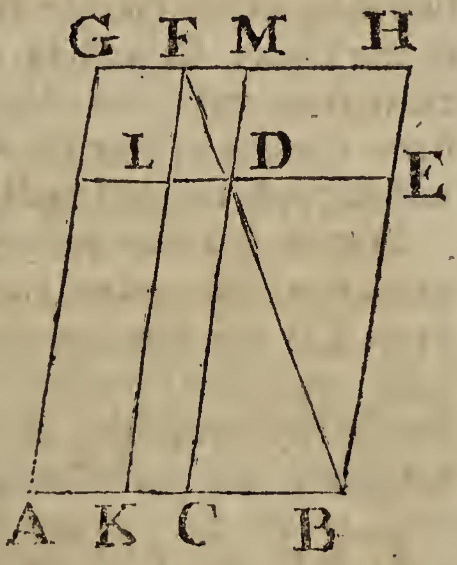
Let AF be any parallelogram applied to AK, any other part of AB than the half, ſo as to be deficient from the parallelogram upon the whole line AB by the parallelogram KH ſimilar, and ſimilarly ſituated to CE; AD is greater than AF.

Firſt, let AK the baſe of AF, be greater than AC the half of AB; and becauſe CE is ſimilar to the parallelogram KH, they are about the ſame diameter[a]: Draw their diameter DB, and complete the ſcheme: Becauſe the parallelogram CF is equal[b] to FE, add KH to both, therefore the whole CH is equal to the whole KE: But CH is equal[c] to CG, becauſe the baſe AC is equal to the baſe CB; therefore CG is equal to KE: To each of theſe add CF; then the whole AF is equal to the gnomon CHL: Therefore CE, or the parallelogram AD, is greater than the parallelogram AF.



a 26. 6.
b 43. 1.
c 36. 1.

Next, let AK the baſe of AF, be leſs than AC, and, the ſame conſtruction being made, the parallelogram DH is equal to DG[c], for HM is equal to MG[d], becauſe BC is equal to CA; wherefore DH is greater than LG: But DH is equal[b] to DK; therefore DK is greater than LG: To each of theſe add AL; then the whole AD is greater than the whole AF. Therefore of all parallelograms applied, &c. Q. E. D.

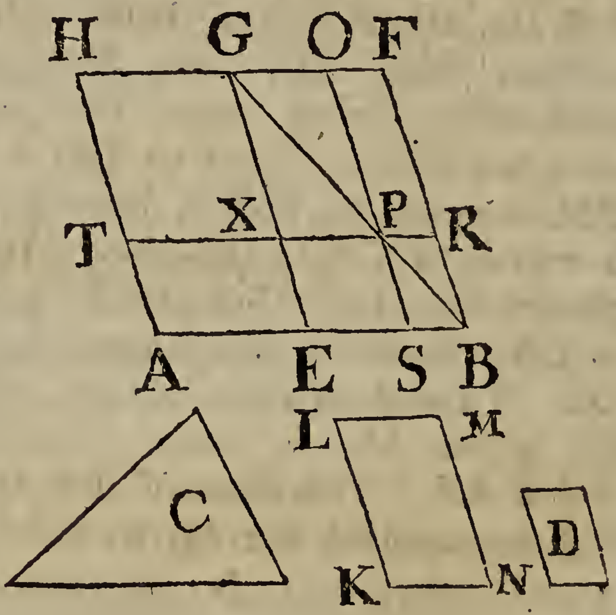


d 34. 1.

PROP. XXVIII. PROB.

See N. TO a given ſtraight line to apply a parallelogram equal to a given rectilineal figure, and deficient by a parallelogram ſimilar to a given parallelogram: But the given rectilineal figure to which the parallelogram to be applied is to be equal, muſt not be greater than the parallelogram applied to half of the given line, having its defect ſimilar to the defect of that which is to be applied; that is, to the given parallelogram.

Let AB be the given ſtraight line, and C the given rectilineal figure, to which the parallelogram to be applied is required to be equal, which figure muſt not be greater than the parallelogram applied to the half of the line having its defect from that upon the whole line ſimilar to the defect of that which is to be applied; and let D be the parallelogram to which this defect is required to be ſimilar. It is required to apply a parallelogram to the ſtraight line AB, which ſhall be equal to the figure C, and be deficient from the parallelogram upon the whole line by a parallelogram ſimilar to D.



Divide AB into two equal
a 10. 1. parts[a] in the point E, and
upon EB deſcribe the paral-
b 18. 6. lelogram EEFG ſimilar[b] and ſimilarly ſituated to D, and complete the parallelogram AG, which muſt either be equal to C, or greater than it, by the determination: And if AG be equal to C, then what was required is already done: For, upon the ſtraight line AB, the parallelogram AG is applied equal to the figure C, and deficient by the parallelogram EF ſimilar to D: But, if AG be not equal to C, it is greater than
c 25. 6. it; and EF is equal to AG; therefore EF alſo is greater than C. Make[c] the parallelogram KLMN equal to the exceſs of
d 21. 6. EF above C, and ſimilar and ſimilarly ſituated to D; but D is ſimilar to EF, therefore[d] alſo KM is ſimilar to EF: Let KL be

be the homologous ſide to EG, and LM to GF: And becauſe EF is equal to C and KM together, EF is greater than KM; therefore the ſtraight line EG is greater than KL, and GF than LM: Make GX equal to LK, and GO equal to LM, and complete the parallelogram XGOP: Therefore XO is equal and ſimilar to KM; but KM is ſimilar to EF; wherefore alſo XO is ſimilar to EF, and therefore XO and EF are about the ſame diameter[e]: Let GPB be their diameter, and complete the e 26. 6. ſcheme: Then becauſe EF is equal to C and KM together, and XO a part of the one is equal to KM a part of the other, the remainder, viz. the gnomon ERO, is equal to the remainder C: And becauſe OR is equal[f] to XS, by adding SR to each, the f 34. 1. whole OB is equal to the whole XB: But XB is equal[g] to TE, g 36. 1. becauſe the baſe AE is equal to the baſe EB; wherefore alſo TE is equal to OB: Add XS to each, then the whole TS is equal to the whole, viz. to the gnomon ERO: But it has been proved that the gnomon ERO is equal to C, and therefore alſo TS is equal to C. Wherefore the parallelogram TS, equal to the given rectilineal figure C, is applied to the given ſtraight line AB deficient by the parallelogram SR, ſimilar to the given one D, becauſe SR is ſimilar to EF[h]. Which was to be done. h 24. 6.

PROP. XXIX. PROB.

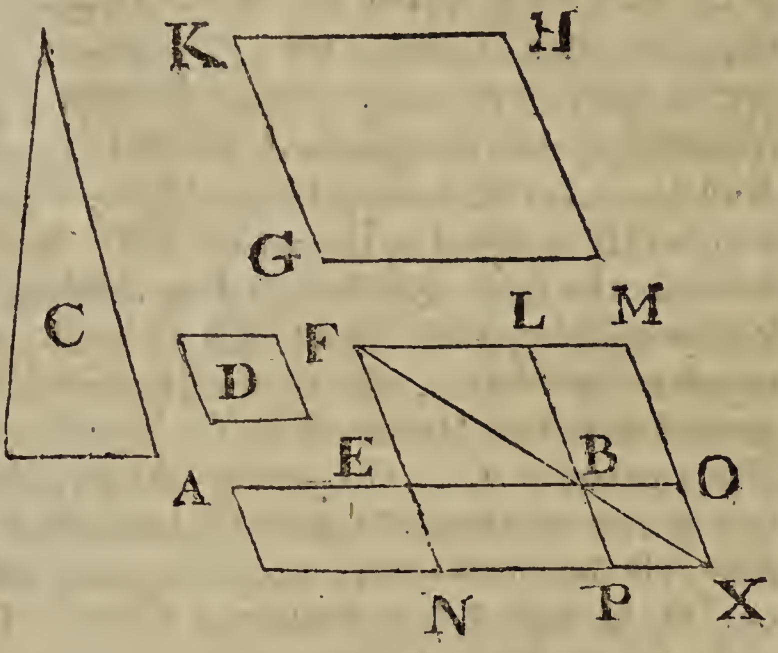
TO a given ſtraight line to apply a parallelogram equal to a given rectilineal figure, exceeding by a parallelogram ſimilar to another given. See N.

Let AB be the given ſtraight line, and C the given rectilineal figure to which the parallelogram to be applied is required to be equal, and D the parallelogram to which the exceſs of the one to be applied above that upon the given line is required to be ſimilar. It is required to apply a parallelogram to the given ſtraight line AB which ſhall be equal to the figure C, exceeding by a parallelogram ſimilar to D.

Divide AB into two equal parts in the point E, and upon EB deſcribe[a] the parallelogram EL ſimilar, and ſimilarly ſitua- a 18. 6.

ted to D: And make [b] the parallelogram GH equal to EL and C together, and ſimilar, and ſimilarly ſituated to D; wherefore GH is ſimilar to EL [c]: Let KH be the ſide homologous to FL, and KG to FE: And becauſe the parallelogram GH is greater than EL, therefore the ſide KH is greater than FL, and KG than FE: Produce FL and FE, and make FLM equal to KH, and FEN to KG, and complete the parallelogram MN. MN is therefore equal and ſimilar to GH; but GH is ſimilar to EL; wherefore MN is ſimilar to EL, and conſequently EL and MN are about the ſame diameter [d]: Draw their diameter FX, and complete the ſcheme. Therefore, ſince GH is equal to EL and C together, and that GH is equal to MN; MN is equal to EL and C: Take away the common part EL; then the remainder, viz. the gnomon NOL is equal to C. And becauſe AE is equal to EB, the parallelogram AN is equal [e] to the parallelogram NB, that is, to BM [f]. Add NO to each; therefore the whole, viz. the parallelogram AX, is equal to the gnomon NOL. But the gnomon NOL is equal to C; therefore alſo AX is equal to C. Wherefore to the ſtraight line AB there is applied the parallelogram AX equal to the given rectilineal C, exceeding by the parallelogram PO, which is ſimilar to D, becauſe PO is ſimilar to EL [g]. Which was to be done.

b 25. 6.
c 21. 6.
d 26. 6.
e 36. 1.
f 43. 1.
g 24. 6.

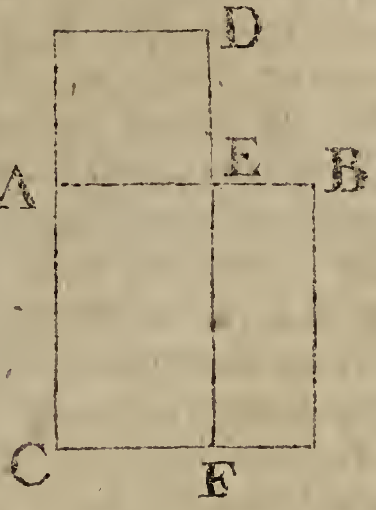


PROP. XXX. PROB.

TO cut a given ſtraight line in extreme and mean ratio.

Let AB be the given ſtraight line; it is required to cut it in extreme and mean ratio.

Upon AB deſcribe [a] the ſquare BC, and to AC apply the parallelogram CD equal to BC, exceeding by the figure AD ſimilar to BC [b]: But BC is a ſquare, therefore alſo AD is a ſquare; and becauſe BC is equal to CD, by taking the common part CE from each, the remainder BF is equal to the remainder AD: And theſe figures are equiangular, therefore their ſides about the equal angles are reciprocally proportional [c]: Wherefore, as FE to ED, ſo AE to EB: But FE is equal to AC [d], that is to AB; and ED is equal to AE: Therefore as BA to AE, ſo is AE to EB: But AB is greater than AE; wherefore AE is greater than EB [e]: Therefore the ſtraight line AB is cut in extreme and mean ratio in E [f]. Which was to be done.

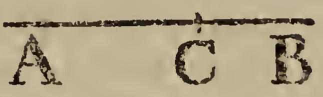


a 46. 1.
b 29. 6.
c 14. 6.
d 34. 1.
e 14. 5.
f 3. def. 6.

Otherwiſe,

Let AB be the given ſtraight line; it is required to cut it in extreme and mean ratio.

Divide AB in the point C, ſo that the rectangle contained by AB, BC be equal to the ſquare of AC [g]: Then, becauſe the rectangle AB, BC is equal to the ſquare of AC, as BA to AC, ſo is AC to CB [h]: Therefore AB is cut in extreme and mean ratio in C [f]. Which was to be done.



g 11. 2.
h 17. 6.

PROP. XXXI. THEOR.

In right angled triangles, the rectilineal figure deſcribed upon the ſide oppoſite to the right angle, is equal to the ſimilar, and ſimilarly deſcribed figures upon the ſides containing the right angle.

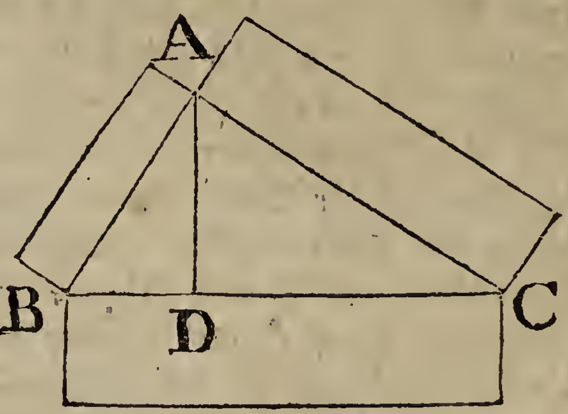
See N.

Let ABC be a right angled triangle, having the right angle BAC: The rectilineal figure deſcribed upon BC is equal to the ſimilar, and ſimilarly deſcribed figures upon BA, AC.

Draw the perpendicular AD; therefore, becauſe in the right angled triangle ABC, AD is drawn from the right angle at A perpendicular to the baſe BC, the triangles ABD, ADC are ſimilar to the whole triangle ABC, and to one another [a], and

a 8. 6.

becauſe the triangle ABC is ſimilar to ADB, as CB to BA, ſo is BA to BD [b]; and becauſe theſe three ſtraight lines are proportionals, as the firſt to the third, ſo is the figure upon the firſt to the ſimilar, and ſimilarly deſcribed figure upon the ſecond [c]: Therefore as CB to BD, ſo is the figure upon CB to the ſimilar and ſimilarly deſcribed figure upon BA: And, inverſely [d], as DB to BC, ſo is the figure upon BA to that upon BC: For the ſame reaſon, as DC to CB, ſo is the figure upon CA to that upon CB. Wherefore as BD and DC together to BC, ſo are the figures upon BA, AC to that upon BC [e]: But BD and DC together are equal to BC. Therefore the figure deſcribed on BC is equal [f] to the ſimilar and ſimilarly deſcribed figures on BA, AC. Wherefore, in right angled triangles, &c. Q. E. D.



b 4. 6.
c 2. Cor. 20. 6.
d B. 5.
e 24. 5.
f A. 5.

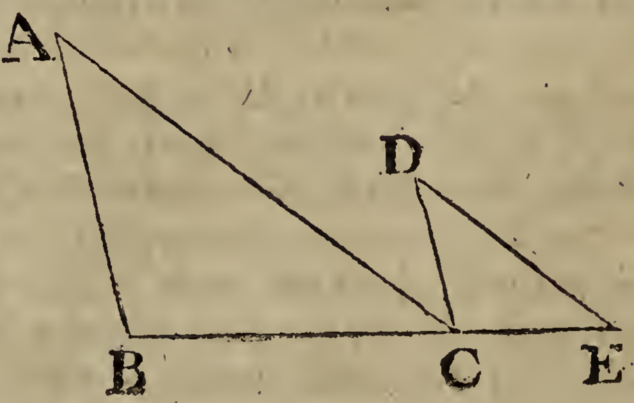
PROP. XXXII. THEOR.

See N.

IF two triangles which have two ſides of the one proportional to two ſides of the other, be joined at one angle, ſo as to have their homologous ſides parallel to one another; the remaining ſides ſhall be in a ſtraight line.

Let ABC, DCE be two triangles which have the two ſides BA, AC proportional to the two CD, DE, viz. BA to AC, as CD to DE; and let AB be parallel to DC, and AC to DE, BC and CE are in a ſtraight line.

Becauſe AB is parallel to DC, and the ſtraight line AC meets them, the alternate angles BAC, ACD are equal [a]; for the ſame reaſon, the angle CDE is equal to the angle ACD; wherefore alſo BAC is equal to CDE: And becauſe



a 29. 1.

the triangles ABC, DCE have one angle at A equal to one at D, and the ſides about theſe angles proportionals, viz. BA to AC, as CD to DE, the triangle ABC is equiangular [b] to DCE: Therefore the angle ABC is equal to the angle DCE: And the angle BAC was proved to be equal to ACD: Therefore the whole angle ACE is equal to the two angles ABC, BAC; add the common angle ACB, then the angles ACE, ACB are equal to the angles ABC, BAC, ACB: But ABC, BAC, ACB are equal to two right angles [c]; therefore alſo the angles ACE, ACB are equal to two right angles: And ſince at the point C, in the ſtraight line AC, the two ſtraight lines BC, CE, which are on the oppoſite ſides of it, make the adjacent angles ACE, ACB equal to two right angles; therefore [d] BC and CE are in a ſtraight line. Wherefore, if two triangles, &c. Q. E. D.

b 6. 6.

c 32. 1.

d 14. 1.

PROP. XXXIII. THEOR.

IN equal circles, angles, whether at the centres or circumferences, have the ſame ratio which the circumferences on which they ſtand have to one another: So alſo have the ſectors.

See N.

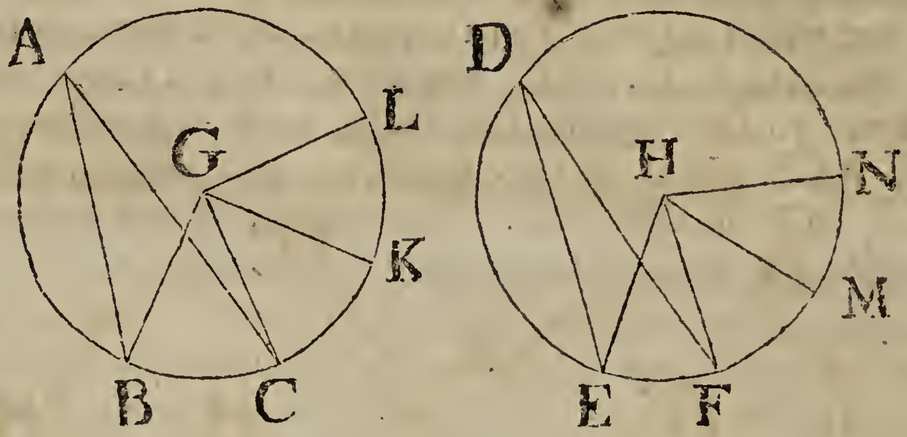
Let ABC, DEF be equal circles; and at their centres the angles BGC, EHF, and the angles BAC, EDF at their circumferences; as the circumference BC to the circumference EF, ſo is the angle BGC to the angle EHF, and the angle BAC to the angle EDF; and alſo the ſector BGC to the ſector EHF.

Take any number of circumferences CK, KL, each equal to BC, and any number whatever FM, MN each equal to EF: And join GK, GL, HM, HN. Becauſe the circumferences BC, CK, KL are all equal, the angles BGC, CGK, KGL are alſo all equal [a]: Therefore what multiple ſoever the circumference BL is of the circumference BC, the ſame multiple is the angle BGL of the angle BGC: For the ſame reaſon, whatever multiple the circumference EN is of the circumference EF, the ſame multiple is the angle EHN of the angle EHF: And

a 27. 3.

Book VI.

And if the circumference BL be equal to the circumference EN, the angle BGL is alſo equal [a] to the angle EHN; and if the circumference BL be greater than EN, likewiſe the angle BGL is greater than EHN; and if leſs, leſs: There being then four magnitudes, the two circumferences BC, EF, and the two angles BGC, EHF; of the circumference BC, and of the angle BGC, have been taken any equimultiples whatever, viz. the circumference BL, and the angle BGL; and of the circumference EF, and of the angle EHF, any equimultiples whatever, viz. the circumference EN, and the angle EHN: And it has been proved, that, if the circumference BL be greater than EN, the angle BGL is greater than EHN; and if equal, equal; and if leſs, leſs: As therefore the circumference BC to the circumference EF, ſo [b] is the angle BGC to the angle EHF: But as the angle BGC is to the angle EHF, ſo is [c] the angle BAC to the angle EDF, for each is double of each [d]: Therefore, as the circumference BC is to EF, ſo is the angle BGC to the angle EHF, and the angle BAC to the angle EDF.



Alſo, as the circumference BC to EF, ſo is the ſector BGC to the ſector EHF. Join BC, CK, and in the circumferences BC, CK take any points X, O, and join BX, XC, CO, OK: Then, becauſe in the triangles GBC, GCK the two ſides BG, GC are equal to the two CG, GK, and that they contain equal angles; the baſe BC is equal [e] ~~to the~~ baſe CK, and the triangle GBC to the triangle GCK: And becauſe the circumference BC is equal to the circumference CK, the remaining part of the whole circumference of the circle ABC, is equal to the remaining part of the whole circumference of the ſame circle: Wherefore the angle BXC is equal to the angle COK [a]; and the ſegment BXC is therefore ſimilar to the ſegment COK [f];

and

[a] 27. 3.

[b] 5. def. 5.

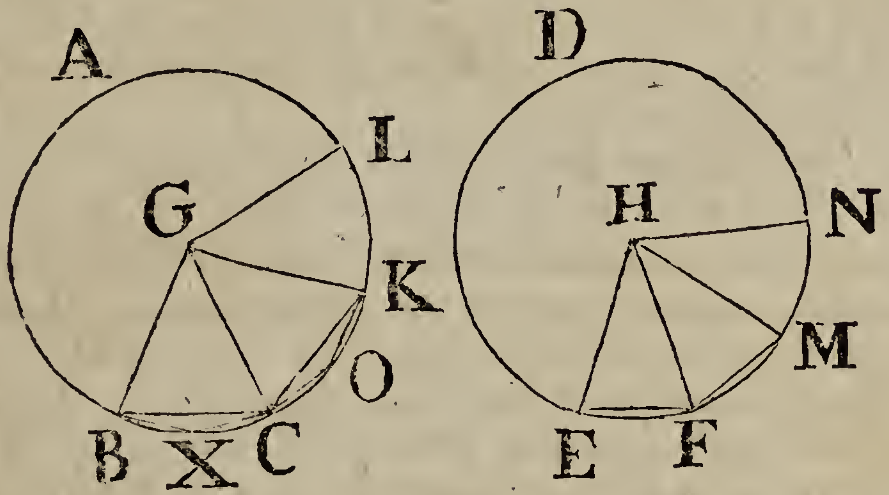
[c] 15. 5.

[d] 20. 3.

[e] 4. 1.

[f] 11. def. 3.

and they are upon equal ſtraight lines BC, CK: But ſimilar ſegments of circles upon equal ſtraight lines, are equal [g] to one another: Therefore the ſegment BXC is equal to the ſegment COK: And the triangle BGC is equal to the triangle CGK; therefore the whole, the ſector BGC, is equal to the whole, the ſector CGK: For the ſame reaſon, the ſector KGL is equal to each of the ſectors BGC, CGK: In the ſame manner, the ſectors EHF, FHM, MHN may be proved equal to one another: Therefore, what multiple ſoever the circumference BL is of the circumference BC, the ſame multiple is the ſector BGL of the ſector BGC: For the ſame reaſon, whatever multiple the circumference EN is of EF, the ſame multiple is the ſector EHN of the ſector EHF: And if the circumference BL be equal to EN, the ſector BGL is equal to the ſector EHN; and if the circumference BL be greater than EN, the ſector BGL is greater than the ſector EHN; and if leſs, leſs: Since, then, there are four magnitudes, the two circumferences BC, EF, and the two ſectors BGC, EHF, and of the circumference BC, and ſector BGC, the circumference BL and ſector BGL are any equal multiples whatever; and of the circumference EF, and ſector EHF, the circumference EN and ſector EHN, are any equimultiples whatever; and that it has been proved, if the circumference BL be greater than EN, the ſector BGL is greater than the ſector EHN; and if equal, equal; and if leſs, leſs. Therefore [b], as the circumference BC is to the circumference EF, ſo is the ſector BGC to the ſector EHF. Wherefore, in equal circles, &c. Q. E. D.



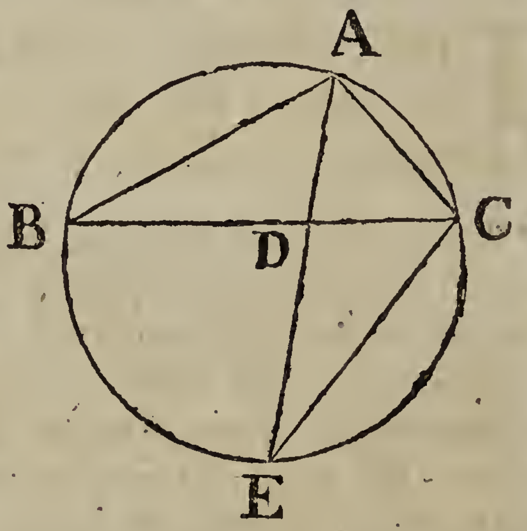
[g] 24. 3.

[b] 5. def 5.

PROP. B. THEOR.

See N. IF an angle of a triangle be biſected by a ſtraight line, which likewiſe cuts the baſe; the rectangle contained by the ſides of the triangle is equal to the rectangle contained by the ſegments of the baſe, together with the ſquare of the ſtraight line biſecting the angle.

Let ABC be a triangle, and let the angle BAC be biſected by the ſtraight line AD; the rectangle BA, AC is equal to the rectangle BD, DC, together with the ſquare of AD.



a 5. 4. Deſcribe the circle[a] ACB about the triangle, and produce
AD to the circumference in E,
and join EC: Then becauſe the
angle BAD is equal to the angle
CAE, and the angle ABD to the
b 21. 3. angle[b] AEC, for they are in the
ſame ſegment; the triangles ABD,
AEC are equiangular to one another: Therefore as BA to AD,
c 4. 6. ſo is[c] EA to AC, and conſequently the rectangle BA, AC is
d 16. 6. equal[d] to the rectangle EA, AD,
e 3. 2. that is[e], to the rectangle ED, DA,
together with the ſquare of AD: But the rectangle ED, DA
f 35. 3. is equal to the rectangle[f] BD, DC. Therefore the rectangle
BA, AC is equal to the rectangle BD, DC, together with the ſquare of AD. Wherefore, if an angle, &c. Q. E. D.

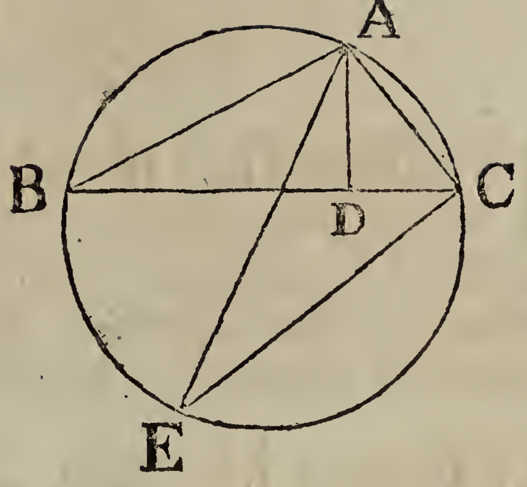
PROP. C. THEOR.

See N. IF from any angle of a triangle a ſtraight line be drawn perpendicular to the baſe; the rectangle contained by the ſides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle deſcribed about the triangle.

Let ABC be a triangle, and AD the perpendicular from the angle A to the baſe BC; the rectangle BA, AC is equal to the rectangle contained by AD and the diameter of the circle deſcribed about the triangle.

Deſcribe

Book VI.



Deſcribe [a] the circle ACB about the triangle, and draw its diameter AE, and join EC: Becauſe the right angle BDA is equal [b] to the angle ECA in a ſemicircle, and the angle ABD to the angle AEC in the ſame ſegment [c]; the triangles ABD, AEC are equiangular: Therefore as [d] BA to AD, ſo is EA to AC; and conſequently the rectangle BA, AC is equal [e] to the rectangle EA, AD. If therefore from an angle, &c. Q. E. D.

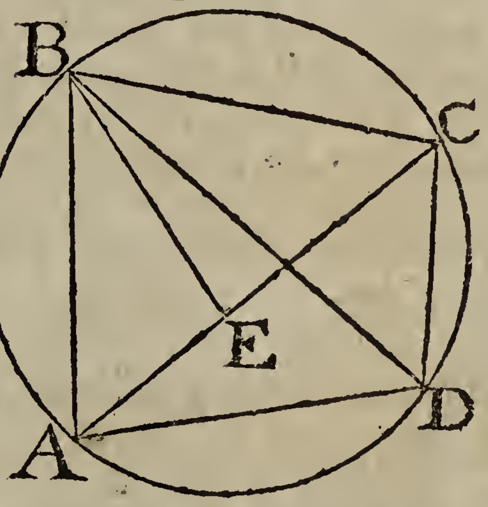
a 5. 4.
b 31. 3.
c 21. 3.
d 4. 6.
e 16. 6.

PROP. D. THEOR.

THE rectangle contained by the diagonals of a quadrilateral inſcribed in a circle, is equal to both the rectangles contained by its oppoſite ſides. See N.

Let ABCD be any quadrilateral inſcribed in a circle, and join AC, BD; the rectangle contained by AC, BD is equal to the two rectangles contained by AB, CD, and by AD, BC*.

Make the angle ABE equal to the angle DBC; add to each of theſe the common angle EBD, then the angle ABD is equal to the angle EBC: And the angle BDA is equal [a] to the angle BCE, becauſe they are in the ſame ſegment; therefore the triangle ABD is equiangular to the triangle BCE: Wherefore [b] as BC is to CE, ſo is BD to DA; and conſequently the rectangle BC, AD is equal [c] to the rectangle BD, CE: Again, becauſe the angle ABE is equal to the angle DBC, and the angle [a] BAE to the angle BDC, the triangle ABE is equiangular to the triangle BCD: As therefore BA to AE, ſo is BD to DC; wherefore the rectangle BA, DC is equal to the rectangle BD, AE: But the rectangle BC, AD has been ſhewn equal to the rectangle BD, CE; therefore the whole rectangle AC, BD [d] is equal to the rectangle AB, DC, together with the rectangle AD, BC. Therefore the rectangle, &c. Q. E. D.



a 21. 3.
b 4. 6.
c 16. 6.
d 1. 2.

---

* This is a Lemma of Cl. Ptolomæus, in page 9. of his μεγαλη συνταξις.

THE ELEMENTS OF EUCLID.

BOOK XI.

DEFINITIONS.

I.

A Solid is that which hath length, breadth, and thickneſs.

II.

That which bounds a ſolid is a ſuperficies.

III.

A ſtraight line is perpendicular, or at right angles to a plane, when it makes right angles with every ſtraight line meeting it in that plane.

IV.

A plane is perpendicular to a plane, when the ſtraight lines drawn in one of the planes perpendicularly to the common ſection of the two planes, are perpendicular to the other plane.

V.

The inclination of a ſtraight line to a plane is the acute angle contained by that ſtraight line, and another drawn from the point in which the firſt line meets the plane, to the point in which a perpendicular to the plane drawn from any point of the firſt line above the plane, meets the ſame plane.

VI.

The inclination of a plane to a plane is the acute angle contained by two ſtraight lines drawn from any the ſame point of their common ſection at right angles to it, one upon one plane, and the other upon the other plane.

VII.

Two planes are ſaid to have the ſame, or a like inclination to one another, which two other planes have, when the ſaid angles of inclination are equal to one another.

VIII.

Parallel planes are ſuch which do not meet one another though produced.

IX.

A ſolid angle is that which is made by the meeting of more than two plane angles, which are not in the ſame plane, in one point. See N.

X.

‘ The tenth definition is omitted for reaſons given in the notes.’ See N.

XI.

Similar ſolid figures are ſuch as have all their ſolid angles equal, each to each, and which are contained by the ſame number of ſimilar planes. See N.

XII.

A pyramid is a ſolid figure contained by planes that are conſtituted betwixt one plane and one point above it in which they meet.

XIII.

A priſm is a ſolid figure contained by plane figures of which two that are oppoſite are equal, ſimilar, and parallel to one another; and the others parallelograms.

XIV.

A ſphere is a ſolid figure deſcribed by the revolution of a ſemicircle about its diameter, which remains unmoved.

XV.

The axis of a ſphere is the fixed ſtraight line about which the ſemicircle revolves.

XVI.

The centre of a ſphere is the ſame with that of the ſemicircle.

XVII.

The diameter of a ſphere is any ſtraight line which paſſes thro' the centre, and is terminated both ways by the ſuperficies of the ſphere.

XVIII.

A cone is a ſolid figure deſcribed by the revolution of a right angled triangle about one of the ſides containing the right angle, which ſide remains fixed.

If the fixed ſide be equal to the other ſide containing the right angle, the cone is called a right angled cone; if it be leſs than the other ſide, an obtuſe angled, and if greater, an acute angled cone.

XIX.

The axis of a cone is the fixed ſtraight line about which the triangle revolves.

XX.

The baſe of a cone is the circle deſcribed by that ſide containing the right angle, which revolves.

XXI.

A cylinder is a ſolid figure deſcribed by the revolution of a right angled parallelogram about one of its ſides which remains fixed.

XXII.

The axis of a cylinder is the fixed ſtraight line about which the parallelogram revolves.

XXIII.

The baſes of a cylinder are the circles deſcribed by the two revolving oppoſite ſides of the parallelogram.

XXIV.

Similar cones and cylinders are thoſe which have their axes and the diameters of their baſes proportionals.

XXV.

A cube is a ſolid figure contained by ſix equal ſquares.

XXVI.

A tetrahedron is a ſolid figure contained by four equal and equilateral triangles.

XXVII.

An octahedron is a ſolid figure contained by eight equal and equilateral triangles.

XXVIII.

A dodecahedron is a ſolid figure contained by twelve equal pentagons which are equilateral and equiangular.

XXIX.

An icoſahedron is a ſolid figure contained by twenty equal and equilateral triangles.

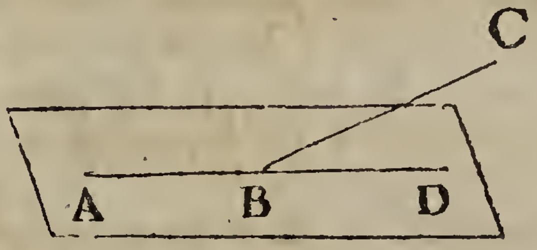
DEF. A.

A parallelepiped is a ſolid figure contained by ſix quadrilateral figures, whereof every oppoſite two are parallel.

PROP. I. THEOR.

ONE part of a ſtraight line cannot be in a plane, and another part above it. See N.

If it be poſſible, let AB, part of the ſtraight line ABC, be in the plane, and the part BC above it: And ſince the ſtraight line AB is in the plane, it can be produced in that plane: Let it be produced to D. And let any plane paſs through the ſtraight line AD, and be turned about it until it paſs thro' the point C; and becauſe the points B, C are in this plane, the ſtraight line BC is in it [a]: Therefore there are two ſtraight lines ABC, ABD in the ſame plane that have a common ſegment AB, which is impoſſible [b]. Therefore, one part, &c. Q. E. D.



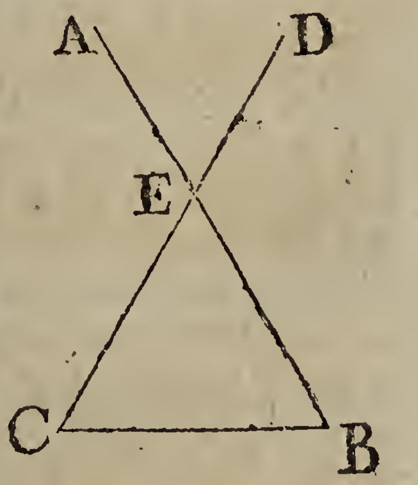
a 7. def. 1.
b Cor. 11. 1.

PROP. II. THEOR.

TWO ſtraight lines which cut one another are in one plane, and three ſtraight lines which meet one another are in one plane. See N.

Let two ſtraight lines AB, CD, cut one another in E; AB, CD are one plane: And three ſtraight lines EC, CB, BE which meet one another, are in one plane.

Let any plane paſs through the ſtraight line EB, and let the plane be turned about EB, produced, if neceſſary, until it paſs through the point C: Then becauſe the points E, C are in this plane, the ſtraight line EC is in it [a]: For the ſame reaſon, the ſtraight line BC is in the ſame; and, by the hypotheſis, EB is in it: Therefore the three ſtraight lines EC, CB, BE are in one plane: But in the plane in which EC, EB are, in the ſame are [b] CD, AB: Therefore AB, CD are in one plane. Wherefore two ſtraight lines, &c. Q. E. D.



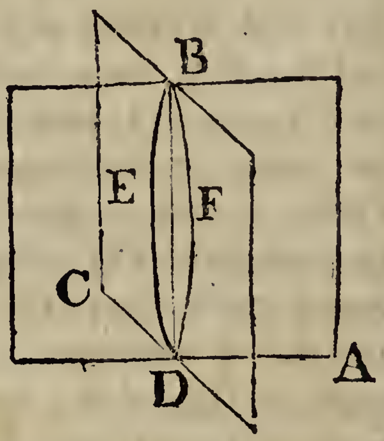
a 7. def. 1.
b 1. 11.

PROP. III. THEOR.

See N. IF two planes cut one another, their common ſection is a ſtraight line.

Let two planes AB, BC, cut one another, and let the line DB be their common ſection: DB is a ſtraight line: If it be not, from the point D to B, draw, in the plane AB, the ſtraight line DEB, and in the plane BC the ſtraight line DFB: Then two ſtraight lines DEB, DFB have the ſame extremities, and therefore include a ſpace betwixt them; which is impoſſible [a]: Therefore BD the common ſection of the planes AB, BC cannot but be a ſtraight line. Wherefore, if two planes, &c. Q. E. D.

a 10. Ax. 1.



PROP. IV. THEOR.

See N. IF a ſtraight line ſtand at right angles to each of two ſtraight lines in the point of their interſection, it ſhall alſo be at right angles to the plane which paſſes through them, that is, to the plane in which they are.

Let the ſtraight line EF ſtand at right angles to each of the ſtraight lines AB, CD in E, the point of their interſection: EF is alſo at right angles to the plane paſſing through AB, CD.

Take the ſtraight lines AE, EB, CE, ED all equal to one another; and through E draw, in the plane in which are AB, CD, any ſtraight line GEH; and join AD, CB; then, from any point F in EF, draw FA, FG, FD, FC, FH, FB: And becauſe the two ſtraight lines AE, ED are equal to the two BE, EC,
a 15. 1. and that they contain equal angles [a] AED, BEC, the baſe AD
b 4. 1. is equal [b] to the baſe BC, and the angle DAE to the angle EBC: And the angle AEG is equal to the angle BEH [a]; therefore the triangles AEG, BEH have two angles of one equal to two angles of the other, each to each, and the ſides AE, EB, adjacent to the equal angles, equal to one another; where-
c 26. 1. fore they ſhall have their other ſides equal [c]: GE is therefore equal

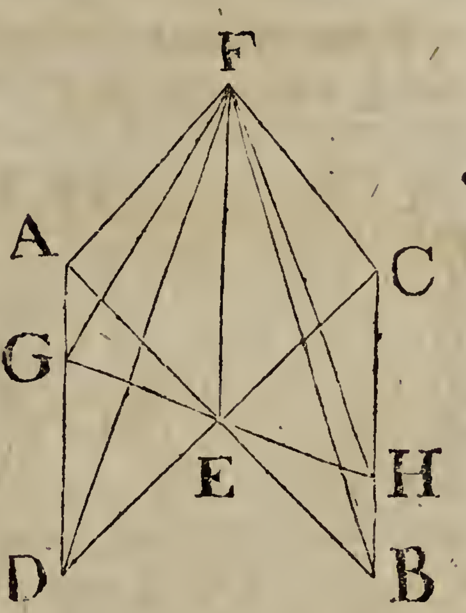
equal to EH, and AG to BH: And becauſe AE is equal to EB, and FE common and at right angles to them, the baſe AF is equal [b] to the baſe FB; for the ſame reaſon, CF is equal to FD: And becauſe AD is equal to BC, and AF to FB, the two ſides FA, AD are equal to the two FB, BC, each to each; and the baſe DF was proved equal to the baſe FC; therefore the angle FAD is equal [d] to the angle FBC: Again, it was proved that GA is equal to BH, and alſo AF to FB; FA, then, and AG, are equal to FB and BH, and the angle FAG has been proved equal to the angle FBH; therefore the baſe GF is equal [b] to the baſe FH: Again, becauſe it was proved, that GE is equal to EH, and EF is common; GE, EF are equal to HE, EF; and the baſe GF is equal to the baſe FH; therefore the angle GEF is equal [d] to the angle HEF; and conſequently each of theſe angles is a right [e] angle. Therefore FE makes right angles with GH, that is, with any ſtraight line drawn through E in the plane paſſing through AB, CD. In like manner, it may be proved, that FE makes right angles with every ſtraight line which meets it in that plane. But a ſtraight line is at right angles to a plane when it makes right angles with every ſtraight line which meets it in that plane [f]: Therefore EF is at right angles to the plane in which are AB, CD. Wherefore, if a ſtraight line, &c. Q. E. D.

[b] 4. 1.

[d] 8. 1.

[e] 10. def. 1.

[f] 3. def. 11.



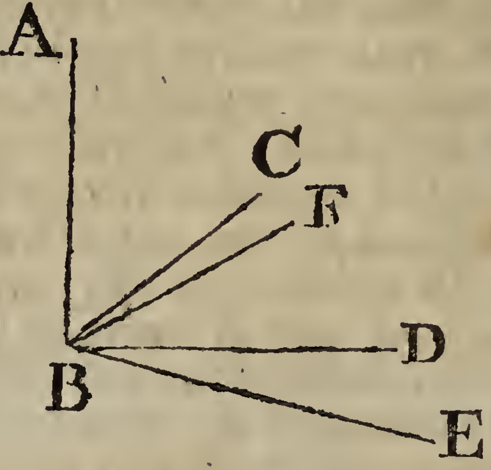
PROP. V. THEOR.

IF three ſtraight lines meet all in one point, and a ſtraight line ſtands at right angles to each of them in that point; theſe three ſtraight lines are in one and the ſame plane.

See N.

Let the ſtraight line AB ſtand at right angles to each of the ſtraight lines BC, BD, BE, in B the point where they meet; BC, BD, BE are in one and the ſame plane.

If not, let, if it be poſſible, BD and BE be in one plane, and BC be above it; and let a plane paſs through AB, BC the common ſection of which with the plane, in which BD and BE are, ſhall be a ſtraight [a] line; let this be BF: Therefore the three ſtraight lines AB, BC, BF are all in one plane, viz. that which paſſes through AB, BC; and becauſe AB ſtands at right angles to each of the ſtraight lines BD, BE, it is alſo at right angles [b] to the plane paſſing through them; and therefore makes right angles [c] with every ſtraight line meeting it in that plane; but BF which is in that plane meets it: Therefore the angle ABF is a right angle; but the angle ABC, by the hypotheſis, is alſo a right angle; therefore the angle ABF is equal to the angle ABC, and they are both in the ſame plane, which is impoſſible: Therefore the ſtraight line BC is not above the plane in which are BD and BE: Wherefore the three ſtraight lines BC, BD, BE are in one and the ſame plane. Therefore, if three ſtraight lines, &c. Q. E. D.



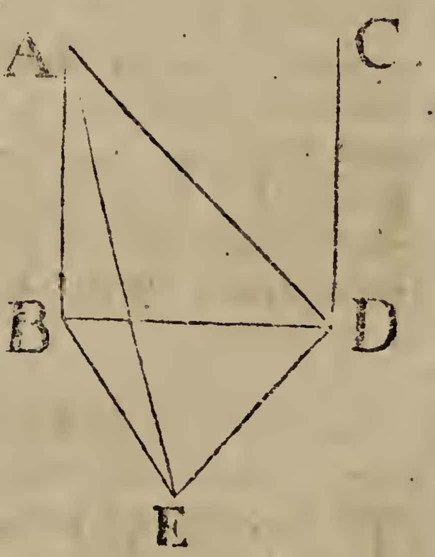
[a] 5. 11.
[b] 4. 11.
[c] 3. def. 11.

PROP. VI. THEOR.

IF two ſtraight lines be at right angles to the ſame plane, they ſhall be parallel to one another.

Let the ſtraight lines AB, CD be at right angles to the ſame plane; AB is parallel to CD.

Let them meet the plane in the points B, D, and draw the ſtraight line BD, to which draw DE at right angles, in the ſame plane; and make DE equal to AB, and join BE, AE, AD. Then, becauſe AB is perpendicular to the plane, it ſhall make right [a] angles with every ſtraight line which meets it, and is in that plane: But BD, BE, which are in that plane, do each of them meet AB. Therefore each of the angles ABD, ABE is a right angle: For the ſame reaſon, each of the angles CDB, CDE is a right angle: And becauſe AB is equal to DE, and BD common, the two ſides AB, BD are equal to the two ED, DB; and they contain right angles; therefore the baſe AD is equal [b] to the baſe BE: Again, becauſe AB is equal to



[a] 3. def. 11.
[b] 4. 1.

to DE, and BE to AD; AB, BE are equal to ED, DA; and, in the triangles ABE, EDA, the baſe AE is common; therefore the angle ABE is equal [c] to the angle EDA: But ABE is a right angle; therefore EDA is alſo a right angle, and ED perpendicular to DA: But it is alſo perpendicular to each of the two BD, DC: Wherefore ED is at right angles to each of the three ſtraight lines BD, DA, DC in the point in which they meet: Therefore theſe three ſtraight lines are all in the ſame plane [d]: But AB is in the plane in which are BD, DA, becauſe any three ſtraight lines which meet one another are in one plane [e]: Therefore AB, BD, DC are in one plane: And each of the angles ABD, BDC is a right angle; therefore AB is parallel [f] to CD. Wherefore, if two ſtraight lines, &c. Q. E. D.

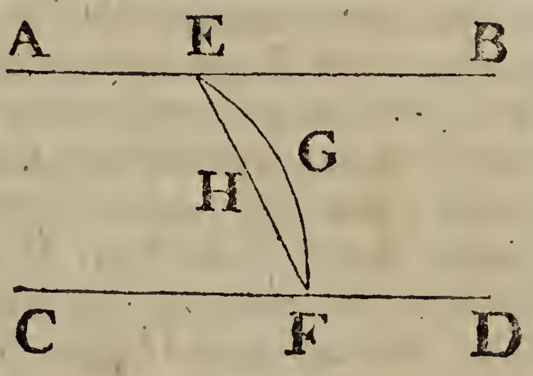
c 8. 1.
d 5. 11.
e 2. 11.
f 28. 1.

PROP. VII. THEOR.

IF two ſtraight lines be parallel, the ſtraight line drawn from any point in the one to any point in the other, is in the ſame plane with the parallels. See N.

Let AB, CD be parallel ſtraight lines, and take any point E in the one, and the point F in the other: The ſtraight line which joins E and F is in the ſame plane with the parallels.

If not, let it be, if poſſible, above the plane, as EGF; and in the plane ABCD in which the parallels are, draw the ſtraight line EHF from E to F; and ſince EGF alſo is a ſtraight line, the two ſtraight lines EHF, EGF include a ſpace between them, which is impoſſible [a]. Therefore the ſtraight line joining the points E, F is not above the plane in which the parallels AB, CD are, and is therefore in that plane. Wherefore, if two ſtraight lines, &c. Q. E. D.



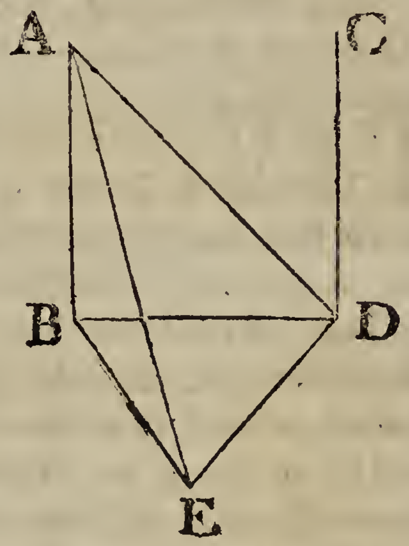
a 10. Ax. 1.

PROP. VIII. THEOR.

IF two ſtraight lines be parallel, and one of them is at right angles to a plane; the other alſo ſhall be at right angles to the ſame plane. See N.

Let AB, CD be two parallel ſtraight lines, and let one of them AB be at right angles to a plane; the other CD is at right angles to the ſame plane.

Let AB, CD meet the plane in the points B, D, and join BD: Therefore[g] AB, CD, BD are in one plane. In the plane to which AB is at right angles, draw DE at right angles to BD, and make DE equal to AB, and join BE, AE, AD. And becauſe AB is perpendicular to the plane, it is perpendicular to every ſtraight line which meets it, and is in that plane[a]: Therefore each of the angles ABD, ABE, is a right angle: And becauſe the ſtraight line BD meets the parallel ſtraight lines AB, CD, the angles ABD, CDB are together equal[b] to two right angles: And ABD is a right angle; therefore alſo CDB is a right angle, and CD perpendicular to BD: And becauſe AB is equal to DE, and BD common, the two AB, BD are equal to the two ED, DB, and the angle ABD is equal to the angle EDB, becauſe each of them is a right angle; therefore the baſe AD is equal[c] to the baſe BE: Again, becauſe AB is equal to DE, and BE to AD; the two AB, BE are equal to the two ED, DA; and the baſe AE is common to the triangles ABE, EDA; wherefore the angle ABE is equal[d] to the angle EDA: And ABE is a right angle; and therefore EDA is a right angle, and ED perpendicular to DA: But it is alſo perpendicular to BD; therefore ED is perpendicular[e] to the plane which paſſes through BD, DA, and ſhall[f] make right angles with every ſtraight line meeting it in that plane: But DC is in the plane paſſing through BD, DA, becauſe all three are in the plane in which are the parallels AB, CD: Wherefore ED is at right angles to DC; and therefore CD is at right angles to DE: But CD is alſo at right angles to DB; CD then is at right angles to the two ſtraight lines DE, DB in the point of their interſection D; and therefore is at right angles[e] to the plane paſſing through DE, DB, which is the ſame plane to which AB is at right angles. Therefore, if two ſtraight lines, &c. Q. E. D.



[g] 7. 11.
[a] 3. def. 11.
[b] 29. 1.
[c] 4. 1.
[d] 8. 1.
[e] 4. 11.
[f] 3. def. 11.

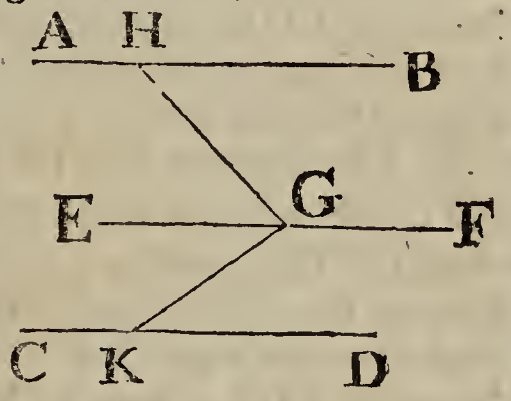
PROP.

PROP. IX. THEOR.

TWO ſtraight lines which are each of them parallel to the ſame ſtraight line, and not in the ſame plane with it, are parallel to one another.

Let AB, CD be each of them parallel to EF, and not in the ſame plane with it; AB ſhall be parallel to CD.

In EF take any point G, from which draw, in the plane paſſing through EF, AB, the ſtraight line GH at right angles to EF; and in the plane paſſing through EF, CD, draw GK at right angles to the ſame EF. And becauſe EF is perpendicular both to GH and GK, EF is perpendicular [a] to the plane HGK paſſing through them: And EF is parallel to AB; therefore AB is at right angles [b] to the plane HGK. For the ſame reaſon, CD is likewiſe at right angles to the plane HGK. Therefore AB, CD are each of them at right angles to the plane HGK. But if two ſtraight lines are at right angles to the ſame plane, they ſhall be parallel [c] to one another. Therefore AB is parallel to CD. Wherefore two ſtraight lines, &c. Q. E. D.



[a] 4. 11.

[b] 8. 11.

[c] 6. 11.

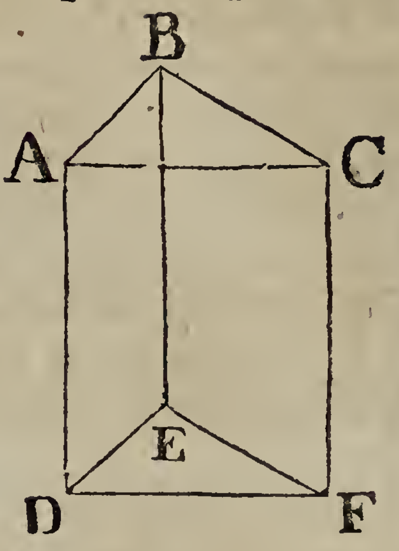
PROP. X. THEOR.

IF two ſtraight lines meeting one another be parallel to two others that meet one another, and are not in the ſame plane with the firſt two; the firſt two and the other two ſhall contain equal angles.

Let the two ſtraight lines AB, BC which meet one another be parallel to the two ſtraight lines DE, EF that meet one another, and are not in the ſame plane with AB, BC. The angle ABC is equal to the angle DEF.

Take BA, BC, ED, EF all equal to one another; and join

AD, CF, BE, AC, DF : Becauſe BA is equal and parallel to
ED, therefore AD is [a] both equal and
a 33. 1. parallel to BE. For the ſame reaſon,
CF is equal and parallel to BE. Therefore AD and CF are each of them equal and parallel to BE. But ſtraight
lines that are parallel to the ſame ſtraight
line, and not in the ſame plane with it,
b 9. 11. are parallel [b] to one another. Therefore
c 1. Ax. 1. AD is parallel to CF ; and it is equal [c]
to it, and AC, DF join them towards
the ſame parts ; and therefore [a] AC is
equal and parallel to DF. And becauſe AB, BC are equal to DE, EF, and the baſe AC to the
d 8. 1. baſe DF ; the angle ABC is equal [d] to the angle DEF. Therefore, if two ſtraight lines, &c. Q. E. D.

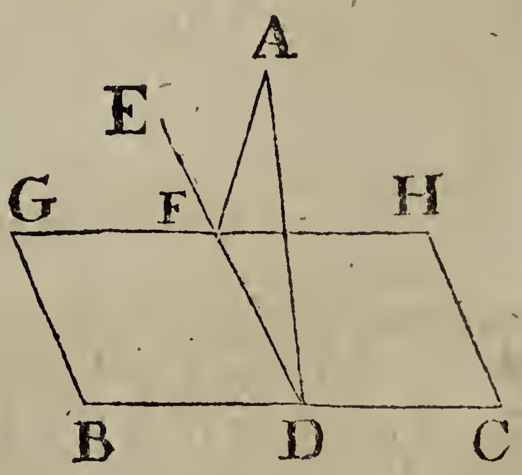


PROP. XI. PROB.

TO draw a ſtraight line perpendicular to a plane, from a given point above it.

Let A be the given point above the plane BH ; it is required to draw from the point A a ſtraight line perpendicular to the plane BH.

In the plane draw any ſtraight line BC, and from the point
a 12. 1. A draw [a] AD perpendicular to BC. If then AD be alſo perpendicular to the plane BH, the thing required is already
done ; but if it be not, from the
b 11. 1. point D draw [b], in the plane BH,
the ſtraight line DE at right angles to BC ; and from the point
A draw AF perpendicular to
c 31. 1. DE ; and through F draw [c] GH
parallel to BC : And becauſe BC
is at right angles to ED and DA,
d 4. 11. BC is at right angles [d] to the plane
paſſing through ED, DA. And
GH is parallel to BC ; but, if
two ſtraight lines be parallel, one of which is at right angles to
e 8. 11. a plane, the other ſhall be at right [e] angles to the ſame plane ;
wherefore GH is at right angles to the plane through ED, DA,
f 3. def. 11. and is perpendicular [f] to every ſtraight line meeting it in that
plane. But AF, which is in the plane through ED, DA, meets
it :

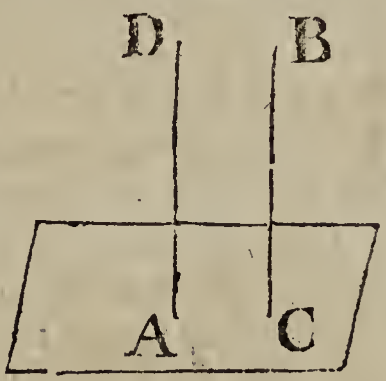


it: Therefore GH is perpendicular to AF; and conſequently AF is perpendicular to GH; and AF is perpendicular to DE: Therefore AF is perpendicular to each of the ſtraight lines GH, DE. But if a ſtraight line ſtands at right angles to each of two ſtraight lines in the point of their interſection, it ſhall alſo be at right angles to the plane paſſing through them. But the plane paſſing through ED, GH is the plane BH; therefore AF is perpendicular to the plane BH; therefore, from the given point A, above the plane BH, the ſtraight line AF is drawn perpendicular to that plane: Which was to be done.

PROP. XII. PROB.

TO erect a ſtraight line at right angles to a given plane, from a point given in the plane.

Let A be the point given in the plane; it is required to erect a ſtraight line from the point A at right angles to the plane.



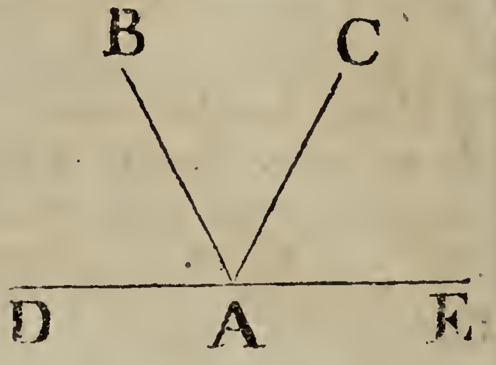
From any point B above the plane draw [a] BC perpendicular to it; and from A draw [b] AD parallel to BC. Becauſe, therefore, AD, CB are two parallel ſtraight lines, and one of them BC is at right angles to the given plane, the other AD is alſo at right angles to it [c]. Therefore a ſtraight line has been erected at right angles to a given plane from a point given in it. Which was to be done.

a 11. 11.
b 31. 1.
c 8. 11.

PROP. XIII. THEOR.

FROM the ſame point in a given plane, there cannot be two ſtraight lines at right angles to the plane, upon the ſame ſide of it: And there can be but one perpendicular to a plane from a point above the plane.

For, if it be poſſible, let the two ſtraight lines AC, AB, be at right angles to a given plane from the ſame point A in the plane, and upon the ſame ſide of it; and let a plane paſs through BA, AC; the common ſection of this with the given plane is a ſtraight [a] line paſſing through A: Let DAE be their common ſection: Therefore the ſtraight lines AB, AC, DAE are in one plane: And becauſe CA is at right angles to the given plane, it ſhall make right angles with every ſtraight line meeting it in that plane. But DAE, which is in that plane, meets CA; therefore CAE is a right angle. For the ſame reaſon BAE is a right angle. Wherefore the angle CAE is equal to the angle BAE; and they are in one plane, which is impoſſible. Alſo, from a point above a plane, there can be but one perpendicular to that plane; for, if there could be two, they would be parallel [b] to one another, which is abſurd. Therefore, from the ſame point, &c. Q. E. D.



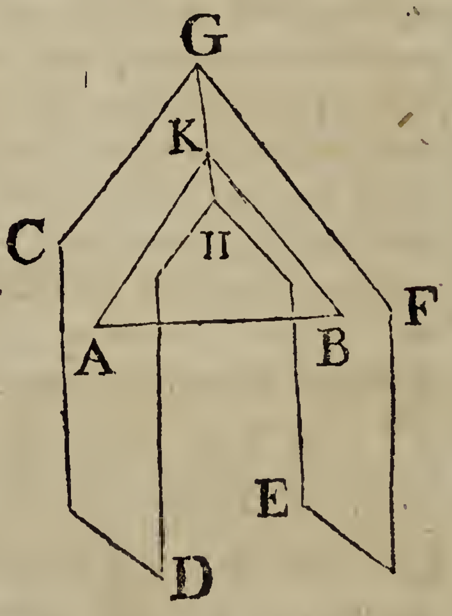
a 3. 11.
b 6. 11.

PROP. XIV. THEOR.

PLANES to which the ſame ſtraight line is perpendicular, are parallel to one another.

Let the ſtraight line AB be perpendicular to each of the planes CD, EF; theſe planes are parallel to one another.

If not, they ſhall meet one another when produced; let them meet; their common ſection ſhall be a ſtraight line GH, in which take any point K, and join AK, BK: Then, becauſe AB is perpendicular to the plane EF, it is perpendicular [a] to the ſtraight line BK which is in that plane. Therefore ABK is a right angle. For the ſame reaſon, BAK is a right angle; wherefore the two angles ABK, BAK of the triangle ABK are equal to two right angles, which is impoſſible [b]: Therefore the planes CD, EF, though produced, do not meet one another; that is, they are parallel [c]. Therefore planes, &c. Q. E. D.



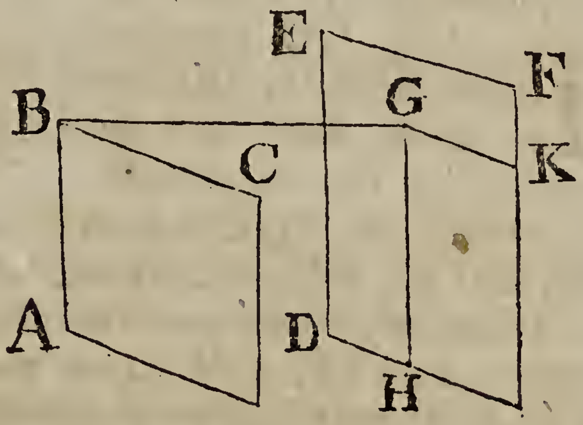
a 3. def. 11.
b 17. 1.
c 8. def. 11.

PROP.

PROP. XV. THEOR.

IF two ſtraight lines meeting one another, be parallel See N.
to two ſtraight lines which meet one another, but
are not in the ſame plane with the firſt two; the plane
which paſſes through theſe is parallel to the plane paſſing
through the others.

Let AB, BC, two ſtraight lines meeting one another, be parallel to DE, EF that meet one another, but are not in the ſame plane with AB, BC: The planes through AB, BC, and DE, EF ſhall not meet, though produced.



From the point B draw BG perpendicular [a] to the plane a 11. 11.
which paſſes through DE, EF, and let it meet that plane in
G; and through G draw GH parallel [b] to ED, and GK pa- b 31. 1.
rallel to EF: And becauſe BG is perpendicular to the plane
through DE, EF, it ſhall
make right angles with every
ſtraight line meeting it in that
plane [c]. But the ſtraight lines c 3. def. 11.
GH, GK in that plane meet
it: Therefore each of the angles BGH, BGK is a right
angle: And becauſe BA is
parallel [d] to GH (for each of d 9. 11.
them is parallel to DE, and
they are not both in the ſame plane with it) the angles GBA,
BGH are together equal [e] to two right angles: And BGH is a e 29. 1.
right angle; therefore alſo GBA is a right angle, and GB perpendicular to BA: For the ſame reaſon, GB is perpendicular to
BC: Since therefore the ſtraight line GB ſtands at right angles
to the two ſtraight lines BA, BC, that cut one another in B;
GB is perpendicular [f] to the plane through BA, BC: And it is f 4. 11.
perpendicular to the plane through DE, EF; therefore BG is
perpendicular to each of the planes through AB, BC, and DE,
EF: But planes to which the ſame ſtraight line is perpendicular,
are parallel [g] to one another: Therefore the plane through AB, g 14. 11.
BC is parallel to the plane through DE, EF. Wherefore, if
two ſtraight lines, &c. Q. E. D.

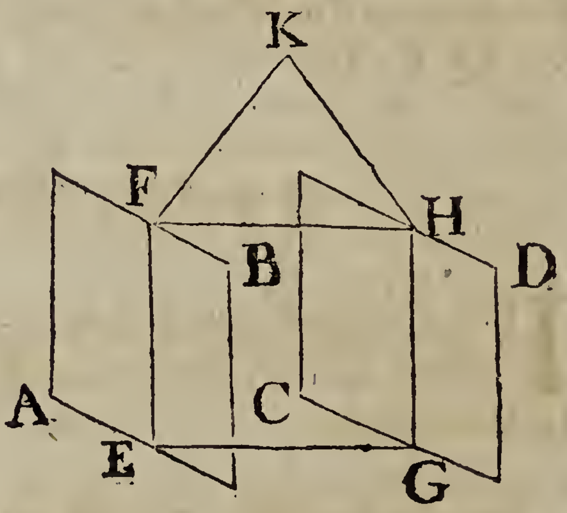
PROP. XVI. THEOR.

See N.

IF two parallel planes be cut by another plane, their common ſections with it are parallels.

Let the parallel planes, AB, CD be cut by the plane EFHG, and let their common ſections with it be EF, GH: EF is parallel to GH.

For, if it is not, EF, GH ſhall meet, if produced, either on the ſide of FH, or EG: Firſt, let them be produced on the ſide of FH, and meet in the point K: Therefore, ſince EFK is in the plane AB, every point in EFK is in that plane; and K is a point in EFK; therefore K is in the plane AB: For the ſame reaſon K is alſo in the plane CD: Wherefore the planes AB, CD produced meet one another; but they do not meet, ſince they are parallel by the hypotheſis: Therefore the ſtraight lines EF, GH do not meet when produced on the ſide of FH: In the ſame manner it may be proved, that EF, GH do not meet when produced on the ſide of EG: But ſtraight lines which are in the ſame plane and do not meet, though produced either way, are parallel: Therefore EF is parallel to GH. Wherefore, if two parallel planes, &c. Q. E. D.



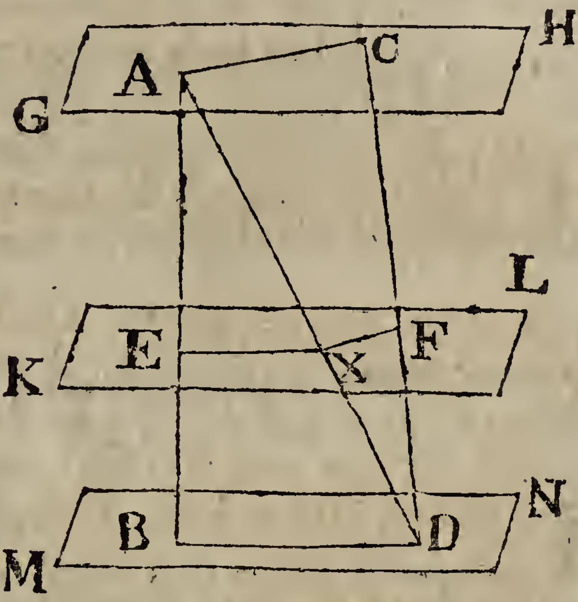
PROP. XVII. THEOR.

IF two ſtraight lines be cut by parallel planes, they ſhall be cut in the ſame ratio.

Let the ſtraight lines AB, CD be cut by the parallel planes GH, KL, MN, in the points A, E, B; C, F, D: As AE is to EB, ſo is CF to FD.

Join AC, BD, AD, and let AD meet the plane KL in the point X; and join EX, XF: Becauſe the two parallel planes KL, MN are cut by the plane EBDX, the common ſections

EX, BD, are parallel [a]. For the ſame reaſon, becauſe the two parallel planes GH, KL are cut by the plane AXFC, the common ſections AC, XF are parallel: And becauſe EX is parallel to BD, a ſide of the triangle ABD, as AE to EB, ſo is [b] AX to XD. Again, becauſe XF is parallel to AC, a ſide of the triangle ADC, as AX to XD, ſo is CF to FD: And it was proved that AX is to XD, as AE to EB; Therefore [c], as AE to EB, ſo is CF to FD. Wherefore, if two ſtraight lines, &c. Q. E. D.



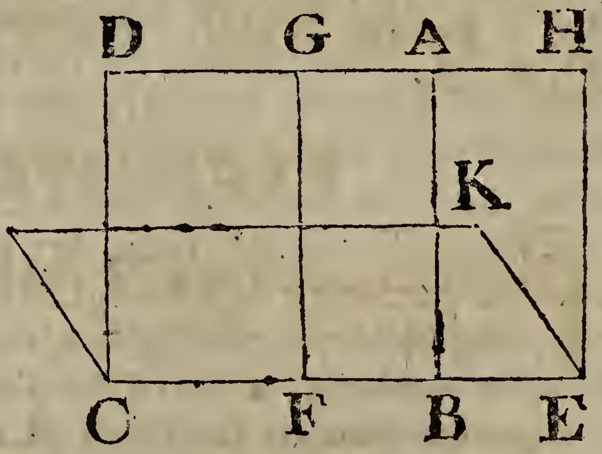
a 16. 11.
b 2. 6.
c 11. 5.

PROP. XVIII. THEOR.

IF a ſtraight line be at right angles to a plane, every plane which paſſes through it ſhall be at right angles to that plane.

Let the ſtraight line AB be at right angles to a plane CK; every plane which paſſes through AB ſhall be at right angles to the plane CK.

Let any plane DE paſs through AB, and let CE be the common ſection of the planes DE, CK; take any point F in CE, from which draw FG in the plane DE at right angles to CE: And becauſe AB is perpendicular to the plane CK, therefore it is alſo perpendicular to every ſtraight line in that plane meeting it [a]: And conſequently it is perpendicular to CE: Wherefore ABF is a right angle; but GFB is likewiſe a right angle; therefore AB is parallel [b] to FG. And AB is at right angles to the plane CK; therefore FG is alſo at right angles to the ſame plane [c]. But one plane is at right angles to another plane when the ſtraight lines drawn in one of the planes, at right angles to their common ſection, are alſo at right



a 3. def. 11.
b 28. 1.
c 8. 11.

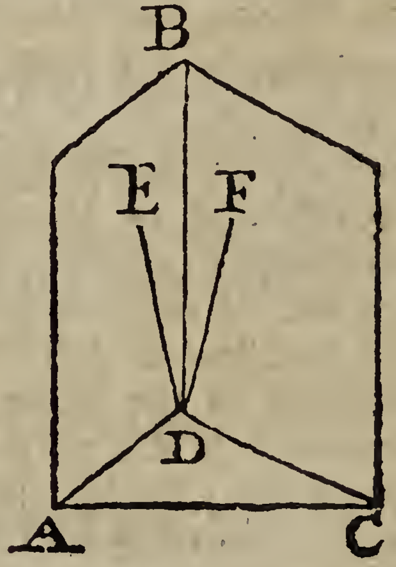
angles to the other plane[d]; and any ſtraight line FG in the plane DE, which is at right angles to CE the common ſection
d 4. def. 11. of the planes, has been proved to be perpendicular to the other plane CK; therefore the plane DE is at right angles to the plane CK. In like manner, it may be proved that all the planes which paſs through AB are at right angles to the plane CK. Therefore, if a ſtraight line, &c. Q. E. D.

PROP. XIX. THEOR.

IF two planes cutting one another be each of them perpendicular to a third plane; their common ſection ſhall be perpendicular to the ſame plane.

Let the two planes AB, BC be each of them perpendicular to a third plane, and let BD be the common ſection of the firſt two; BD is perpendicular to the third plane.

If it be not, from the point D draw, in the plane AB, the ſtraight line DE at right angles to AD the common ſection of the plane AB with the third plane; and in the plane BC draw DF at right angles to CD the common ſection of the plane BC with the third plane. And becauſe the plane AB is perpendicular to the third plane, and DE is drawn in the plane AB at right angles to AD their common ſection, DE is perpendicular to the
a 4. def. 11. third plane[a]. In the ſame manner it may be proved that DF is perpendicular to the third plane. Wherefore, from the point D two ſtraight lines ſtand at right angles to the third plane, upon the ſame ſide of it, which is im-
b 13. 11. poſſible[b]: Therefore, from the point D there cannot be any ſtraight line at right angles to the third plane, except BD the common ſection of the planes AB, BC. BD therefore is perpendicular to the third plane. Wherefore, if two planes, &c. Q. E. D.

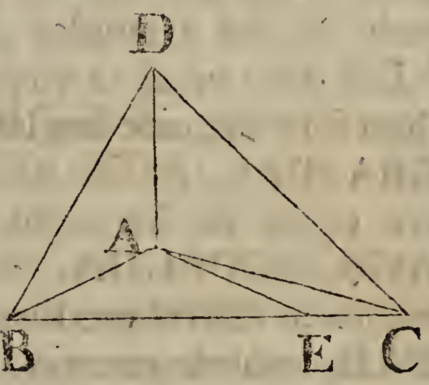


PROP. XX. THEOR.

IF a ſolid angle be contained by three plane angles, any two of them are greater than the third. See N.

Let the ſolid angle at A be contained by the three plane angles BAC, CAD, DAB. Any two of them are greater than the third.

If the angles BAC, CAD, DAB be all equal, it is evident that any two of them are greater than the third. But if they are not, let BAC be that angle which is not leſs than either of the other two, and is greater than one of them DAB; and at the point A in the ſtraight line AB, make, in the plane which paſſes through BA, AC, the angle BAE equal [a] to the angle DAB; and make AE equal to AD, and through E draw a 23. 1. BEC cutting AB, AC in the points B, C, and join DB, DC. And becauſe DA is equal to AE, and AB is common, the two DA, AB are equal to the two EA, AB, and the angle DAB is equal to the angle EAB: Therefore the baſe DB is equal [b] to the baſe BE. b 4. 1. And becauſe BD, DC are greater [c] than CB, and one of them BD has been proved equal to BE a part of CB, therefore c 20. 1. the other DC is greater than the remaining part EC. And becauſe DA is equal to AE, and AC common, but the baſe DC greater than the baſe EC; therefore the angle DAC is greater [d] than the angle EAC; and, by the conſtruction, the angle DAB d 25. 1. is equal to the angle BAE; wherefore the angles DAB, DAC are together greater than BAE, EAC, that is, than the angle BAC. But BAC is not leſs than either of the angles DAB, DAC; therefore BAC, with either of them, is greater than the other. Wherefore, if a ſolid angle, &c. Q. E. D.



PROP. XXI. THEOR.

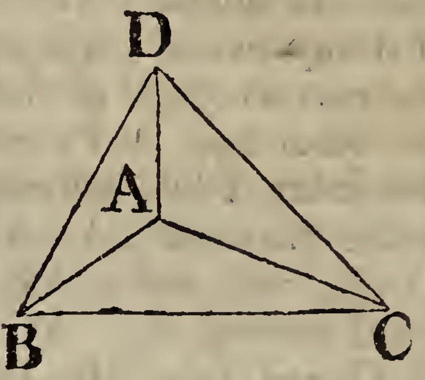
EVERY ſolid angle is contained by plane angles which together are leſs than four right angles.

Firſt, Let the ſolid angle at A be contained by three plane angles BAC, CAD, DAB. Theſe three together are leſs than four right angles.

Take in each of the ſtraight lines AB, AC, AD any points B, C, D, and join BC, CD, DB: Then, becauſe the ſolid angle at B is contained by the three plane angles CBA, ABD, DBC,

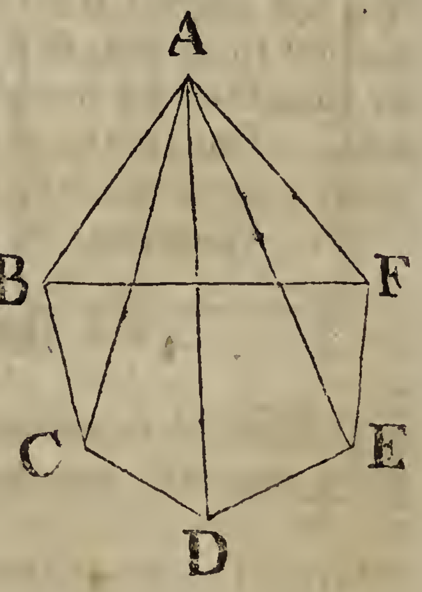
a 20. 11. any two of them are greater[a] than the third; therefore the angles CBA, ABD are greater than the angle DBC: For the ſame reaſon, the angles BCA, ACD are greater than the angle DCB; and the angles CDA, ADB greater than BDC: Wherefore the ſix angles CBA, ABD, BCA, ACD, CDA, ADB are greater than the three angles DBC, BCD, CDB: But the three angles DBC, BCD, CDB are equal to two

b 32. 1. right angles[b]: Therefore the ſix angles CBA, ABD, BCA, ACD, CDA, ADB are greater than two right angles: And becauſe the three angles of each of the triangles ABC, ACD, ADB are eqnal to two right angles, therefore the nine angles of theſe three triangles, viz. the angles CBA, BAC, ACB, ACD, CDA. DAC, ADB, DBA, BAD are equal to ſix right angles: Of theſe the ſix angles CBA, ACB, ACD, CDA, ADB, DBA are greater than two right angles: Therefore the remaining three angles BAC, DAC, BAD, which contain the ſolid angle at A, are leſs than four right angles.



Next, let the ſolid angle at A be contained by any number of plane angles BAC, CAD, DAE, EAF, FAB; theſe together are leſs than four right angles.

Let the planes in which the angles are, be cut by a plane, and let the common ſection of it with thoſe planes be BC, CD, DE, EF, FB: And becauſe the ſolid angle at B is contained by three plane angles CBA, ABF, FBC, of which any two are greater[a] than the third, the angles CBA, ABF are greater than the angle FBC: For the ſame reaſon, the two plane angles at each of the points C, D, E, F, viz. the angles which are at the baſes of the triangles having the common vertex A, are greater than the third angle at the ſame point, which is one of the angles of the polygon BCDEF: Therefore all the angles at the baſes of the triangles are together greater



greater than all the angles of the polygon: And becauſe all the angles of the triangles are together equal to twice as many right angles as there are triangles [b]; that is, as there are ſides b 32. 1. in the polygon BCDEF; and that all the angles of the polygon, together with four right angles, are likewiſe equal to twice as many right angles as there are ſides in the polygon [c]; there- c 1. Cor. 32. 1. fore all the angles of the triangles are equal to all the angles of the polygon together with four right angles. But all the angles at the baſes of the triangles are greater than all the angles of the polygon, as has been proved. Wherefore the remaining angles of the triangles, viz. thoſe at the vertex, which contain the ſolid angle at A, are leſs than four right angles. Therefore every ſolid angle, &c. Q. E. D.

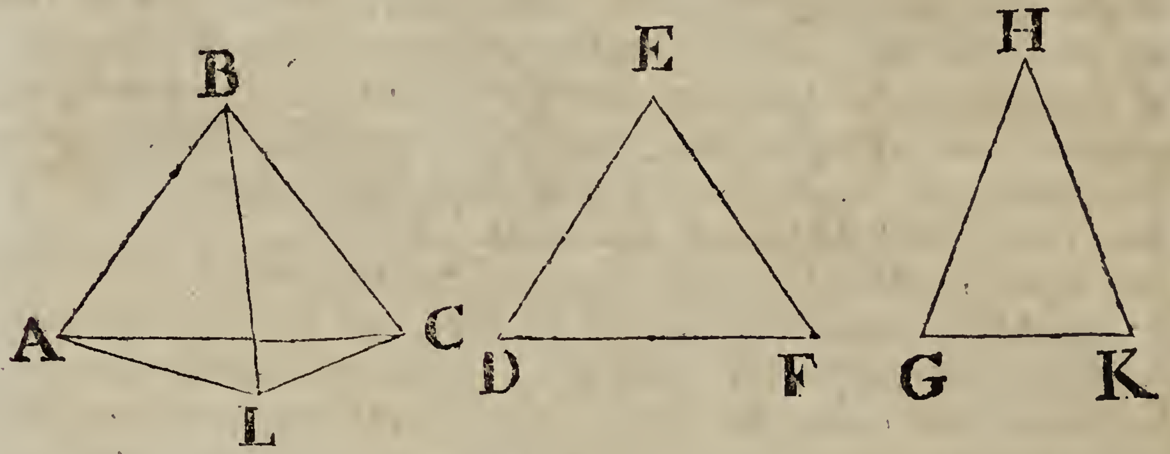
PROP. XXII. THEOR.

IF every two of three plane angles be greater than the third, and if the ſtraight lines which contain them be all equal; a triangle may be made of the ſtraight lines that join the extremities of thoſe equal ſtraight lines. See N.

Let ABC, DEF, GHK be three plane angles, whereof every two are greater than the third, and are contained by the equal ſtraight lines AB, BC, DE, EF, GH, HK; if their extremities be joined by the ſtraight lines AC, DF, GK, a triangle may be made of three ſtraight lines equal to AC, DF, GK; that is, every two of them are together greater than the third.

If the angles at B, E, H are equal: AC, DF, GK are alſo equal [a], and any two of them greater than the third: But if a 4. 1. the angles are not all equal, let the angle ABC be not leſs than either of the two at E, H; therefore the ſtraight line AC is not leſs than either of the other two DF, GK [b]; and it is b 4. Cor. 24. 1. plain that AC, together with either of the other two, muſt be greater than the third: Alſo DF with GK are greater than AC: For, at the point B in the ſtraight line AB make [c] the c 23. 1.

angle ABL equal to the angle GHK, and make BL equal to one of the ſtraight lines AB, BC, DE, EF, GH, HK, and join AL, LC: Then becauſe AB, BL are equal to GH, HK, and the angle ABL to the angle GHK, the baſe AL is equal to the baſe GK: And becauſe the angles at E, H are greater than the angle ABC, of which the angle at H is equal to ABL, therefore the remaining angle at E is greater than the angle LBC:



And becauſe the two ſides LB, BC are equal to the two DE, EF, and that the angle DEF is greater than the angle LBC, the baſe DF is greater [d] than the baſe LC: And it has been proved that GK is equal to AL; therefore DF and GK are greater than AL and LC: But AL and LC are greater [e] than AC; much more then are DF and GK greater than AC. Wherefore every two of theſe ſtraight lines AC, DF, GK are greater than the third; and, therefore, a triangle may be made [f], the ſides of which ſhall be equal to AC, DF, GK. Q. E. D.

d 24. 1.
e 20. 1.
f 22. 1.

PROP. XXIII. PROB.

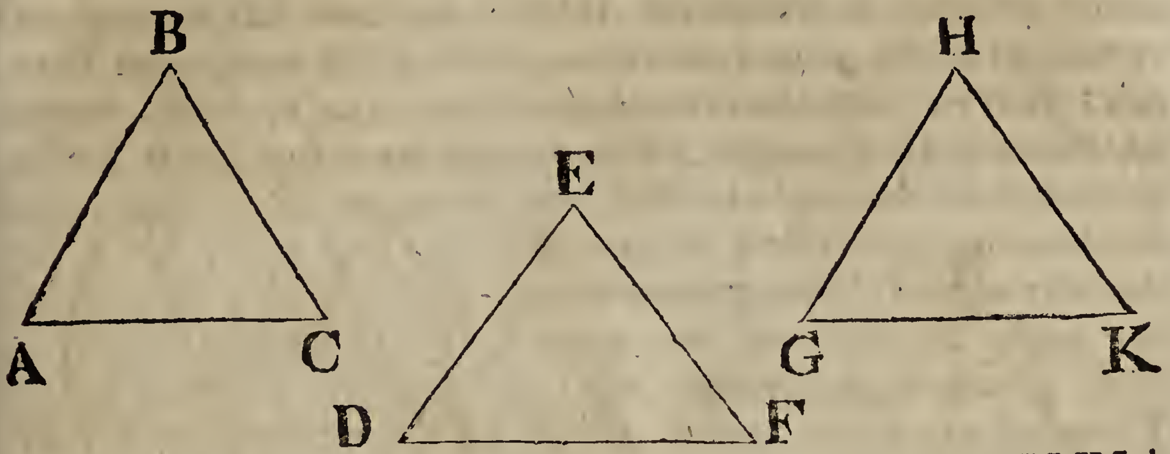
See N.

TO make a ſolid angle which ſhall be contained by three given plane angles, any two of them being greater than the third, and all three together leſs than four right angles.

Let the three given plane angles be ABC, DEF, GHK, any two of which are greater than the third, and all of them together leſs than four right angles. It is required to make a ſolid angle contained by three plane angles equal to ABC, DEF, GHK, each to each.

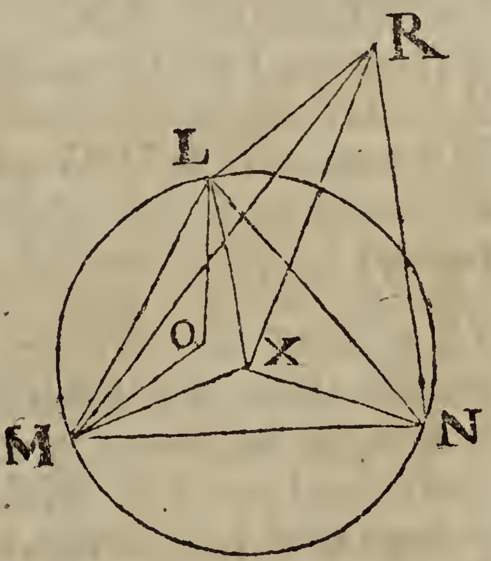
From the ſtraight lines containing the angles, cut off AB, BC, DE, EF, GH, HK, all equal to one another; and join AC, DF, GK: Then a triangle may be made [a] of three ſtraight lines equal to AC, DF, GK. Let this be the triangle LMN [b], ſo that AC be equal to LM, DF to MN, and GK to LN; and about the triangle LMN deſcribe [c] a circle, and find its centres X, which will either be within the triangle, or in one of its ſides, or without it.

a 22. 11.
b 22. 1.
c 5. 4.

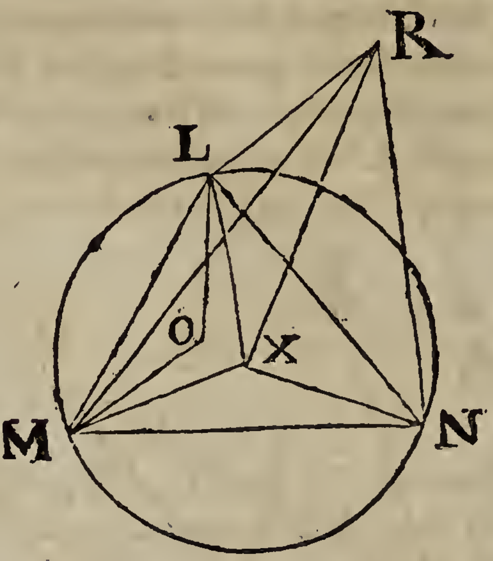


Firſt, Let the centre X be within the triangle, and join LX, MX, NX: AB is greater than LX: If not, AB muſt either be equal to, or leſs than LX; firſt, let it be equal: Then becauſe AB is equal to LX, and that AB is alſo equal to BC, and LX to XM, AB and BC are equal to LX and XM, each to each; and the baſe AC is, by conſtruction, equal to the baſe LM; wherefore the angle ABC is equal to the angle LXM [d]. For the ſame reaſon, the angle DEF is equal to the angle MXN, and the angle GHK to the angle NXL: Therefore the three angles ABC, DEF, GHK are equal to the three angles LXM, MXN, NXL: But the three angles LXM, MXN, NXL are equal to four right angles [e]: therefore alſo the three angles ABC, DEF, GHK are equal to four right angles: But, by the hypotheſis, they are leſs than four right angles, which is abſurd; therefore AB is not equal to LX: But neither can AB be leſs than LX: For, if poſſible, let it be leſs, and upon the ſtraight line LM, on the ſide of it on which is the centre X, deſcribe the triangle LOM, the ſides LO, OM of which are equal to AB, BC; and becauſe the baſe LM is equal to the baſe

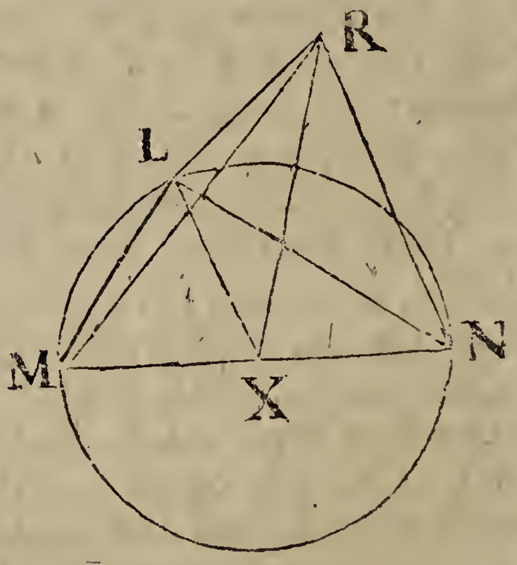
d 8. 1.
e 2. Cor. 15. 1.



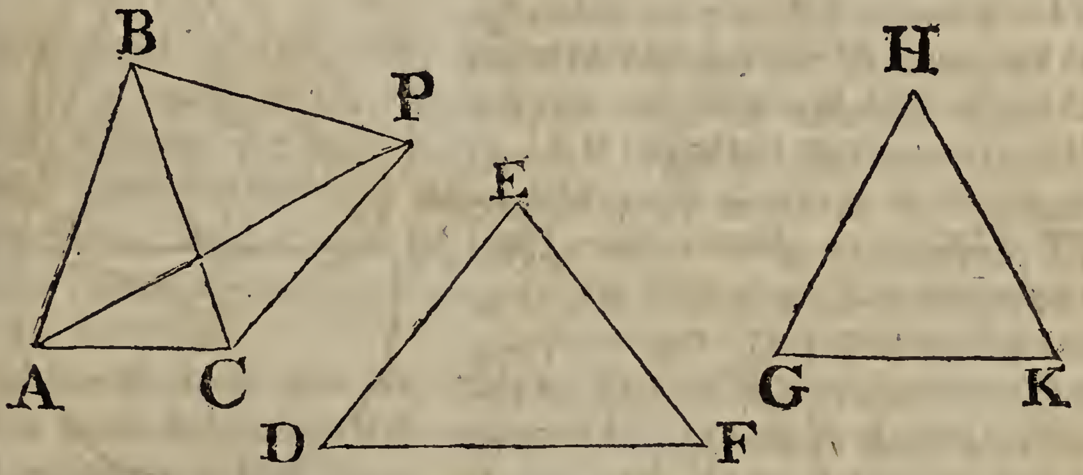
baſe AC, the angle LOM is equal to the angle ABC [d]: And AB, that is, LO, by the hypotheſis, is leſs than LX; where- d 8. 1. fore LO, OM fall within the triangle LXM; for, if they fell upon its ſides, or without it, they would be equal to, or greater than f 21. 1. LX, XM [f]: Therefore the angle LOM, that is, the angle ABC, is greater than the angle LXM [f]: In the ſame manner it may be proved that the angle DEF is greater than the angle MXN, and the angle GHK greater than the angle NXL. Therefore the three angles ABC, DEF, GHK are greater than the three angles LXM, MXN, NXL; that is, than four right angles: But the ſame angles ABC, DEF, GHK are leſs than four right angles; which is abſurd: Therefore AB is not leſs than LX, and it has been proved that it is not equal to LX; wherefore AB is greater than LX.

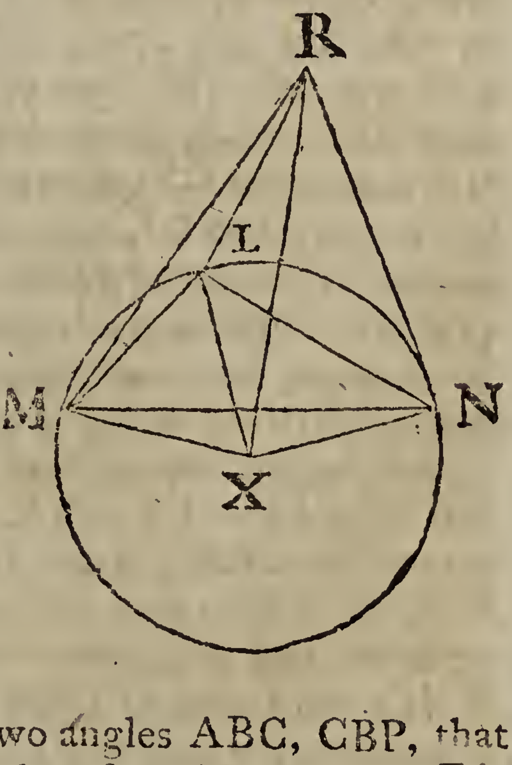


Next, Let the centre X of the circle fall in one of the ſides of the triangle, viz. in MN, and join XL: In this caſe alſo AB is greater than LX. If not, AB is either equal to LX, or leſs than it: Firſt, let it be equal to XL: Therefore AB and BC, that is, DE, and EF, are equal to MX and XL, that is, to MN: But, by the conſtruction, MN is equal to DF; therefore DE, EF are equal to DF, which is im- † 20. 1. poſſible [†]: Wherefore AB is not equal to LX; nor is it leſs; for then, much more, an abſurdity would follow: Therefore AB is greater than LX.



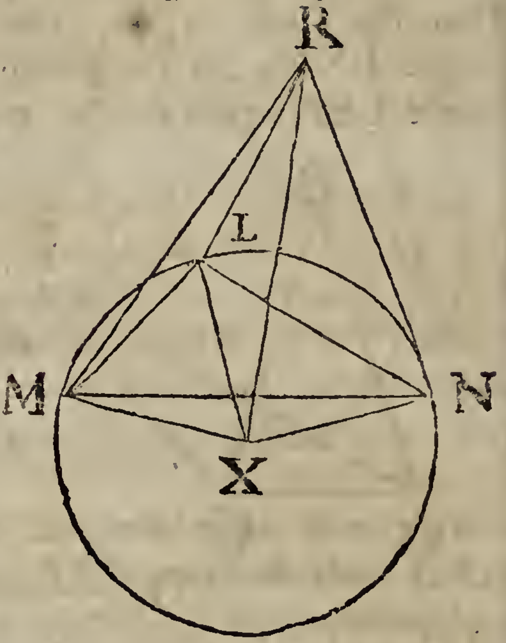
But, let the centre X of the circle fall without the triangle LMN, and join LX, MX, NX. In this caſe likewiſe AB is greater than LX: If not, it is either equal to, or leſs than LX: Firſt, let it be equal; it may be proved in the ſame manner, as in the firſt caſe, that the angle ABC is equal to the angle MXL, and GHK to LXN; therefore the whole angle MXN is equal to the two angles, ABC, GHK: But ABC and GHK are together greater than the angle DEF; therefore alſo the angle MXN is greater than DEF. And becauſe DE, EF are equal to MX, XN, and the baſe DF to the baſe MN, the angle MXN is equal [d] to the angle DEF: And it has been proved, that it is greater than DEF, which is abſurd. d 8. 1. Therefore AB is not equal to LX. Nor yet is it leſs; for then, as has been proved in the firſt caſe, the angle ABC is greater than the angle MXL, and the angle GHK greater than the angle LXN. At the point B in the ſtraight line CB make the angle CBP equal to the angle GHK, and make BP equal to HK, and join CP, AP. And becauſe CB is equal to GH; CB, BP are equal to GH, HK, each to each, and they contain equal angles; wherefore the baſe CP is equal to the baſe GK, that is, to LN. And in the iſoſceles triangles ABC, MXL, becauſe the angle ABC is greater than the angle MXL, therefore the angle MLX at the baſe is greater [g] than the angle ACB at the baſe. g 32. 1. For the ſame reaſon, becauſe the angle GHK, or CBP, is greater than the angle LXN, the angle XLN is greater than the angle CBP. Therefore the whole angle MLX is greater than the whole angle ACP. And becauſe ML, LN are equal to AC, CP, each to each, but the angle MLN is greater than the angle ACP, the baſe MN is greater [h] than the baſe AP. h 24. 1. And MN is equal to DF; therefore alſo DF is greater than AP. Again, becauſe DE, EF are equal to AB, BP, but the baſe DF greater than the baſe AP, the angle DEF is greater [k] than the angle ABP. k 25. 1. And ABP is equal to the two angles ABC, CBP, that is, to the two angles ABC, GHK; therefore the angle DEF is greater than the two angles ABC, GHK; but it is alſo leſs than theſe which is impoſſible. Therefore AB is not leſs than LX;





Book XI. LX; and it has been proved that it is not equal to it; therefore AB is greater than LX.

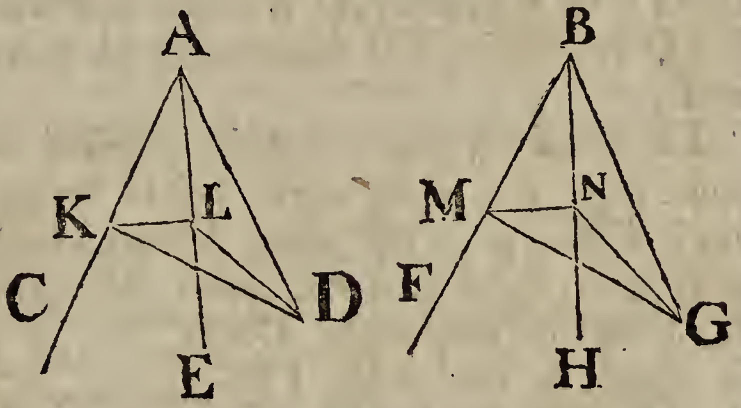
a 12. 11. From the point X erect [a] XR at right angles to the plane of the circle LMN. And becauſe it has been proved in all the caſes, that AB is greater than LX, find a ſquare equal to the exceſs of the ſquare of AB above the ſquare of LX, and make RX equal to its ſide, and join RL, RM, RN. Becauſe RX is perpendicular to the plane of the circle LMN, it b 3. def. 11. is [b] perpendicular to each of the ſtraight lines LX, MX, NX. And becauſe LX is equal to MX, and XR common, and at right angles to each of them, the baſe RL is equal to the baſe RM. For the ſame reaſon, RN is equal to each of the two RL, RM. Therefore the three ſtraight lines RL, RM, RN are all equal. And becauſe the ſquare of XR is equal to the exceſs of the ſquare of AB above the ſquare c 47. 1. of LX; therefore the ſquare of AB is equal to the ſquares of LX, XR. But the ſquare of RL is equal [c] to the ſame ſquares, becauſe LXR is a right angle. Therefore the ſquare of AB is equal to the ſquare of RL, and the ſtraight line AB to RL. But each of the ſtraight lines BC, DE, EF, GH, HK is equal to AB, and each of the two RM, RN is equal to RL. Wherefore AB, BC, DE, EF, GH, HK are each of them equal to each of the ſtraight lines RL, RM, RN. And becauſe RL, RM, are equal to AB, BC, and the baſe LM to the baſe AC; d 8. 1. the angle LRM is equal [d] to the angle ABC. For the ſame reaſon, the angle MRN is equal to the angle DEF, and NRL to GHK. Therefore there is made a ſolid angle at R, which is contained by three plane angles LRM, MRN, NRL, which are equal to the three given plane angles ABC, DEF, GHK, each to each. Which was to be done.



PROP. A. THEOR.

IF each of two ſolid angles be contained by three plane angles equal to one another, each to each; the planes in which the equal angles are, have the ſame inclination to one another. See N.

Let there be two ſolid angles at the points A, B; and let the angle at A be contained by the three plane angles CAD, CAE, EAD; and the angle at B by the three plane angles FBG, FBH, HBG; of which the angle CAD is equal to the angle FBG, and CAE to FBH, and EAD to HBG: The planes in which the equal angles are, have the ſame inclination to one another.



In the ſtraight line AC take any point K, and in the plane CAD from K draw the ſtraight line KD at right angles to AC, and in the plane CAE the ſtraight line KL at right angles to the ſame AC: Therefore the angle DKL is the inclination [a] of the plane CAD to the plane CAE: In BF take BM equal to AK, and from the point M draw, in the planes FBG, FBH, the ſtraight lines MG, MN at right angles to BF; therefore the angle GMN is the inclination [a] of the plane FBG to the plane FBH: Join LD, NG; and becauſe in the triangles KAD, MBG, the angles KAD, MBG are equal, as alſo the right angles AKD, BMG, and that the ſides AK, BM, adjacent to the equal angles, are equal to one another; therefore KD is equal [b] to MG, and AD to BG: For the ſame reaſon, in the triangles KAL, MBN, KL is equal to MN, and AL to BN: And in the triangles LAD, NBG, LA, AD are equal to NB, BG, and they contain equal angles; therefore the baſe LD is equal [c] to the baſe NG. Laſtly, in the triangles KLD, MNG, the ſides DK, KL are equal to GM, MN, and the baſe LD to the baſe NG; therefore the angle DKL is equal [d] to the angle GMN: But the angle DKL is the inclination of the plane CAD to the plane CAE, and the angle GMN is the inclination

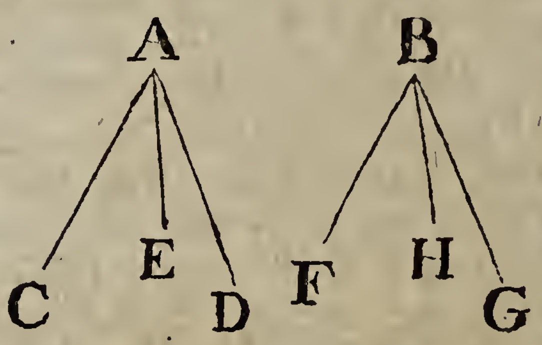
a 6. def. 11.
b 26. 1.
c 4. 1.
d 8. 1.

tion of the plane FBG to the plane FBH, which planes have therefore the ſame inclination [a] to one another: And in the
a 7. def. 11. ſame manner it may be demonſtrated, that the other planes in which the equal angles are, have the ſame inclination to one another. Therefore, if two ſolid angles, &c. Q. E. D.

PROP. B. THEOR.

See N. IF two ſolid angles be contained, each by three plane angles which are equal to one another, each to each, and alike ſituated; theſe ſolid angles are equal to one another.

Let there be two ſolid angles at A and B, of which the ſolid angle at A is contained by the three plane angles CAD, CAE, EAD; and that at B, by the three plane angles FBG, FBH, HBG; of which CAD is equal to FBG; CAE to FBH; and EAD to HBG: The ſolid angle at A is equal to the ſolid angle at B.



Let the ſolid angle at A be applied to the ſolid angle at B; and, firſt, the plane angle CAD being applied to the plane angle FBG, ſo as the point A may coincide with the point B, and the ſtraight line AC with BF; then AD coincides with BG, becauſe the angle CAD is equal to the angle FBG: And becauſe the inclination of the plane CAE to the plane
a A. 11. CAD is equal [a] to the inclination of the plane FBH to the plane FBG, the plane CAE coincides with the plane FBH, becauſe the planes CAD, FBG coincide with one another: And becauſe the ſtraight lines AC, BF coincide, and that the angle CAE is equal to the angle FBH; therefore AE coincides with BH, and AD coincides with BG; wherefore the plane EAD coincides with the plane HBG: Therefore the ſolid angle A coincides with the ſolid angle B, and conſeqently they are e-
b 8. A. 1. qual [b] to one another. Q. E. D.

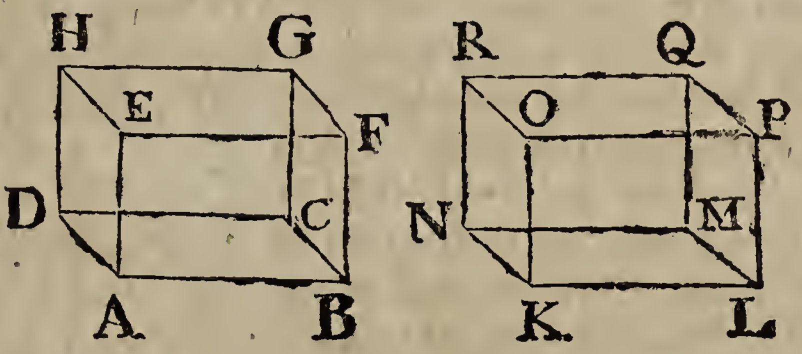
PROP. C. THEOR.

SOLID figures contained by the ſame number of equal and ſimilar planes alike ſituated, and having none of their ſolid angles contained by more than three plane angles; are equal and ſimilar to one another. See N.

Let AG, KQ be two ſolid figures contained by the ſame number of ſimilar and equal planes, alike ſituated, viz. let the plane AC be ſimilar and equal to the plane KM, the plane AF to KP; BG to LQ; GD to QN; DE to NO; and laſtly, FH ſimilar and equal to PR: The ſolid figure AG is equal and ſimilar to the ſolid figure KQ.

Becauſe the ſolid angle at A is contained by the three plane angles BAD, BAE, EAD, which, by the hypotheſis, are equal to the plane angles LKN, LKO, OKN, which contain the ſolid angle at K, each to each; therefore the ſolid angle at A is equal [a] to the ſolid angle at K: In the ſame manner, the other ſolid angles of the figures are equal to one another. If, then, the ſolid figure AG be applied to the ſolid figure KQ, firſt, the plane figure AC being applied to the plane figure KM; the ſtraight line AB coinciding with KL, the figure AC muſt coincide with the figure KM, becauſe they are equal and ſimilar: Therefore the ſtraight lines AD, DC, CB coincide with KN, NM, ML, each with each; and the points A, D, C, B, with the points K, N, M, L: And the ſolid angle at A coincides with [a] the ſolid angle at K; wherefore the plane AF coincides with the plane KP, and the figure AF with the figure KP, becauſe they are equal and ſimilar to one another: Therefore the ſtraight lines AE, EF, FB, coincide with KO, OP, PL; and the points E, F, with the points O, P. In the ſame manner, the figure AH coincides with the figure KR, and the ſtraight line DH with NR, and the point H with the point R: And becauſe the ſolid angle at B is equal to the ſolid angle at L, it may be proved, in the ſame manner, that the figure BG coincides with the

a B. 11.



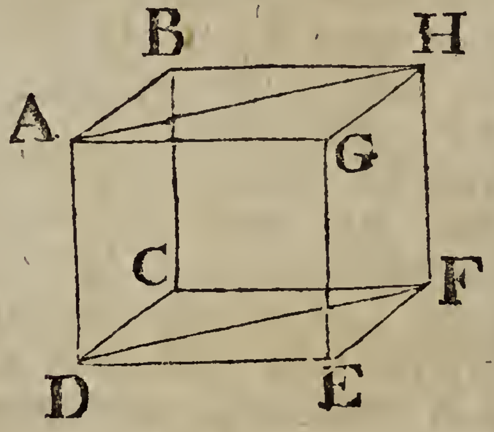
the figure LQ, and the ſtraight line CG with MQ, and the point G with the point Q: Since, therefore all the planes and ſides of the ſolid figure AG coincide with the planes and ſides of the ſolid figure KQ, AG is equal and ſimilar to KQ: And, in the ſame manner, any other ſolid figures whatever contained by the ſame number of equal and ſimilar planes, alike ſituated, and having none of their ſolid angles contained by more than three plane angles, may be proved to be equal and ſimilar to one another. Q. E. D.

PROP. XXIV. THEOR.

See N. IF a ſolid be contained by ſix planes, two and two of which are parallel; the oppoſite planes are ſimilar and equal parallelograms.

Let the ſolid CDGH be contained by the parallel planes AC, GF; BG, CE; FB, AE: Its oppoſite planes are ſimilar and equal parallelograms.

Becauſe the two parallel planes BG, CE, are cut by the
a 16. 11. plane AC, their common ſections AB, CD, are parallel [a]. Again, becauſe the two parallel planes BF, AE, are cut by the plane AC, their common ſections AD, BC, are parallel [a]: And AB is parallel to CD; therefore AC is a parallelogram. In like manner, it may be proved that each of the figures CE, FG, GB, BF, AE is a parallelogram: Join AH, DF; and becauſe AB is parallel to DC, and BH to CF; the two ſtraight lines AB, BH, which meet one another, are parallel to DC and CF which meet one another, and are not in the ſame plane with the other two; wherefore they con-
b 10. 11. tain equal angles [b]; the angle ABH is therefore equal to the angle DCF: And becauſe AB, BH, are equal to DC, CF, and the angle ABH equal to the angle DCF; therefore the baſe
c 4. 1. AH is equal [c] to the baſe DF, and the triangle ABH to the tri-
d 34. 1. angle DCF: And the parallelogram BG is double [d] of the triangle ABH, and the parallelogram CE double of the triangle DCF; therefore the paralellogram BG is equal and ſimilar to the parallelogram CE. In the ſame manner it may be proved, that the parallelogram AC is equal and ſimilar



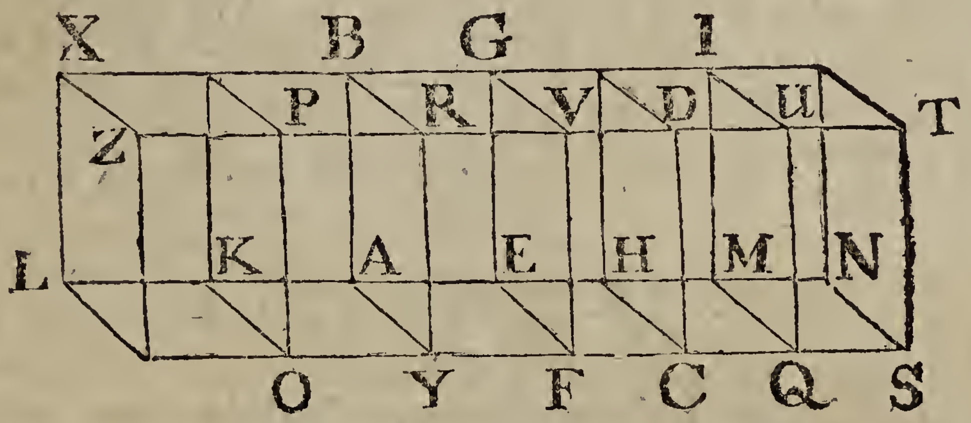
milar to the parallelogram GF, and the parallelogram AE to BF. Therefore, if a ſolid, &c. Q. E. D. Book XI.

PROP. XXV. THEOR.

IF a ſolid parallelepiped be cut by a plane parallel to two of its oppoſite planes; it divides the whole into two ſolids, the baſe of one of which ſhall be to the baſe of the other, as the one ſolid is to the other. See N.

Let the ſolid parallelepiped ABCD be cut by the plane EV, which is parallel to the oppoſite planes AR, HD, and divides the whole into the two ſolids ABFV, EGCD; as the baſe AEFY of the firſt is to the baſe EHCF of the other, ſo is the ſolid ABFV to the ſolid EGCD.

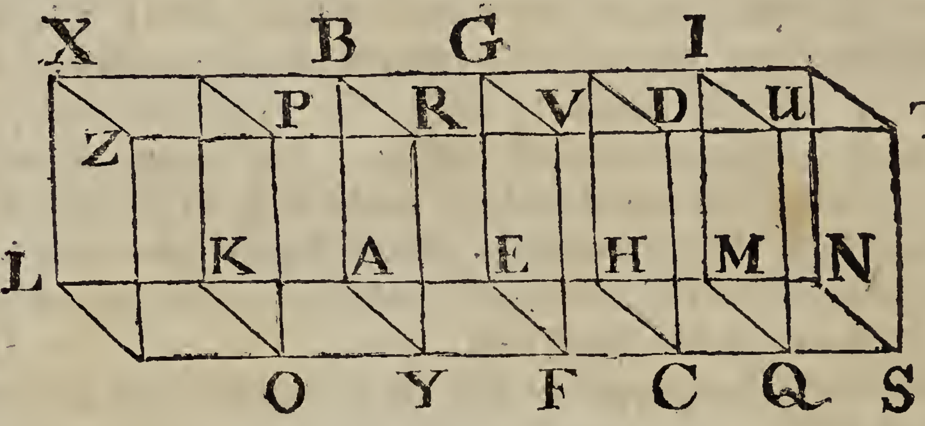
Produce AH both ways, and take any number of ſtraight lines HM, MN, each equal to EH, and any number AK, KL each equal to EA, and complete the parallelograms LO, KY, HQ, MS, and the ſolids LP, KR, HU, MT: Then, becauſe the ſtraight lines LK, KA, AE are all equal, the parallelograms



LO, KY, AF are equal [a]: And likewiſe the parallelograms KX, KB, AG [a]; as alſo [b] the parallelograms LZ, KP, AR, becauſe they are oppoſite planes: For the ſame reaſon, the parallelograms EC, HQ, MS, are equal [a]; and the parallelograms HG, HI, IN, as alſo [b] HD, MU, NT: Therefore three planes of the ſolid LP, are equal and ſimilar to three planes of the ſolid KR, as alſo to three planes of the ſolid AV: But the three planes oppoſite to theſe three are equal and ſimilar [b] to them in the ſeveral ſolids, and none of their ſolid angles are contained by more than three plane angles: Therefore the three ſolids LP, KR, AV are equal [c] to one another: For the ſame reaſon, the three ſolids ED, HU, MT are equal to one another: There-

a 36. 1.
b 24. 11.
c C. 11.

fore what multiple ſoever the baſe LF is of the baſe AF, the ſame multiple is the ſolid LV of the ſolid AV: For the ſame reaſon whatever multiple the baſe NF is of the baſe HF, the ſame multiple is the ſolid NV of the ſolid ED: And if the baſe
c C. 11. LF be equal to the baſe NF, the ſolid LV is equal[c] to the ſolid NV; and if the baſe LF be greater than the baſe NF, the ſolid LV is greater than the ſolid NV; and if leſs, leſs: Since then there are four magnitudes, viz. the two baſes AF, FH,



and the two ſolids AV, ED, and of the baſe AF and ſolid AV, the baſe LF and ſolid LV are any equimultiples whatever; and of the baſe FH and ſolid ED, the baſe FN and ſolid NV are any equimultiples whatever; and it has been proved, that if the baſe LF is greater than the baſe FN, the ſolid LV is greater than the ſolid NV; and if equal, equal; and if leſs, leſs.
d 5. def. 5. Therefore[d] as the baſe AF is to the baſe FH, ſo is the ſolid AV to the ſolid ED. Wherefore, if a ſolid, &c. Q. E. D.

PROP. XXVI. PROB.

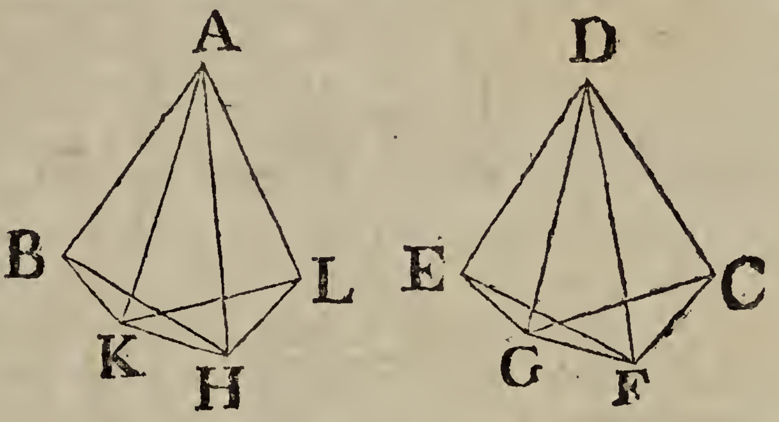
See N. AT a given point in a given ſtraight line, to make a ſolid angle equal to a given ſolid angle contained by three plane angles.

Let AB be a given ſtraight line, A a given point in it, and D a given ſolid angle contained by the three plane angles EDC, EDF, FDC: It is required to make at the point A in the ſtraight line AB a ſolid angle equal to the ſolid angle D.

In the ſtraight line DF take any point F, from which draw
a 11. 11. a[a] FG perpendicular to the plane EDC, meeting that plane in G; join DG, and at the point A in the ſtraight line AB
b 23. 1. make[b] the angle BAL equal to the angle EDC, and in the plane BAL make the angle BAK equal to the angle EDG;
c 12. 11 then make AK equal to DG, and from the point K erect[c] KH at

at right angles to the plane BAL; and make KH equal to GF, and join AH: Then the ſolid angle at A, which is contained by the three plane angles BAL, BAH, HAL, is equal to the ſolid angle at D contained by the three plane angles EDC, EDF, FDC.

Take the equal ſtraight lines AB, DE, and join HB, KB, FE, GE: And becauſe FG is perpendicular to the plane EDC, it makes right angles [d] with every ſtraight line meeting it in d 3. def. 11. that plane: Therefore each of the angles FGD, FGE, is a right angle: For the ſame reaſon, HKA, HKB are right angles: And becauſe KA, AB are equal to GD, DE, each to each, and contain equal angles, therefore the baſe BK is equal [e] to the e 4. 1. baſe EG: And KH is equal to GF, and HKB, FGE, are right angles, therefore HB is equal to FE: Again, becauſe AK, KH are equal to DG, GF, and contain right angles, the baſe AH is equal to the baſe DF; and AB is equal to DE: therefore HA, AB are equal to FD, DE, and the baſe HB is equal to the baſe FE, therefore the angle BAH is equal [f] to f 8. 1. the angle EDF: For the ſame reaſon, the angle HAL is equal to the angle FDC. Becauſe if AL and DC be made equal, and KL, HL, GC, FC be joined, ſince the whole angle BAL is equal to the whole EDC, and the parts of them BAK, EDG are, by the conſtruction, equal; therefore the remaining angle KAL is equal to the remaining angle GDC: And becauſe KA, AL are equal to GD, DC, and contain equal angles, the baſe KL is equal [e] to the baſe GC: And KH is equal to GF, ſo that LK, KH are equal to CG, GF, and they contain right angles; therefore the baſe HL is equal to the baſe FC: Again, becauſe HA, AL are equal to FD, DC, and the baſe HL to the baſe FC, the angle HAL is equal [f] to the angle FDC: Therefore, becauſe the three plane angles BAL, BAH, HAL, which contain the ſolid angle at A, are equal to the three plane angles EDC, EDF, FDC, which contain the ſolid angle at D, each to each, and are ſituated in the ſame order, the ſolid angle at g B. 11. A is equal [g] to the ſolid angle at D. Therefore, at a given point in a given ſtraight line, a ſolid angle has been made equal to a given ſolid angle contained by three plain angles. Which was to be done.

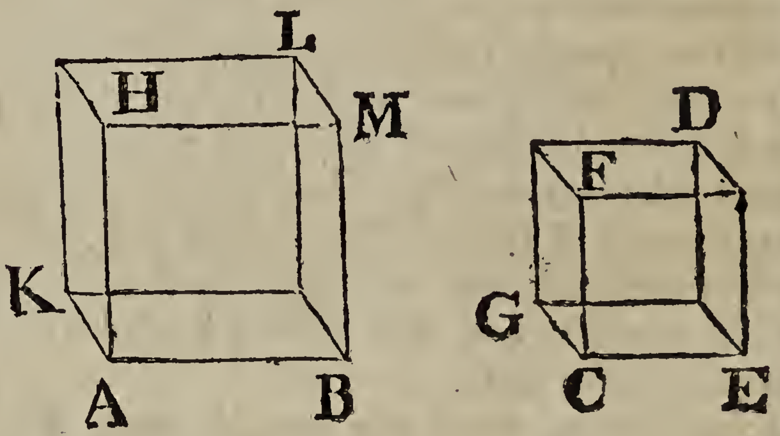


PROP. XXVI. PROB.

TO deſcribe from a given ſtraight line a ſolid parallelepiped ſimilar, and ſimilarly ſituated to one given.

Let AB be the given ſtraight line, and CD the given ſolid parallelepiped. It is required from AB to deſcribe a ſolid parallelepiped ſimilar and ſimilarly ſituated to CD.

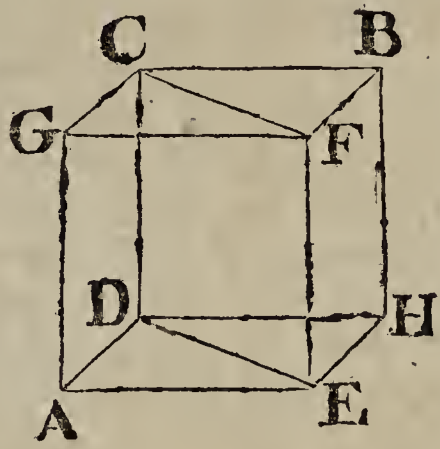
a 26. 11. At the point A of the given ſtraight line AB, make [a] a ſolid angle equal to the ſolid angle at C, and let BAK, KAH, HAB be the three plane angles which contain it, ſo that BAK be equal to the angle ECG, and KAH to GCF, and HAB to
b 12. 6. FCE: And as EC to CG, ſo make [b] BA to AK; and as GC to
c 22. 5. CF, ſo make [b] KA to AH; wherefore, ex æquali [c], as EC to CF, ſo is BA to AH: Complete the parallelogram BH, and the ſolid AL: And becauſe, as EC to CG, ſo BA to AK, the ſides about the equal angles ECG, BAK are proportionals; therefore the parallelogram BK is ſimilar to EG. For the ſame reaſon, the parallelogram KH is ſimilar to GF, and HB to FE. Wherefore three parallelograms of the ſolid AL are ſimilar to three of the ſolid
d 24. 11. CD; and the three oppoſite ones in each ſolid are equal [d] and ſimilar to theſe, each to each. Alſo, becauſe the plane angles which contain the ſolid angles of the figures are equal, each to each, and ſituated in the ſame order, the ſolid angles are e-
e B. 11. qual [e] each to each. Therefore the ſolid AL is ſimilar [f] to the
f 11. def. 11. ſolid CD. Wherefore from a given ſtraight line AB a ſolid parallelepiped AL has been deſcribed ſimilar, and ſimilarly ſituated to the given one CD. Which was to be done.



PROP. XXVIII. THEOR.

IF a ſolid parallelepiped be cut by a plane paſſing through the diagonals of two of the oppoſite planes; it ſhall be cut in two equal parts. See N.

Let AB be a ſolid parallelepiped, and DE, CF the diagonals of the oppoſite parallelograms AH, GB, viz. thoſe which are drawn betwixt the equal angles in each: And becauſe CD, FE are each of them parallel to GA, and not in the ſame plane with it, CD, FE are parallel[a]; wherefore the diagonals CF, a 9. 11. DE are in the plane in which the parallels are, and are themſelves parallels[b]: b 16. 11. And the plane CDEF ſhall cut the ſolid AB into two equal parts.



Becauſe the triangle CGF is equal[c] c 34. 1. to the triangle CBF, and the triangle DAE to DHE; and that the parallelogram CA is equal[d] d 24. 11. and ſimilar to the oppoſite one BE; and the parallelogram GE to CH: Therefore the priſm contained by the two triangles CGF, DAE, and the three parallelograms CA, GE, EC, is equal[e] e C. 11. to the priſm contained by the two triangles CBF, DHE, and the three parallelograms BE, CH, EC; becauſe they are contained by the ſame number of equal and ſimilar planes, alike ſituated, and none of their ſolid angles are contained by more than three plane angles. Therefore the ſolid AB is cut into two equal parts by the plane CDEF. Q. E. D.

‘ N. B. The inſiſting ſtraight lines of a parallelepiped, mentioned in the next and ſome following propoſitions, are the ſides of the parallelograms betwixt the baſe and the oppoſite plane parallel to it.’

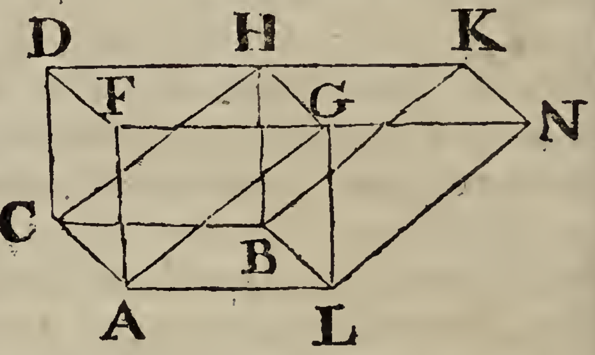
PROP. XXIX. THEOR.

SOLID parallelepipeds upon the ſame baſe, and of the ſame altitude, the inſiſting ſtraight lines of which are terminated in the ſame ſtraight lines of the plane oppoſite to the baſe, are equal to one another. See N.

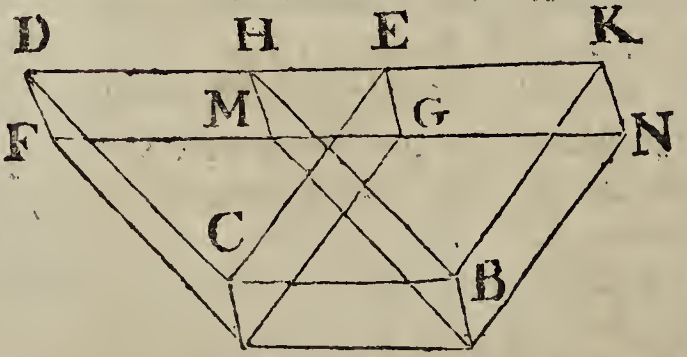
See the figures below.

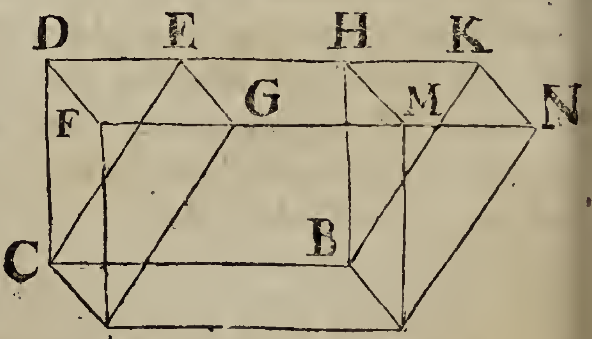
Let the ſolid parallelepipeds AH, AK be upon the ſame baſe AB, and of the ſame altitude, and let their inſiſting ſtraight lines AF, AG, LM, LN, be terminated in the ſame ſtraight line FN, and CD, CE, BH, BK be terminated in the ſame ſtraight line DK; the ſolid AH is equal to the ſolid AK.

Firſt, let the parallelograms DG, HN, which are oppoſite to the baſe AB, have a common ſide HG: Then, becauſe the ſolid AH is cut by the plane AGHC paſſing through the diagonals AG, CH of the oppoſite planes ALGF, CBHD, AH is a 28. 11. cut into two equal parts [a] by the plane AGHC: Therefore the ſolid AH is double of the priſm which is contained betwixt the triangles ALG, CBH: For the ſame reaſon, becauſe the ſolid AK is cut by the plane LGHB through the diagonals LG, BH of the oppoſite planes ALNG, CBKH, the ſolid AK is double of the ſame priſm which is contained betwixt the triangles ALG, CBH. Therefore the ſolid AH is equal to the ſolid AK.



But, let the parallelograms DM, EN oppoſite to the baſe, have no common ſide: Then, becauſe CH, CK are parallelograms, b 34. 1. CB is equal [b] to each of the oppoſite ſides DH, EK; wherefore DH is equal to EK: Add, or take away the common part HE; then DE is equal to HK: Wherefore alſo the tri- c 38. 1. angle CDE is equal [c] to the triangle BHK: And the parallelo- d 36. 1. gram DG is equal [d] to the parallelogram HN: For the ſame reaſon, the triangle AFG is equal to the triangle LMN, and e 24. 11. the parallelogram CF is equal [e] to the parallelogram BM, and CG to BN; for they are oppoſite. Therefore the priſm which is contained by the two triangles AFG, CDE, and the three f C. 11. parallelograms AD, DG, GC is equal [f] to the priſm, contained by the two triangles LMN, BHK, and the three parallelograms BM, MK, KL. If therefore the priſm LMNBHK be taken





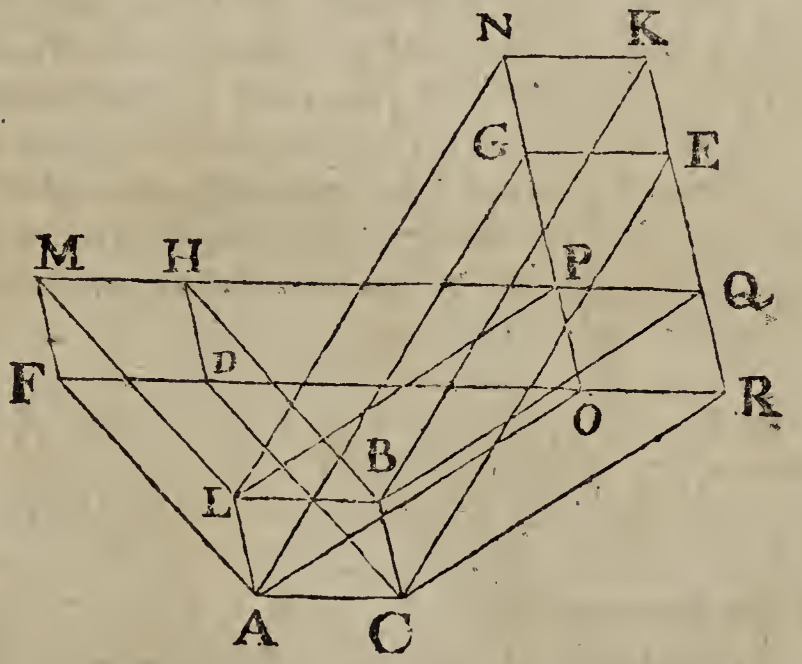
taken from the ſolid of which the baſe is the parallelogram AB, and in which FDKN is the one oppoſite to it; and if from this ſame ſolid there be taken the priſm AFGCDE; the remaining ſolid, viz. the parallelepiped AH, is equal to the remaining parallelepiped AK. Therefore ſolid parallelepipeds, &c. Q. E. D.

PROP. XXX. THEOR.

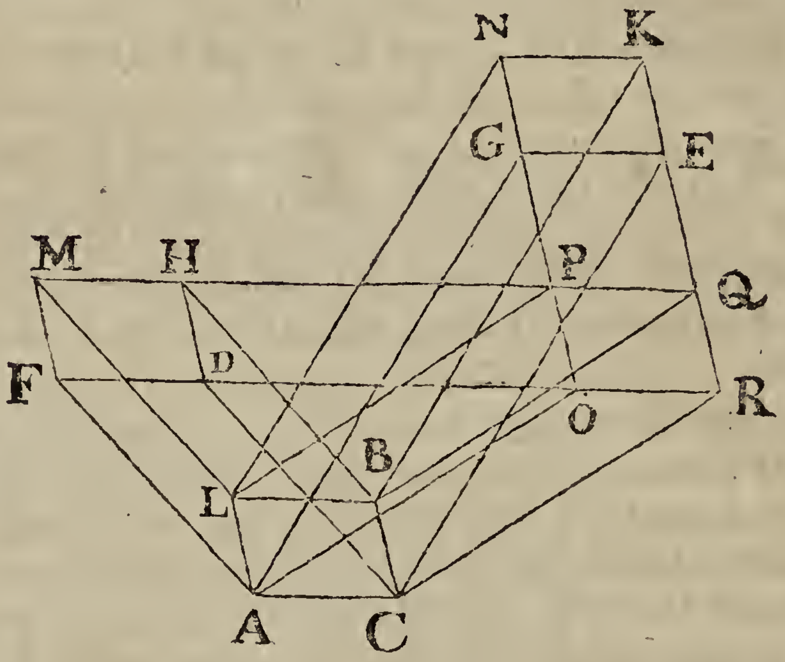
SOLID parallelepipeds upon the ſame baſe, and of the ſame altitude, the inſiſting ſtraight lines of which are not terminated in the ſame ſtraight lines in the plane oppoſite to the baſe, are equal to one another. See N.

Let the parallelepipeds CM, CN, be upon the ſame baſe AB, and of the ſame altitude, but their inſiſting ſtraight lines AF, AG, LM, LN, CD, CE, BH, BK, not terminated in the ſame ſtraight lines: The ſolids CM, CN are equal to one another.

Produce FD, MH, and NG, KE, and let them meet one another in the points O, P, Q, R; and join AO, LP, BQ, CR: And becauſe the plane LBHM is parallel to the oppoſite plane ACDF, and that the plane LBHM is that in which are the parallels LB, MHPQ, in which alſo is the figure BLPQ; and the plane ACDF is that in which are the parallels AC, FDOR, in which alſo is the figure CAOR; therefore the figures BLPQ, CAOR are in parallel planes: In like manner, becauſe the plane ALNG is parallel to the oppoſite plane CBKE, and that the plane ALNG is that in which are the parallels



Book XI. AL, OPGN, in which alſo is the figure ALPO; and the plane CBKE is that in which are the parallels CB, RQEK, in which alſo is the figure CBQR; therefore the figures ALPO, CBQR are in parallel planes: and the planes ACBL, ORQP are parallel; therefore the ſolid CP is a parallelepiped: But the ſolid CM, of which the baſe is ACBL, to which FDHM is the a 29. 11. oppoſite parallelogram, is equal [a] to the ſolid CP, of which the



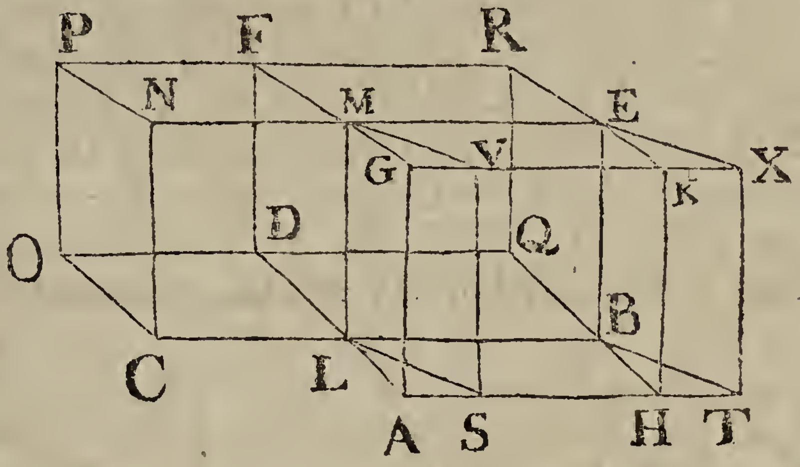
baſe is the parallelogram ACBL, to which ORQP is the one oppoſite; becauſe they are upon the ſame baſe, and their inſiſting ſtraight lines AF, AO, CD, CR; LM, LP, BH, BQ are in the ſame ſtraight lines FR, MQ: And the ſolid CP is equal [a] to the ſolid CN; for they are upon the ſame baſe ACBL, and their inſiſting ſtraight lines AO, AG, LP, LN; CR, CE, BQ, BK are in the ſame ſtraight lines ON, RK: Therefore the ſolid CM is equal to the ſolid CN. Wherefore ſolid parallelepipeds, &c. Q. E. D.

PROP. XXXI. THEOR.

See N. SOLID parallelepipeds which are upon equal baſes, and of the ſame altitude, are equal to one another.

Let the ſolid parallelepipeds AE, CF, be upon equal baſes AB, CD, and be of the ſame altitude; the ſolid AE is equal to the ſolid CF.

Firſt, Let the inſiſting ſtraight lines be at right angles to the baſes AB, CD, and let the baſes be placed in the ſame plane, and ſo as that the ſides CL, LB be in a ſtraight line; therefore the ſtraight line LM, which is at right angles to the plane in which the baſes are, in the point L, is common[a] to the two ſolids AE, CF; let the other inſiſting lines of the ſolids be AG, HK, BE; DF, OP, CN: And firſt, let the angle ALB be equal to the angle CLD; then AL, LD are in a ſtraight line[b]. Produce OD, HB, and let them meet in Q, and complete the ſolid parallelepiped LR, the baſe of which is the parallelogram LQ, and of which LM is one of its inſiſting ſtraight lines: Therefore, becauſe the parallelogram AB is equal to CD, as the baſe AB is to the baſe LQ, ſo is[c] the baſe CD to the ſame LQ: And becauſe the ſolid parallelepiped AR is cut by the plane LMEB, which is parallel to the oppoſite planes AK, DR; as the baſe AB is to the baſe LQ, ſo is[d] the ſolid AE to the ſolid LR: For the ſame reaſon, becauſe the ſolid parallelepiped CR is cut by the plane LMFD, which is parallel to the oppoſite planes CP, BR; as the baſe CD to the baſe LQ, ſo is the ſolid CF to the ſolid LR: But as the baſe AB to the baſe LQ, ſo the baſe CD to the baſe LQ, as before was proved: Therefore as the ſolid AE to the ſolid LR, ſo is the ſolid CF to the ſolid LR; and therefore the ſolid AE is equal[e] to the ſolid CF.

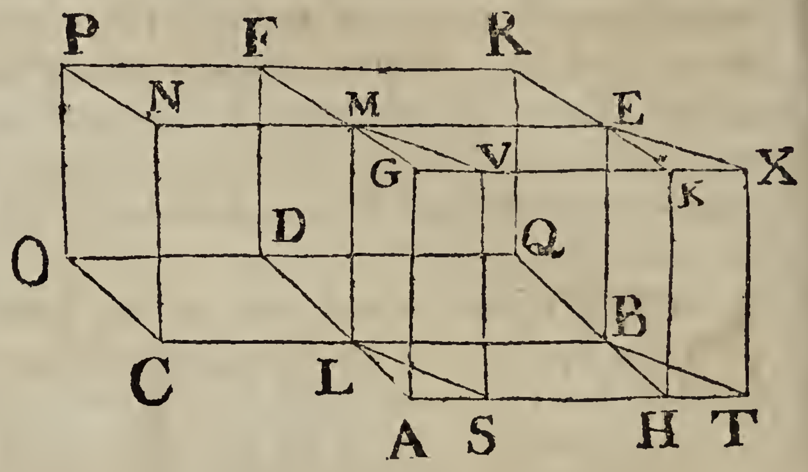


a 13. 11.
b 14. 1.
c 7. 5.
d 25. 11.
e 9. 5.

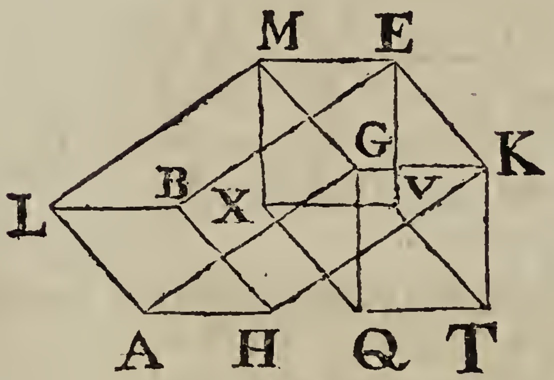
But let the ſolid parallelepipeds SE, CF be upon equal baſes SB, CD, and be of the ſame altitude, and let their inſiſting ſtraight lines be at right angles to the baſes; and place the baſes SB, CD in the ſame plane, ſo that CL, LB be in a ſtraight line; and let the angles SLB, CLD be unequal; the ſolid SE is alſo in this caſe equal to the ſolid CF: Produce DL, TS until they meet in A, and from B draw BH parallel to DA; and let HB, OD produced meet in Q, and complete the ſolids AE, LR: Therefore the ſolid AE, of which the baſe is the parallelogram LE, and AK the one oppoſite to it, is equal[f] to the ſolid SE, of which the baſe is LE, and to which SX is oppoſite; for they are upon the ſame baſe LE, and of the ſame altitude, and their inſiſting ſtraight lines, viz. LA, LS, BH, BT; MG, MV, EK, EX are in the ſame ſtraight lines AT, GX: And be-

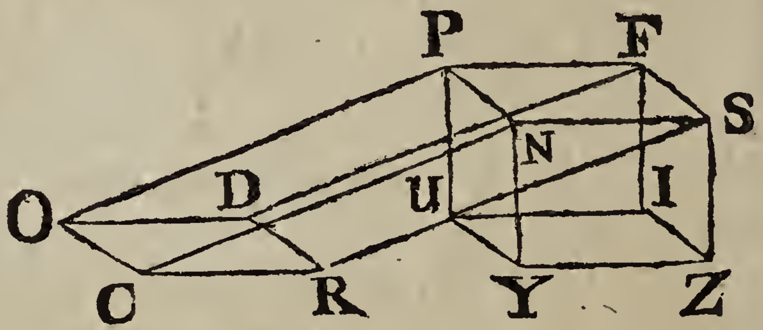
f 29. 11.

Book XI. cauſe the parallelogram AB is equal [g] to SB, for they are upon the ſame baſe LB, and between the ſame parallels LB, AT;
g 35. 1. and that the baſe SB is equal to the baſe CD; therefore the baſe AB is equal to the baſe CD, and the angle ALB is equal to the angle CLD: Therefore, by the firſt caſe, the ſolid AE is equal to the ſolid CF; but the ſolid AE is equal to the ſolid SE, as was demonſtrated; therefore the ſolid SE is equal to the ſolid CF.



But, if the inſiſting ſtraight lines AG, HK, BE, LM; CN, RS, DF, OP, be not at right angles to the baſes AB, CD; in this caſe likewiſe the ſolid AE is equal to the ſolid CF: From the points G, K, E, M; N, S, F, P, draw the ſtraight lines
h 11. 11. GQ, KT, EV, MX; NY, SZ, FI, PU, perpendicular [h] to the plane in which are the baſes AB, CD; and let them meet it in the points Q, T, V, X; Y, Z, I, U, and join QT, TV, VX, XQ; YZ, ZI, IU, UY: Then, becauſe GQ, KT, are at right





i 6. 11. angles to the ſame plane, they are parallel [i] to one another: And MG, EK are parallels; therefore the plane MQ, ET, of which one paſſes through MG, GQ, and the other through EK, KT which are parallel to MG, GQ, and not in the ſame
k 15. 11. plane with them, are parallel [k] to one another: For the ſame reaſon, the planes MV, GT are parallel to one another: Therefore the ſolid QE is a parallelepiped: In like manner, it may be proved, that the ſolid YF is a parallelepiped: But, from what has been demonſtrated, the ſolid EQ is equal to the ſolid FY, becauſe they are upon equal baſes MK, PS, and of the ſame altitude, and have their inſiſting ſtraight lines at right angles

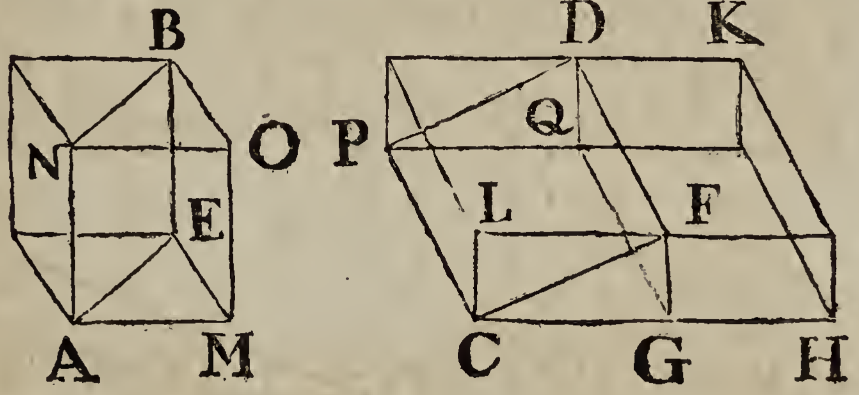
to the baſes: And the ſolid EQ is equal [1] to the ſolid AE; and the ſolid FY to the ſolid CF; becauſe they are upon the ſame baſes and of the ſame altitude: Therefore the ſolid AE is equal to the ſolid CF: Wherefore ſolid parallelepipeds, &c. Q. E. D. [1] 29. or 30. 11.

PROP. XXXII. THEOR.

SOLID parallelepipeds which have the ſame altitude, are to one another as their baſes. See N.

Let AB, CD be ſolid parallelepipeds of the ſame altitude: They are to one another as their baſes; that is, as the baſe AE to the baſe CF, ſo is the ſolid AB to the ſolid CD.

To the ſtraight line FG apply the parallelogram FH equal [a] to AE, ſo that the angle FGH be equal to the angle LCG; and complete the ſolid parallelepiped GK upon the baſe FH, one of whoſe inſiſting lines is FD, whereby the ſolids CD, GK muſt be of the ſame altitude: Therefore the ſolid AB is equal [b] to the ſolid GK, becauſe they are upon equal baſes AE, FH, and are of the ſame altitude: And becauſe the ſolid parallelepiped CK is cut by the plane DG which is parallel to its oppoſite planes, the baſe HF is [c] to the baſe FC, as the ſolid HD to the ſolid DC: But the baſe HF is equal to the baſe AE, and the ſolid GK to the ſolid AB: Therefore, as the baſe AE to the baſe CF, ſo is the ſolid AB to the ſolid CD. Wherefore ſolid parallelepipeds, &c. Q. E. D.



[a] Cor. 45. 1.

[b] 31. 11.

[c] 25. 11.

Cor. From this it is manifeſt that priſms upon triangular baſes, of the ſame altitude, are to one another as their baſes.

Let the priſms, the baſes of which are the triangles AEM, CFG, and NBO, PDQ the triangles oppoſite to them, have the ſame altitude; and complete the parallelograms AE, CF, and the ſolid parallelepipeds AB, CD, in the firſt of which let MO, and in the other let GQ be one of the inſiſting lines. And becauſe the ſolid parallelepipeds AB, CD have the ſame altitude, they are to one another as the baſe AE is to the baſe

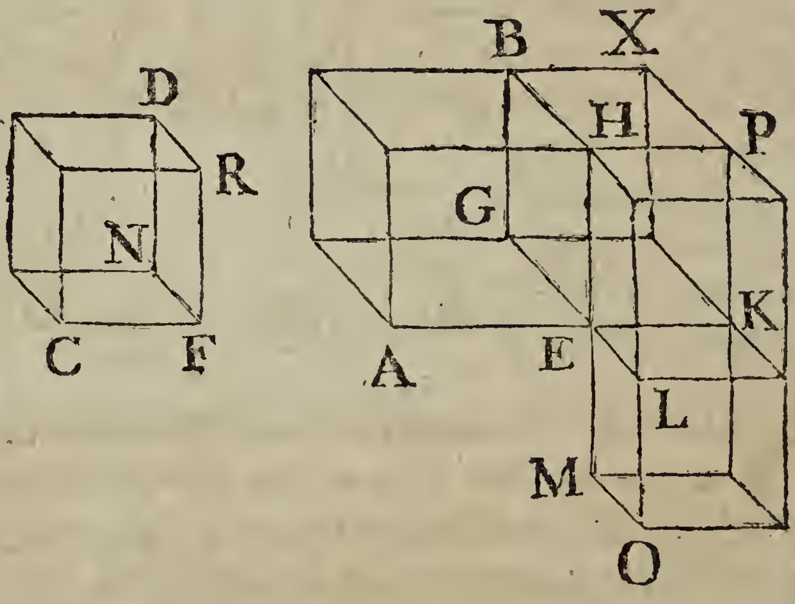
CF; wherefore the priſms, which are their halves [d] are to one another, as the baſe AE to the baſe CF; that is, as d 28. 11. the triangle AEM to the triangle CFG.

PROP. XXXIII. THEOR.

SIMILAR ſolid parallelepipeds are one to another in the triplicate ratio of their homologous ſides.

Let AB, CD be ſimilar ſolid parallelepipeds, and the ſide AE homologous to the ſide CF: The ſolid AB has to the ſolid CD, the triplicate ratio of that which AE has to CF.

Produce AE, GE, HE, and in theſe produced take EK equal to CF, EL equal to FN, and EM equal to FR; and complete the parallelogram KL, and the ſolid KO: Becauſe KE, EL are equal to CF, FN, and the angle KEL equal to the angle CFN, becauſe it is equal to the angle AEG which is equal to CFN, by reaſon that the ſolids AB, CD are ſimilar; therefore the parallelogram KL is ſimilar and equal to the parallelogram CN: For the ſame reaſon, the parallelogram MK is ſimilar and equal to CR, and alſo OE to FD. Therefore three parallelograms of the ſolid KO are equal and ſimilar to three parallelograms of the ſolid CD: And the three oppoſite ones in each ſolid a 24. 11. are equal [a] and ſimilar to theſe: Therefore the ſo- b C. 11. lid KO is equal [b] and ſimilar to the ſolid CD: Complete the parallelogram GK, and complete the ſolids EX, LP upon the baſes GK, KL, ſo that EH be an inſiſting ſtraight line in each of them, whereby they muſt be of the ſame altitude with the ſolid AB: And becauſe the ſolids AB, CD are ſimilar, and, by permutation, as AE is to CF, ſo is EG to FN, and ſo is EH to FR; and FC is equal to EK, and FN to EL, and FR to EM: Therefore, as AE to EK, ſo is EG to EL, and ſo is HE c 1. 6. to EM: But, as AE to EK, ſo [c] is the parallelogram AG to the parallelogram GK; and as GE to EL, ſo is [c] GK to KL and



and as HE to EM, ſo [c] is PE to KM: Therefore as the parallelogram AG to the parallogram GK, ſo is GK to KL, and PE to KM: But as AG, to GK, ſo [d] is the ſolid AB to the ſolid EX; and as GK to KL, ſo [d] is the ſolid EX to the ſolid PL; and as PE to KM, ſo [d] is the ſolid PL to the ſolid KO: And therefore as the ſolid AB to the ſolid EX, ſo is EX to PL, and PL to KO: But if four magnitudes be continual proportionals, the firſt is ſaid to have to the fourth the triplicate ratio of that which it has to the ſecond: Therefore the ſolid AB has to the ſolid KO, the triplicate ratio of that which AB has to EX: But as AB is to EX, ſo is the parallelogram AG to the parallelogram GK, and the ſtraight line AE to the ſtraight line EK. Wherefore the ſolid AB has to the ſolid KO, the triplicate ratio of that which AE has to EK. And the ſolid KO is equal to the ſolid CD, and the ſtraight line EK is equal to the ſtraight line CF. Therefore the ſolid AB has to the ſolid CD, the triplicate ratio of that which the ſide AE has to the homologous ſide CF, &c. Q. E. D.

[c] 1. 6.

[d] 25. 11.

Cor. From this it is manifeſt, that, if four ſtraight lines be continual proportionals, as the firſt is to the fourth, ſo is the ſolid parallelepiped deſcribed from the firſt to the ſimilar ſolid ſimilarly deſcribed from the ſecond; becauſe the firſt ſtraight line has to the fourth the triplicate ratio of that which it has to the ſecond.

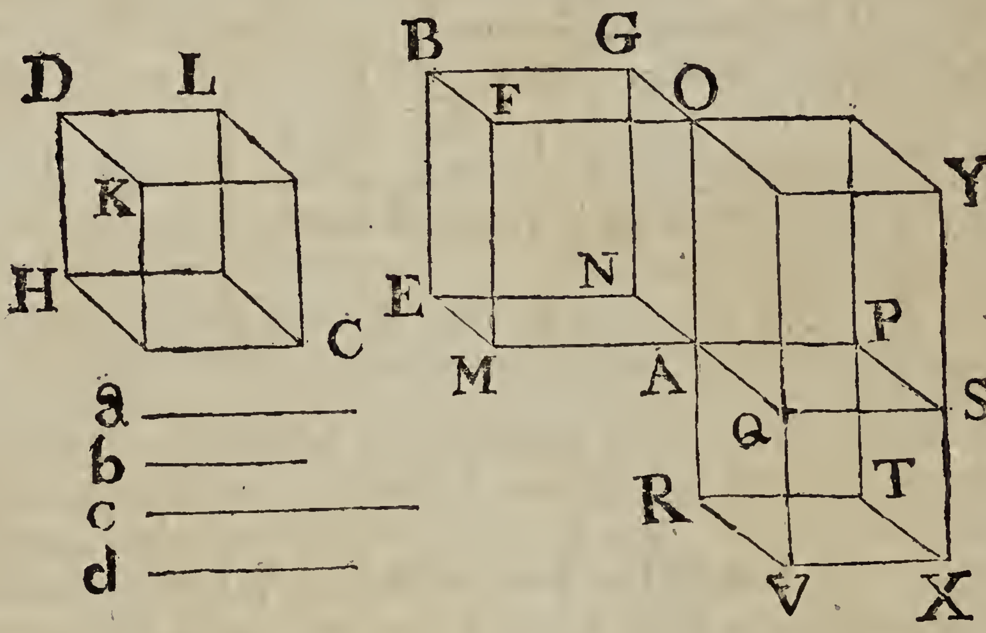
PROP. D. THEOR.

SOLID parallelepipeds contained by parallelograms equiangular to one another, each to each, that is, of which the ſolid angles are equal, each to each, have to one another the ratio which is the ſame with the ratio compounded of the ratios of their ſides.

See N.

Let AB, CD be ſolid parallelepipeds, of which AB is contained by the parallelograms AE, AF, AG equiangular, each to each, to the parallelograms CH, CK, CL, which contain the ſolid CD. The ratio which the ſolid AB has to the ſolid CD is the ſame with that which is compounded of the ratios of the ſides AM to DL, AN to DK, and AO to DH.

Produce MA, NA, OA to P, Q, R, ſo that AP be equal to DL, AQ to DK, and AR to DH; and complete the ſolid parallelepiped AX contained by the parallelograms AS, AT, AV ſimilar and equal to CH, CK, CL, each to each. There-
a C. 11. fore the ſolid AX is equal [a] to the ſolid CD. Complete likewiſe the ſolid AY, the baſe of which is AS, and of which AO is one of its inſiſting ſtraight lines. Take any ſtraight line a, and as MA to AP, ſo make a to b; and as NA to AQ, ſo make b to c; and as AO to AR, ſo c to d: Then, becauſe the parallelogram AE is equiangular to AS, AE is to AS, as the ſtraight line a to c, as is demonſtrated in the 23. Prop. Book 6. and the ſolids AB, AY, being betwixt the parallel planes BOY, EAS, are of the ſame altitude. Therefore the ſolid AB
b 32. 11. is to the ſolid AY, as [b] the baſe AE to the baſe AS; that is, as the ſtraight line a is to c. And the ſolid AY is to the ſolid



c 25. 11. AX, as [c] the baſe OQ is to the baſe QR; that is, as the ſtraight line OA to AR; that is, as the ſtraight line c to the ſtraight line d. And becauſe the ſolid AB is to the ſolid AY, as a is to c, and the ſolid AY to the ſolid AX, as c is to d; ex æquali, the ſolid AB is to the ſolid AX, or CD which is equal to it, as the ſtraight line a is to d. But the ratio of a to d is ſaid to
d def. A. 5. be compounded [d] of the ratios of a to b, b to c, and c to d, which are the ſame with the ratios of the ſides MA to AP, NA to AQ, and OA to AR, each to each. And the ſides AP, AQ, AR are equal to the ſides DL, DK, DH, each to each. Therefore the ſolid AB has to the ſolid CD the ratio which is the ſame with that which is compounded of the ratios of the ſides AM to DL, AN to DK, and AO to DH. Q. E. D.

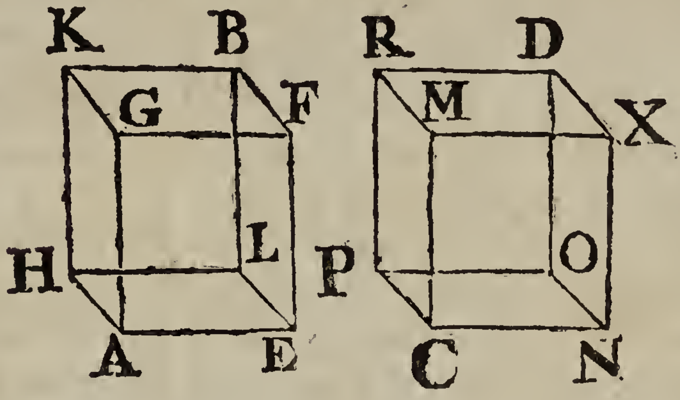
PROP.

PROP. XXXIV. THEOR.

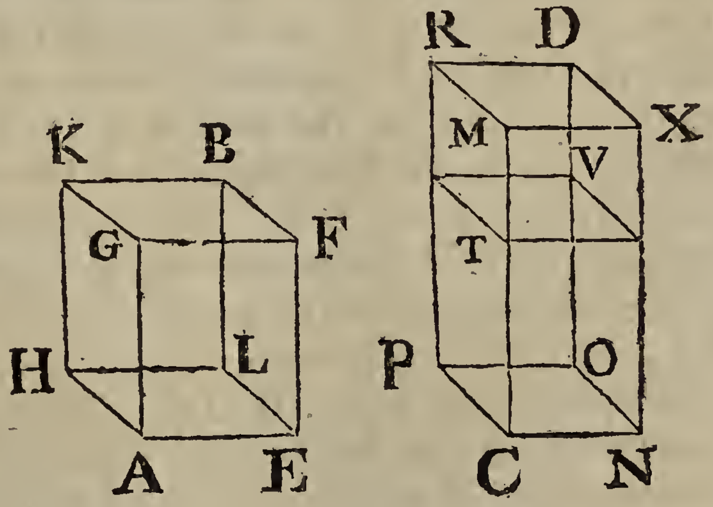
THE baſes and altitudes of equal ſolid parallelepipeds, are reciprocally proportional; and if the baſes and altitudes be reciprocally proportional, the ſolid parallelepipeds are equal. See N.

Let AB, CD be equal ſolid parallelepipeds; their baſes are reciprocally proportional to their altitudes; that is, as the baſe EH is to the baſe NP, ſo is the altitude of the ſolid CD to the altitude of the ſolid AB.

Firſt, Let the inſiſting ſtraight lines AG, EF, LB, HK; CM, NX, OD, PR be at right angles to the baſes. As the baſe EH to the baſe NP, ſo is CM to AG. If the baſe EH be equal to the baſe NP, then becauſe the ſolid AB is likewiſe equal to the ſolid CD, CM ſhall be equal to AG. Becauſe if the baſes EH, NP be equal, but the altitudes AG, CM be not equal, neither ſhall the ſolid AB be equal to the ſolid CD. But the ſolids are equal, by the hypotheſis. Therefore the altitude CM is not unequal to the altitude AG; that is, they are equal. Wherefore as the baſe EH to the baſe NP, ſo is CM to AG.



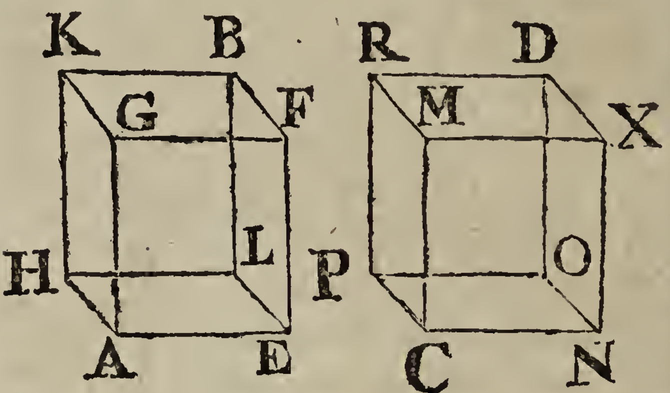
Next, Let the baſes EH, NP not be equal, but EH greater than the other: Since then the ſolid AB is equal to the ſolid CD, CM is therefore greater than AG: For, if it be not, neither alſo in this caſe, would the ſolids AB, CD be equal, which, by the hypotheſis, are equal. Make then CT equal to AG, and complete the ſolid parallelepiped CV of which the baſe is NP, and altitude CT. Becauſe the ſolid AB is equal to the ſolid CD, therefore the ſolid AB is to the ſolid



ſolid CV, as [a] the ſolid CD to the ſolid CV. But as the ſolid AB to the ſolid CV, ſo [b] is the baſe EH to the baſe NP; for the ſolids AB, CV are of the ſame altitude; and as the ſolid CD to CV, ſo [c] is the baſe MP to the baſe PT, and ſo [d] is the ſtraight line MC to CT; and CT is equal to AG. Therefore, as the baſe EH to the baſe NP, ſo is MC to AG. Wherefore, the baſes of the ſolid parallelepipeds AB, CD are reciprocally proportional to their altitudes.

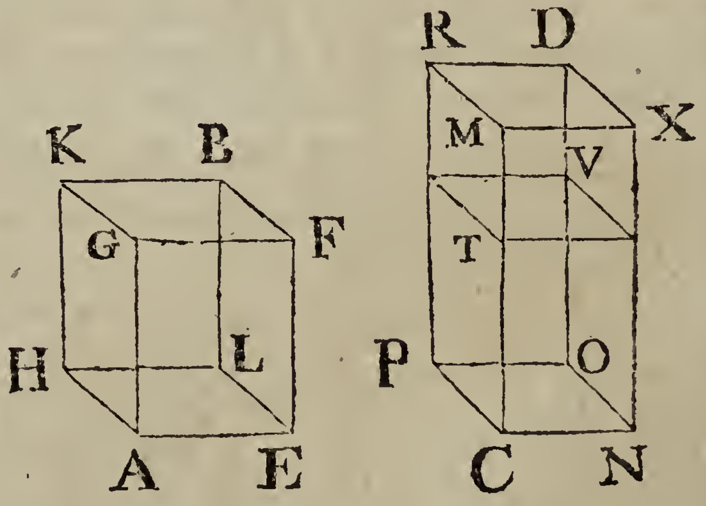
a 7. 5.
b 32. 11.
c 25. 11.
d 1. 6.

Let now the baſes of the ſolid parallelepipeds AB, CD be reciprocally proportional to their altitudes; viz. as the baſe EH to the baſe NP, ſo the altitude of the ſolid CD to the altitude of the ſolid AB; the ſolid AB is equal to the ſolid CD. Let the inſiſting lines be, as before, at right angles to the baſes. Then, if the baſe EH be equal to the baſe NP, ſince EH is to NP, as the altitude of the ſolid CD is to the altitude of the ſolid AB, therefore the altitude of CD is equal [e] to the altitude of AB. But ſolid parallelepipeds upon equal baſes, and of the ſame altitude, are equal [f] to one another; therefore the ſolid AB is equal to the ſolid CD.



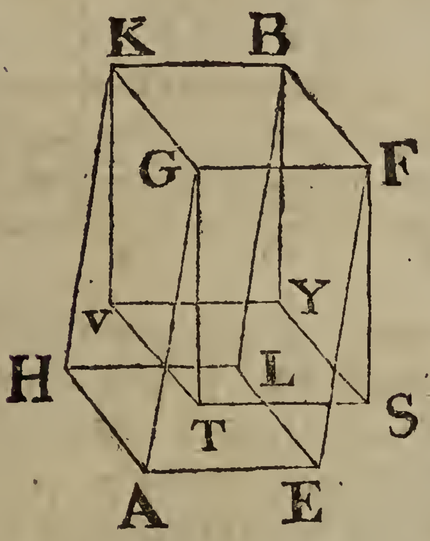
e A. 5.
f 31. 11.

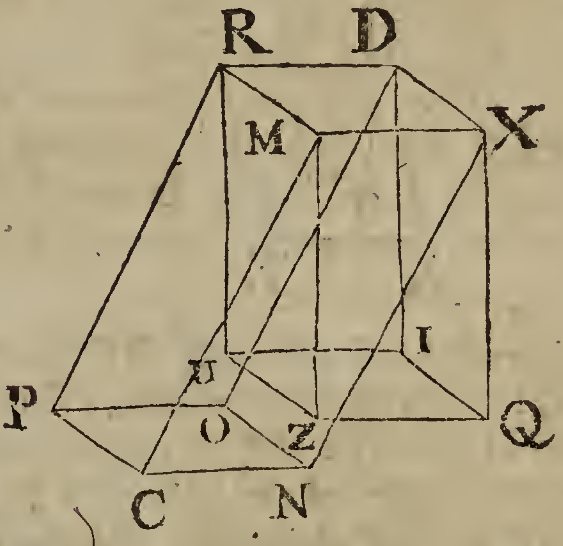
But let the baſes EH, NP be unequal, and let EH be the greater of the two. Therefore, ſince as the baſe EH to the baſe NP, ſo is CM the altitude of the ſolid CD to AG the altitude of AB, CM is greater [e] than AG. Again, Take CT equal to AG, and complete, as before, the ſolid CV. And, becauſe the baſe EH is to the baſe NP, as CM to AG, and that AG is equal to CT, therefore the baſe EH is to the baſe NP, as MC to CT. But as the baſe EH is to NP, ſo [b] is the ſolid AB to the ſolid CV; for the ſolids AB, CV are of the ſame altitude; and as MC to CT, ſo is the baſe MP to the baſe PT,



PT, and the ſolid CD to the ſolid [c] CV: And therefore as the ſolid AB to the ſolid CV. ſo is the ſolid CD to the ſolid CV; c 25. 11. that is, each of the ſolids AB, CD has the ſame ratio to the ſolid CV; and therefore the ſolid AB is equal to the ſolid CD.

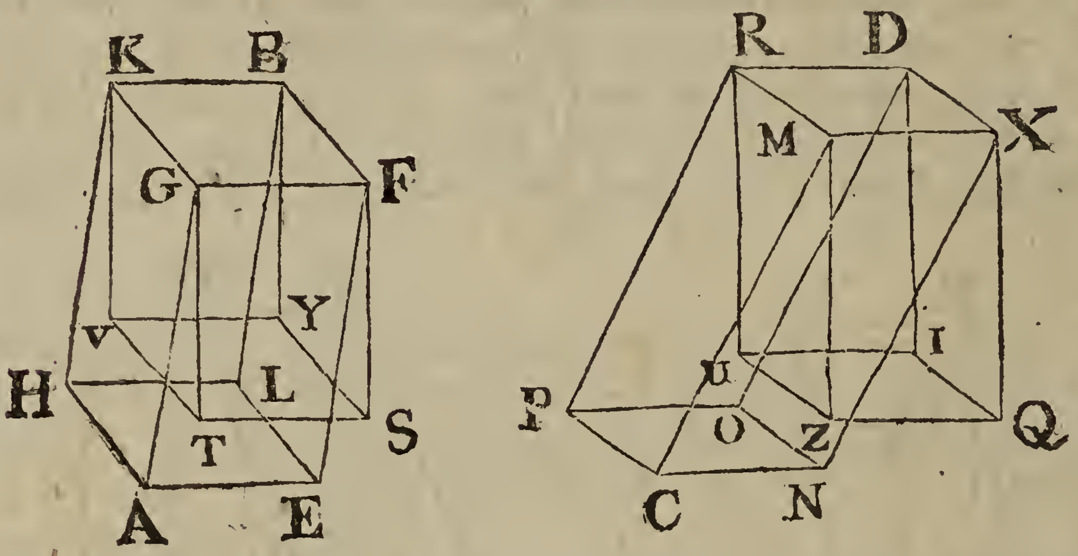
Second general caſe. Let the inſiſting ſtraight lines FE, BL, GA, KH; XN, DO, MC, RP not be at right angles to the baſes of the ſolids; and from the points F, B, K, G; X, D, R, M draw perpendiculars to the planes in which are the baſes EH, NP meeting thoſe planes in the points S, Y, V, T; Q, I, U, Z; and complete the ſolids FV, XU, which are parallelepipeds, as was proved in the laſt part of prop. 31. of this book. In this caſe, likewiſe, if the ſolids AB, CD be equal, their baſes are reciprocally proportional to their altitudes, viz. the baſe EH to the baſe NP, as the altitude of the ſolid CD to the altitude of the ſolid AB. Becauſe the ſolid AB is equal to the ſolid CD, and that the ſolid BT is equal [g] to the g 29. or 30. 11. ſolid BA, for they are upon the ſame baſe FK, and of the





ſame altitude; and that the ſolid DC is equal [g] to the ſolid DZ, being upon the ſame baſe XR, and of the ſame altitude; therefore the ſolid BT is equal to the ſolid DZ: But the baſes are reciprocally proportional to the altitudes of equal ſolid parallelepipeds of which the inſiſting ſtraight lines are at right angles to their baſes, as before was proved: Therefore as the baſe FK to the baſe XR, ſo is the altitude of the ſolid DZ to the altitude of the ſolid BT: And the baſe FK is equal to the baſe EH, and the baſe XR to the baſe NP: Wherefore, as the baſe EH to the baſe NP, ſo is the altitude of the ſolid DZ to the altitude of the ſolid BT: But the altitudes of the ſolids DZ, DC, as alſo of the ſolids BT, BA are the ſame. Therefore as the baſe EH to the baſe NP, ſo is the altitude of the ſolid CD to the altitude of the ſolid AB; that is, the baſes of the ſolid parallelepipeds AB, CD are reciprocally proportional to their altitudes.

Next, Let the baſes of the ſolids AB, CD be reciprocally proportional to their altitudes, viz. the baſe EH to the baſe NP, as the altitude of the ſolid CD to the altitude of the ſolid AB; the ſolid AB is equal to the ſolid CD: The ſame conſtruction being made; becauſe, as the baſe EH to the baſe NP, ſo is the altitude of the ſolid CD to the altitude of the ſolid AB; and that the baſe EH is equal to the baſe FK; and NP to XR; therefore the baſe FK is to the baſe XR, as the altitude of the ſolid CD to the altitude of AB: But the altitudes of the ſolids AB, BT are the ſame, as alſo of CD and DZ; therefore as the baſe FK to the baſe XR, ſo is the altitude of the ſolid DZ to the altitude of the ſolid BT: Wherefore the baſes of the ſolids BT, DZ are reciprocally proportional to their altitudes; and their inſiſting ſtraight lines are at right angles to the baſes; wherefore, as was before proved, the ſolid BT is equal to the ſolid DZ: But BT is equal g to the ſolid BA, and DZ to the ſolid DC, becauſe they are upon the ſame baſes, and of the ſame altitude. Therefore the ſolid AB is equal to the ſolid CD. Q. E. D.



g 29. or 30. 11.

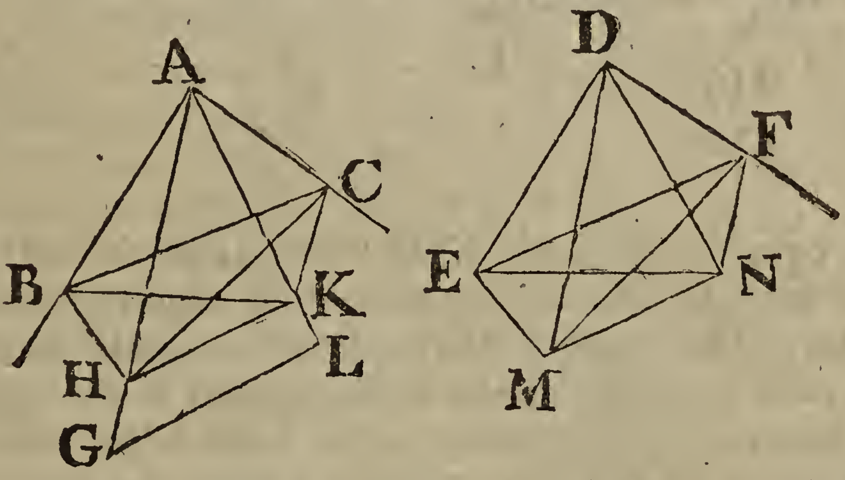
PROP.

PROP. XXXV. THEOR.

See N.

IF, from the vertices of two equal plane angles, there be drawn two ſtraight lines elevated above the planes in which the angles are, and containing equal angles with the ſides of thoſe angles, each to each; and if in the lines above the planes there be taken any points, and from them perpendiculars be drawn to the planes in which the firſt named angles are: And from the points in which they meet the planes, ſtraight lines be drawn to the vertices of the angles firſt named; theſe ſtraight lines ſhall contain equal angles with the ſtraight lines which are above the planes of the angles.

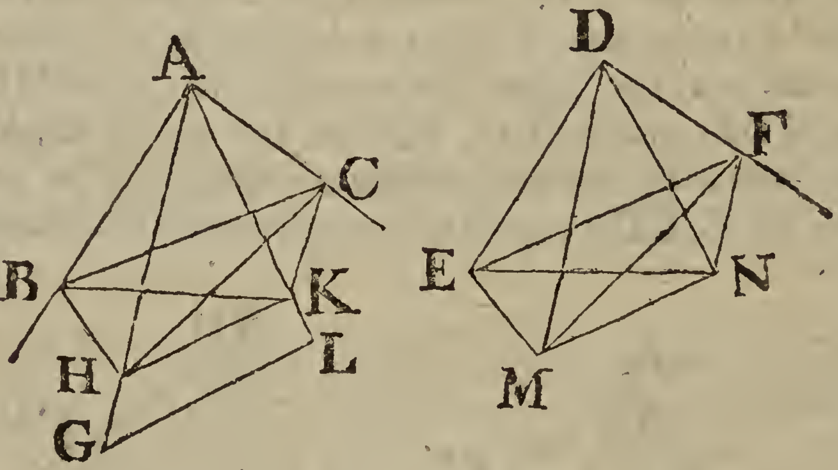
Let BAC, EDF be two equal plane angles; and from the points A, D let the ſtraight lines AG, DM be elevated above the planes of the angles, making equal angles with their ſides each to each, viz. the angle GAB equal to the angle MDE, and GAC to MDF; and in AG, DM let any points G, M be taken, and from them let perpendiculars GL, MN be drawn to the planes BAC, EDF meeting theſe planes in the points L, N; and join LA, ND: The angle GAL is equal to the angle MDN.



Make AH equal to DM, and through H draw HK parallel to GL: But GL is perpendicular to the plane BAC; wherefore HK is perpendicular [a] to the ſame plane: From the points K, N, to the ſtraight lines AB, AC, DE, DF, draw perpendiculars KB, KC, NE, NF; and join HB, BC, ME, EF:

a 8. 11.

Becauſe HK is perpendicular to the plane BAC, the plane b 18. 11. HBK which paſſes through HK is at right angles[b] to the plane BAC; and AB is drawn in the plane BAC at right angles to the common ſection BK of the two planes; therefore AB is c 4. def. 11. perpendicular[c] to the plane HBK, and makes right angles[d] d 3. def. 11. with every ſtraight line meeting it in that plane: But BH meets it in that plane; therefore ABH is a right angle: For the ſame reaſon, DEM is a right angle, and is therefore equal to the angle ABH: And the angle HAB is equal to the angle MDE. Therefore in the two triangles HAB, MDE there are two angles in one equal to two angles in the other, each to each, and one ſide equal to one ſide, oppoſite to one of the equal angles in each, viz. HA equal to DM; therefore the remaining ſides 26. 1. are equal[e], each to each: Wherefore AB is equal to DE. In the ſame manner, if HC and MF be joined, it may be demonſtrated that AC is equal to DF: Therefore, ſince AB is equal to DE, BA and AC are equal to ED and DF; and the angle



BAC is equal to the angle EDF; wherefore the baſe BC is e- f 4. 1. qual[f] to the baſe EF, and the remaining angles to the remaining angles: The angle ABC is therefore equal to the angle DEF: And the right angle ABK is equal to the right angle DEN, whence the remaining angle CBK is equal to the remaining angle FEN: For the ſame reaſon, the angle BCK is equal to the angle EFN: Therefore in the two triangles BCK, EFN, there are two angles in one equal to two angles in the other, each to each, and one ſide equal to one ſide adjacent to the equal angles in each, viz. BC equal to EF; the other ſides, therefore, are equal to the other ſides; BK then is equal to EN: And AB is equal to DE; wherefore AB, BK are equal to DE, EN; and they contain right angles; wherefore the baſe AK is equal to the baſe DN: And ſince AH is equal to

DM, the ſquare of AH is equal to the ſquare of DM: But the ſquares of AK, KH are equal to the ſquare [g] of AH, becauſe AKH is a right angle: And the ſquares of DN, NM are equal to the ſquare of DM, for DNM is a right angle: Wherefore the ſquares of AK, KH are equal to the ſquares of DN, NM; and of thoſe the ſquare of AK is equal to the ſquare of DN: Therefore the remaining ſquare of KH is equal to the remaining ſquare of NM; and the ſtraight line KH to the ſtraight line NM: And becauſe HA, AK are equal to MD, DN each to each, and the baſe HK to the baſe MN as has been proved; therefore the angle HAK is equal [h] to the angle MDN. Q. E. D.

Cor. From this it is manifeſt, that if, from the vertices of two equal plane angles, there be elevated two equal ſtraight lines containing equal angles with the ſides of the angles, each to each; the perpendiculars drawn from the extremities of the equal ſtraight lines to the planes of the firſt angles are equal to one another.

Another Demonſtration of the Corollary.

Let the plane angles BAC, EDF be equal to one another, and let AH, DM be two equal ſtraight lines above the planes of the angles, containing equal angles with BA, AC; ED, DF, each to each, viz. the angle HAB equal to MDE, and HAC equal to the angle MDF; and from H, M let HK, MN be perpendiculars to the planes BAC, EDF: HK is equal to MN.

Becauſe the ſolid angle at A is contained by the three plane angles BAC, BAH, HAC, which are, each to each, equal to the three plane angles EDF, EDM, MDF containing the ſolid angle at D; the ſolid angles at A and D are equal: And therefore coincide with one another; to wit, if the plane angle BAC be applied to the plane angle EDF, the ſtraight line AH coincides with DM, as was ſhown in prop. B of this book: And becauſe AH is equal to DM, the point H coincides with the point M: Wherefore HK which is perpendicular to the plane BAC coincides with MN [i] which is perpendicular to the plane EDF, becauſe theſe planes coincide with one another: Therefore HK is equal to MN. Q. E. D.

[g] 47. 1.

[h] 8. 1.

[i] 13. 11.

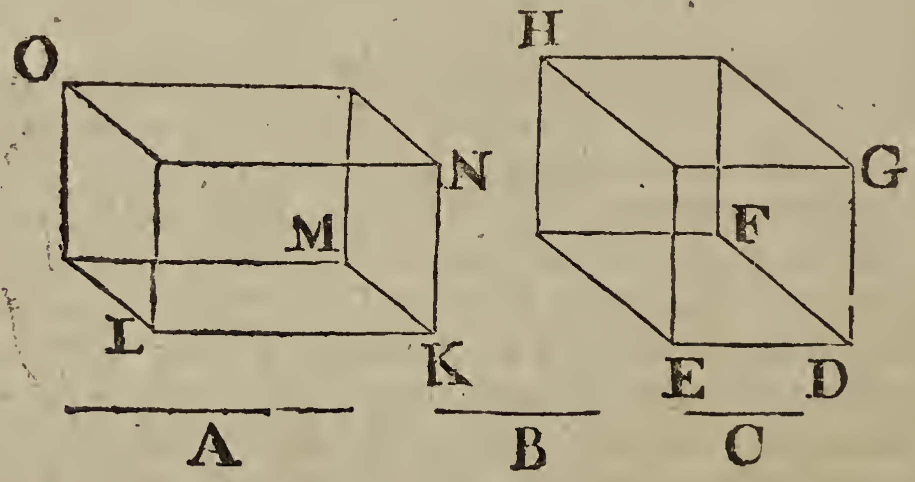
PROP. XXXVI. THEOR.

See N. IF three ſtraight lines be proportionals, the ſolid parallelepiped deſcribed from all three as its ſides, is equal to the equilateral parallelepiped deſcribed from the mean proportional, one of the ſolid angles of which is contained by three plane angles equal, each to each, to the three plane angles containing one of the ſolid angles of the other figure.

Let A, B, C be three proportionals, viz. A to B, as B to C. The ſolid deſcribed from A, B, C is equal to the equilateral ſolid deſcribed from B, equiangular to the other.

Take a ſolid angle D contained by three plane angles EDF, FDG, GDE; and make each of the ſtraight lines ED, DF, DG equal to B, and complete the ſolid parallelepiped DH: Make LK equal to A, and at the point K in the ſtraight line LK make[a] a ſolid angle contained by the three plane angles LKM, MKN, NKL equal to the angles EDF, FDG, GDE, each to each; and make KN equal to B, and KM, equal to C; and complete the ſolid parallelepiped KO: And becauſe, as A is to B, ſo is B to C, and that A is equal to LK, and B to each of the ſtraight lines DE, DF, and C to KM; therefore LK is to ED, as DF to KM; that is, the ſides about the equal angles are reciprocally proportional; therefore the parallelogram LM is equal[b] to EF: And becauſe EDF, LKM are two equal plane angles, and the two equal ſtraight lines DG, KN are drawn from their vertices above their planes, and contain equal angles with their ſides; therefore the perpendiculars from the points G, N, to the planes EDF, LKM are equal

a 26. 11.



b 14. 6.

qual[c] to one another: Therefore the ſolids KO, DH are of the ſame altitude; and they are upon equal baſes LM, EF, and therefore they are equal[d] to one another: But the ſolid KO is deſcribed from the three ſtraight lines A, B, C, and the ſolid DH from the ſtraight line B. If therefore three ſtraight lines, &c. Q. E. D.

c Cor. 35. 11.
d 31. 11.

PROP. XXXVII. THEOR.

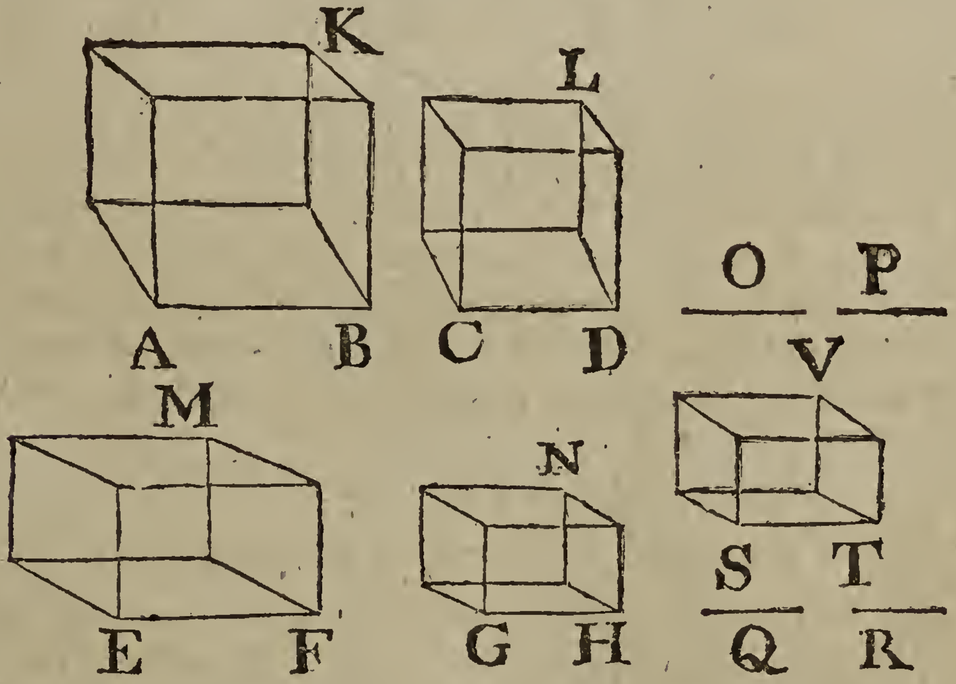
IF four ſtraight lines be proportionals, the ſimilar ſolid parallelepipeds ſimilarly deſcribed from them ſhall alſo be proportionals. And if the ſimilar parallelepipeds ſimilarly deſcribed from four ſtraight lines be proportionals, the ſtraight lines ſhall be proportionals.

See N.

Let the four ſtraight lines AB, CD, EF, GH be proportionals, viz. as AB to CD, ſo EF to GH; and let the ſimilar parallelepipeds AK, CL, EM, GN be ſimilarly deſcribed from them. AK is to CL, as EM to GN.

Make[a] AB, CD, O, P continual proportionals, as alſo EF, GH, Q, R: And becauſe as AB is to CD, ſo EF to GH; and

a 11. 6.

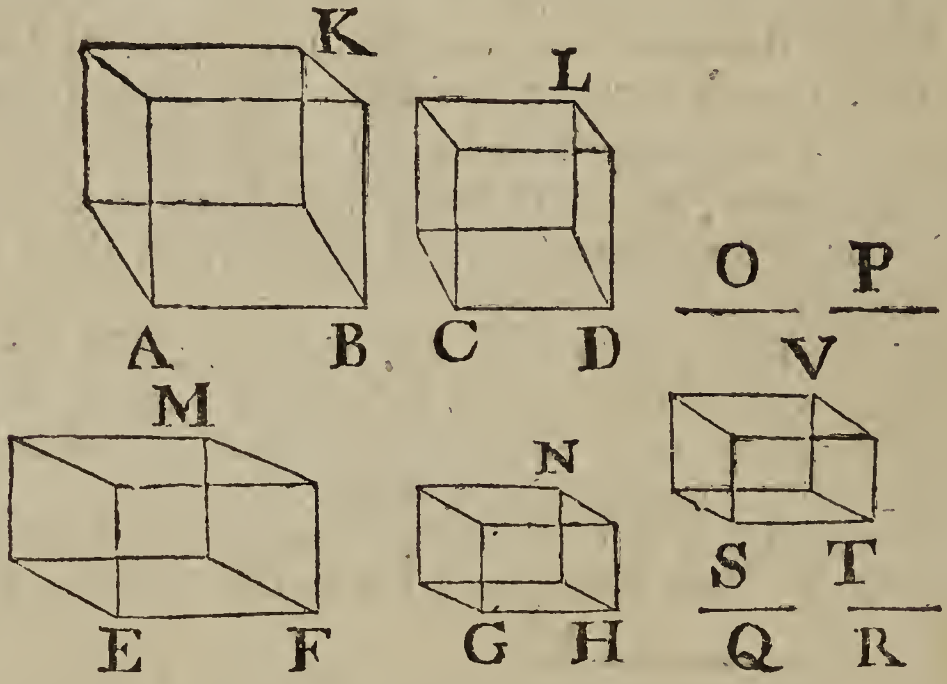


that CD is[b] to O, as GH to Q, and O to P, as Q to R; therefore, ex æquali[c], AB is to P, as EF to R: But as AB to P, ſo[d] is the ſolid AK to the ſolid CL; and as EF to R, ſo[d] is the ſolid EM to the ſolid GN: Therefore[b] as the ſolid AK to the ſolid CL, ſo is the ſolid EM to the ſolid GN.

b 11. 5.
c 22. 5.
d Cor. 33. 11.

Q 4

But let the ſolid AK be to the ſolid CL, as the ſolid EM to the ſolid GN: The ſtraight line AB is to CD, as EF to GH.
e 27. 11. Take AB to CD, as EF to ST, and from ST deſcribe [e] a ſolid parallelepiped SV ſimilar and ſimilarly ſituated to either of the ſolids EM, GN: And becauſe AB is to CD, as EF to ST, and that from AB, CD the ſolid parallelepipeds AK, CL are ſimilarly deſcribed; and in like manner the ſolids EM, SV from the ſtraight lines EF, ST; therefore AK is to CL, as EM to SV: But, by the hypotheſis, AK is to CL, as EM to GN:
f 9. 5. Therefore GN is equal [f] to SV: But it is likewiſe ſimilar and ſimilarly ſituated to SV; therefore the planes which contain the ſolids GN, SV are ſimilar and equal, and their homologous ſides GH, ST equal to one another: And becauſe as AB to CD, ſo EF to ST, and that ST is equal to GH; AB is to CD, as EF to GH. Therefore if four ſtraight lines, &c. Q. E. D.



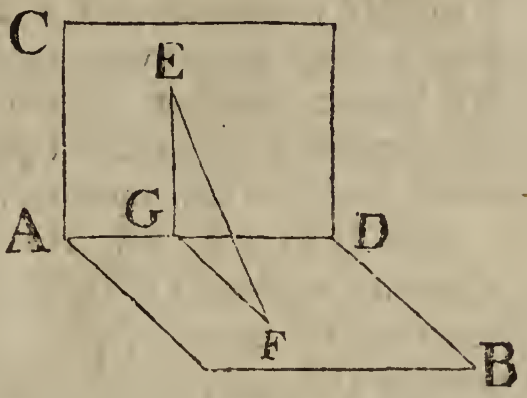
PROP. XXXVIII. THEOR.

See N. " IF a plane be perpendicular to another plane, and " a ſtraight line be drawn from a point in one of " the planes perpendicular to the other plane, this " ſtraight line ſhall fall on the common ſection of the " planes."

" Let the plane CD be perpendicular to the plane AB, and " let AD be their common ſection; if any point E be taken in " the plane CD, the perpendicular drawn from E to the plane " AB ſhall fall on AD.

“ For, if it does not, let it, if poſſible, fall elſewhere, as EF; and let it meet the plane AB in the point F; and from F draw [a], in the plane AB a perpendicular FG to DA, which is alſo perpendicular [b] to the plane CD; and join EG: Then becauſe FG is perpendicular to the plane CD, and the ſtraight line EG, which is in that plane, meets it; therefore FGE is a right angle [c]: But EF is alſo at right angles to the plane AB; and therefore EFG is a right angle: Wherefore two of the angles of the triangle EFG are equal together to two right angles; which is abſurd: Therefore the perpendicular from the point E to the plane AB, does not fall elſewhere than upon the ſtraight line AD: It therefore falls upon it. If therefore a plane,” &c. Q. E. D.

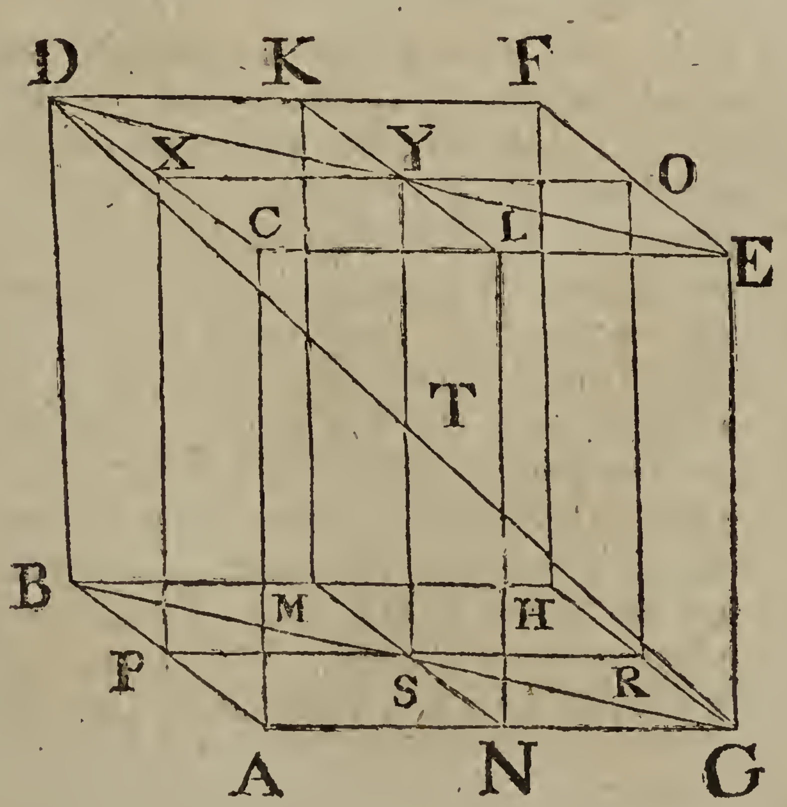
a 12. 1.
b 4. def. 11.
c 3. def. 11.



PROP. XXXIX. THEOR.

See N.

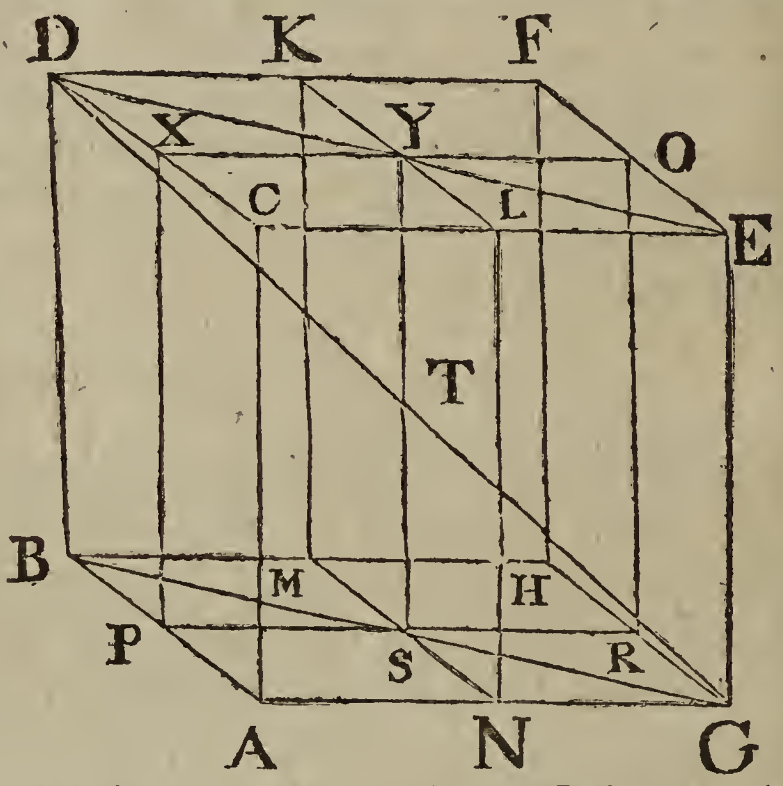
IN a ſolid parallelepiped, if the ſides of two of the oppoſite planes be divided each into two equal parts, the common ſection of the planes paſſing through the points of diviſion, and the diameter of the ſolid parallelepiped cut each other into two equal parts.



Let the ſides of the oppoſite planes CF, AH of the ſolid parallelepiped AF, be divided each into two equal parts in the points K, L, M, N; X, O, P, R; and join KL, MN, XO, PR: And becauſe DK, CL are equal and parallel, KL is parallel [a] to DC: For the ſame reaſon, MN is parallel to BA: And

a 33. 1.

BA is parallel to DC; therefore, becauſe KL, BA are each of them parallel to DC, and not in the ſame plane with it, KL is b 9. 11. parallel[b] to BA: And becauſe KL, MN are each of them parallel to BA, and not in the ſame plane with it, KL is parallel [b] to MN; wherefore KL, MN are in one plane. In like manner, it may be proved, that XO, PR are in one plane. Let YS be the common ſection of the planes KN, XR; and DG the diameter of the ſolid parallelepiped AF: YS and DG do meet, and cut one another into two equal parts.



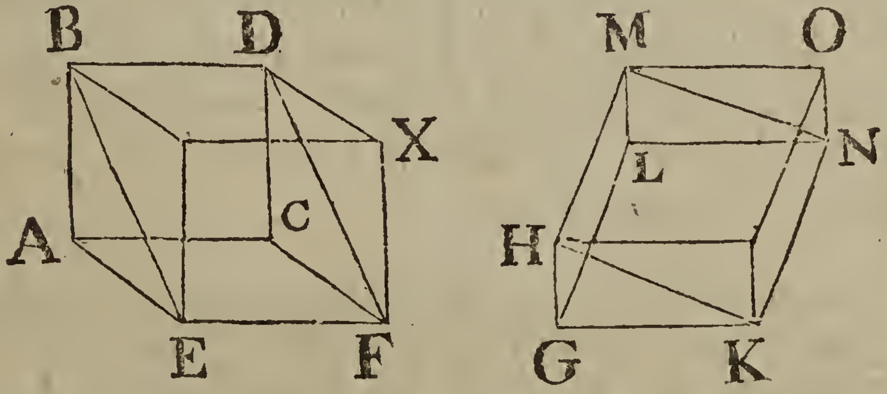
Join DY, YE, BS, SG. Becauſe DX is parallel to OE, the c 29. 1. alternate angles DXY, YOE are equal [c] to one another: And becauſe DX is equal to OE, and XY to YO, and contain equal angles, the baſe DY d 4. 1. is equal [d] to the baſe YE, and the other angles are equal; therefore the angle XYD is equal to the angle OYE, and DYE is a ſtraight e 14. 1. [e] line: For the ſame reaſon BSG is a ſtraight line, and BS equal to SG: And becauſe CA is equal and parallel to DB, and alſo a 33. 1. equal and parallel to EG; therefore DB is equal and parallel [b] to EG: And DE, BG join their extremities; therefore DE is equal and parallel [a] to BG: And DG, YS are drawn from points in the one, to points in the other; and are therefore in one plane: Whence it is manifeſt, that DG, YS muſt meet one another; let them meet in T: And becauſe DE is parallel to BG, the alternate angles EDT, BGT are equal [c]; f 15. 1. and the angle DTY is equal [f] to the angle GTS: Therefore in the triangles DTY, GTS there are two angles in the one equal to two angles in the other, and one ſide equal to one ſide, oppoſite to two of the equal angles, viz. DY to GS; for they are the halves of DE, BG: Therefore the remaining ſides are g 26. 1. equal [g], each to each. Wherefore DT is equal to TG, and YT equal to TS. Wherefore, if in a ſolid, &c. Q. E. D.

PROP. XL. THEOR.

IF there be two triangular priſms of the ſame altitude, the baſe of one of which is a parallelogram, and the baſe of the other a triangle; if the parallelogram be double of the triangle, the priſms ſhall be equal to one another.

Let the priſms ABCDEF, GHKLMN be of the ſame altitude, the firſt whereof is contained by the two triangles ABE, CDF, and the three parallelograms AD, DE, EC; and the other by the two triangles GHK, LMN and the three parallelograms LH, HN, NG; and let one of them have a parallelogram AF, and the other a triangle GHK for its baſe; if the parallelogram AF be double of the triangle GHK, the priſm ABCDEF is equal to the priſm GHKLMN.

Complete the ſolids AX, GO; and becauſe the parallelogram AF is double of the triangle GHK; and the parallelogram HK double [a] of the ſame triangle; therefore the parallelogram AF is equal to HK. But ſolid parallelepipeds upon equal baſes, and of the ſame altitude, are equal [b] to one another. Therefore the ſolid AX is equal to the ſolid GO; and the priſm ABCDEF is half [c] of the ſolid AX; and the priſm GHKLMN half [c] of the ſolid GO. Therefore the priſm ABCDEF is equal to the priſm GHKLMN. Wherefore, if there be two, &c. Q. E. D.



a 34. 1.
b 31. 11.
c 28. 11.

THE ELEMENTS OF EUCLID.

See N.

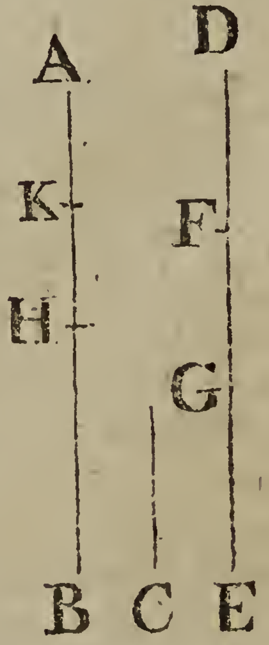
BOOK XII.

LEMMA I.

Which is the firſt propoſition of the tenth book, and is neceſſary to ſome of the propoſitions of this book.

IF from the greater of two unequal magnitudes, there be taken more than its half, and from the remainder more than its half; and ſo on: There ſhall at length remain a magnitude leſs than the leaſt of the propoſed magnitudes.

Let AB and C be two unequal magnitudes, of which AB is the greater. If from AB there be taken more than its half, and from the remainder more than its half, and ſo on; there ſhall at length remain a magnitude leſs than C.



For C may be multiplied ſo as at length to become greater than AB. Let it be ſo multiplied, and let DE its multiple be greater than AB, and let DE be divided into DF, FG, GE, each equal to C. From AB take BH greater than its half, and from the remainder AH take HK greater than its half, and ſo on, until there be as many diviſions in AB as there are in DE: And let the diviſions in AB be AK, KH, HB; and the diviſions in ED be DF, FG GE. And becauſe DE is greater than AB, and that EG taken from DE is not greater than its half, but BH taken from AB is greater than its half; therefore the remainder GD is greater than the remainder HA. Again, becauſe GD is greater than HA, and that GF is not greater than the half of GD, but HK is greater than the half of HA; therefore the remainder FD is greater than the remainder AK. And FD is equal to C, therefore C is greater than AK; that is, AK is leſs than C. Q. E. D.

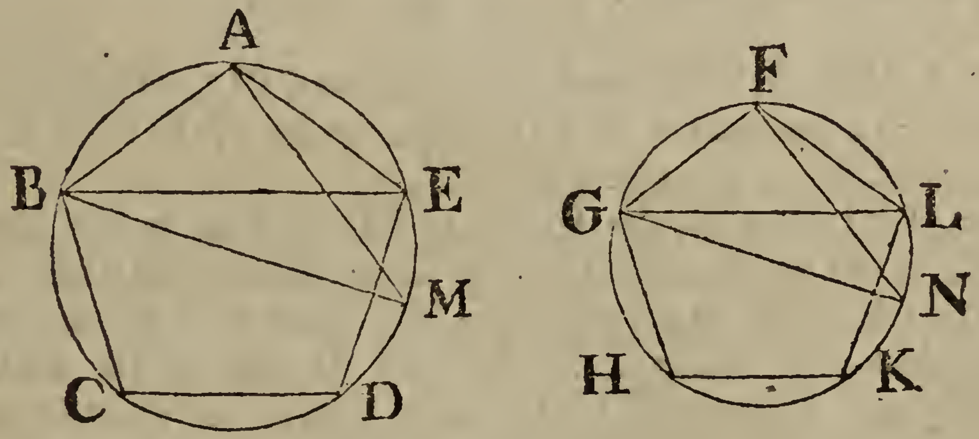
And if only the halves be taken away, the ſame thing may in the ſame way be demonſtrated.

PROP. I. THEOR.

SIMILAR polygons inſcribed in circles, are to one another as the ſquares of their diameters.

Let ABCDE, FGHKL be two circles, and in them the ſimilar polygons ABCDE, FGHKL; and let BM, GN be the diameters of the circles: As the ſquare of BM is to the ſquare of GN, ſo is the polygon ABCDE to the polygon FGHKL.

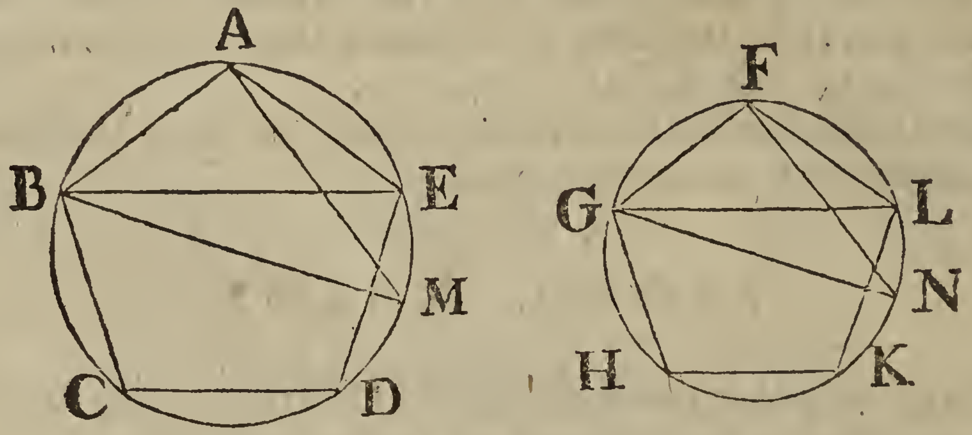
Join BE, AM, GL, FN: And becauſe the polygon ABCDE is ſimilar to the polygon FGHKL, and ſimilar polygons are divided into ſimilar triangles; the triangles ABE, FGL, are ſimilar and equiangular [b]; and therefore the angle AEB is equal to the angle FLG: But AEB is equal [c] to AMB, becauſe they ſtand upon the ſame circumference; and the angle FLG is for the ſame reaſon, equal to the angle FNG: Therefore alſo the angle AMB is equal to FNG: And the right angle BAM is equal to the right [d] angle GFN; wherefore the remaining angles in the triangles ABM, FGN are equal, and they are equiangular to one another:



b 6. 6.
c 21. 3.
d 31. 3.

e 4. 6.
f 10. def. 5. & 22. 5.
g 20. 6.

another: Therefore as BM to GN, ſo [e] is BA to GF; and therefore the duplicate ratio of BM to GN, is the ſame [f] with the duplicate ratio of BA to GF: But the ratio of the ſquare of BM to the ſquare of GN, is the duplicate [g] ratio of that which BM has to GN; and the ratio of the polygon ABCDE to the polygon FGHKL is the duplicate [g] of that which BA has to GF: Therefore as the ſquare of BM to the ſquare of GN, ſo is the polygon ABCDE to the polygon FGHKL. Wherefore ſimilar polygons, &c. Q. E. D.



PROP. II. THEOR.

See N.

CIRCLES are to one another as the ſquares of their diameters.

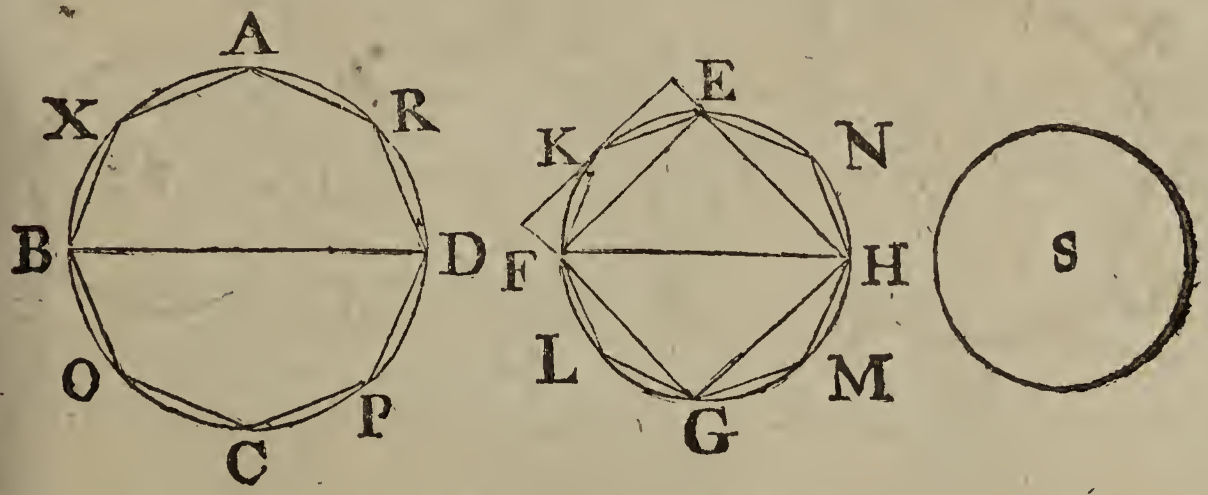
Let ABCD, EFGH be two circles, and BD, FH their diameters: As the ſquare of BD to the ſquare of FH, ſo is the circle ABCD, to the circle EFGH.

For, if it be not ſo, the ſquare of BD ſhall be to the ſquare of FH, as the circle ABCD is to ſome ſpace either leſs than the circle EFGH, or greater than it *. Firſt let it be to a ſpace S leſs than the circle EFGH; and in the circle EFGH deſcribe the ſquare EFGH: This ſquare is greater than half of the circle EFGH; becauſe if, through the points E, F, G, H, there be drawn tangents to the circle, the ſquare

---

* For there is ſome ſquare equal to the circle ABCD; let P be the ſide of it, and to three ſtraight lines BD, FH and P, there can be a fourth proportional; let this be Q: Therefore the ſquares of theſe four ſtraight lines are proportionals; that is, to the ſquares of BD, FH and the circle ABCD, it is poſſible there may be a fourth proportional. Let this be S. And in like manner are to be underſtood ſome things in ſome of the following propoſitions.

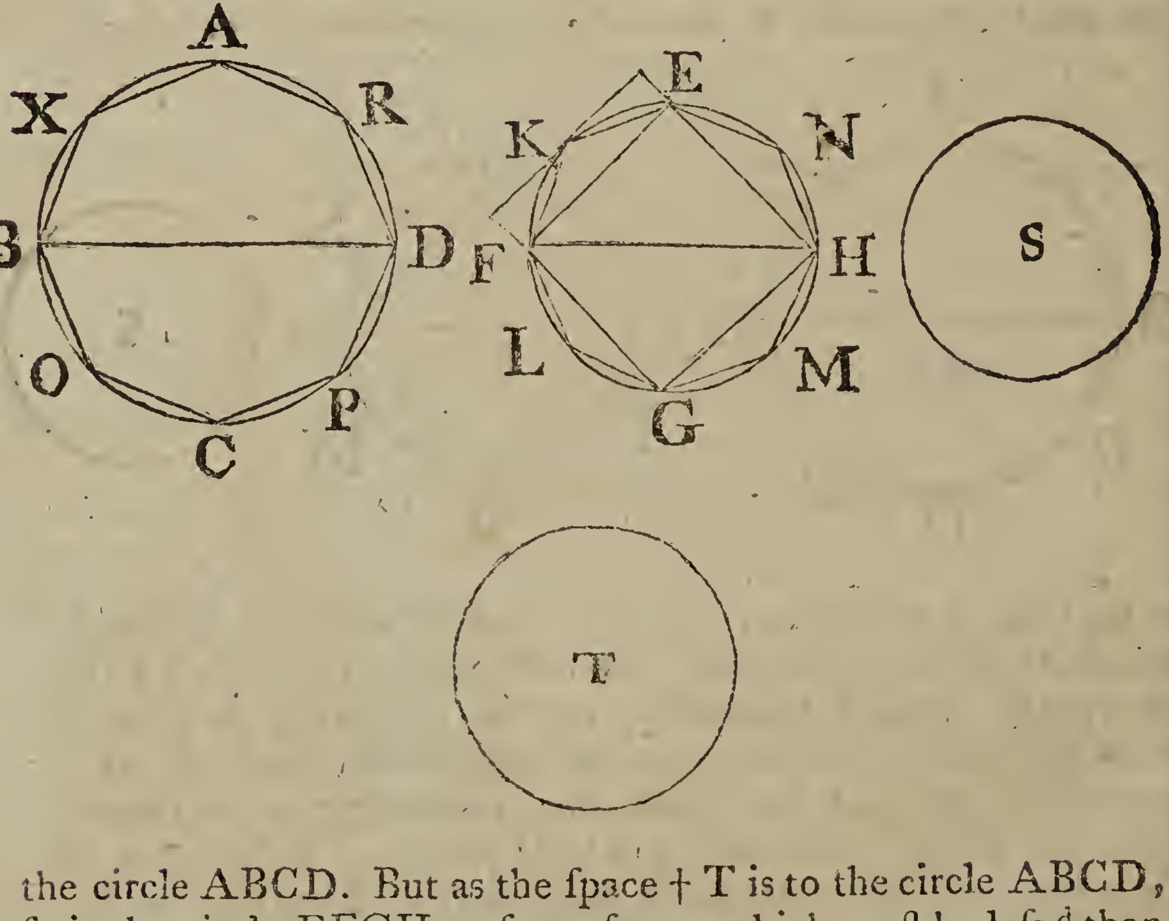
ſquare EFGH is half[a] of the ſquare deſcribed about the circle; and the circle is leſs than the ſquare deſcribed about it; there- a 41. 1. fore the ſquare EFGH is greater than half of the circle. Divide the circumferences EF, FG, GH, HE, each into two equal parts in the points K, L, M, N, and join EK, KF, FL, LG, GM, MH, HN, NE: Therefore each of the triangles EKF, FLG, GMH, HNE is greater than half of the ſegment of the circle it ſtands in; becauſe, if ſtraight lines touching the circle be drawn through the points K, L, M, N, and parallelograms upon the ſtraight lines EF, FG, GH, HE, be completed; each of the triangles EKF, FLG, GMH, HNE ſhall be the half[a] a 41. 1. of the parallelogram in which it is: But every ſegment is leſs than the parallelogram in which it is: Wherefore each of the triangles EKF, FLG, GMH, HNE is greater than half the ſegment of the circle which contains it: And if theſe circumferences before named be divided each into two equal parts, and their extremities be joined by ſtraight lines, by continuing



to do this, there will at length remain ſegments of the circle which, together, ſhall be leſs than the exceſs of the circle EFGH above the ſpace S: Becauſe, by the preceding Lemma, if from the greater of two unequal magnitudes there be taken more than its half, and from the remainder more than its half, and ſo on, there ſhall at length remain a magnitude leſs than the leaſt of the propoſed magnitudes. Let then the ſegments EK, KF, FL, LG, GM, MH, HN, NE be thoſe that remain and are together leſs than the exceſs of the circle EFGH above S: Therefore the reſt of the circle, viz. the polygon EKFLGMHN, is greater than the ſpace S. Deſcribe likewiſe in the circle ABCD the polygon AXBOCPDR ſimilar to the polygon EKFLGMHN: As therefore, the ſquare of BD is to the ſquare of FH, ſo[b] is the polygon AXBOCPDR to the b 1. 12. polygon EKFLGMHN: But the ſquare of BD is alſo to the ſquare of FH, as the circle ABCD is to the ſpace S: Therefore as the circle ABCD is to the ſpace S, ſo is [c] the polygon AXBOCPDR to the polygon EKFLGMHN: But the circle ABCD is greater than the polygon contained in it; wherefore the ſpace S is greater [d] than the polygon EKFLGMHN: But it is likewiſe leſs, as has been demonſtrated; which is impoſſible. Therefore the ſquare of BD is not to the ſquare of FH, as the circle ABCD is to any ſpace leſs than the circle EFGH. In the ſame manner, it may be demonſtrated, that neither is the ſquare of FH to the ſquare of BD, as the circle EFGH is to any ſpace leſs than the circle ABCD. Nor is the ſquare of BD to the ſquare of FH, as the circle ABCD is to any ſpace greater than the circle EFGH: For, if poſſible, let it be ſo to T, a ſpace greater than the circle EFGH: Therefore inverſely as the ſquare of FH to the ſquare of BD, ſo is the ſpace T to the circle ABCD. But as the ſpace † T is to the circle ABCD, ſo is the circle EFGH to ſome ſpace, which muſt be leſs [d] than the circle ABCD, becauſe the ſpace T is greater, by hypotheſis, than the circle EFGH. Therefore as the ſquare of FH is to the

[c] 11. 5.

[d] 14. 5.



---

† For as in the foregoing note, at *, it was explained how it was poſſible there could be a fourth proportional to the ſquares of BD, FH, and the circle ABCD, which was named S. So in like manner there can be a fourth proportional to this other ſpace, named T, and the circles ABCD, EFGH. And the like is to be underſtood in ſome of the following propoſitions.

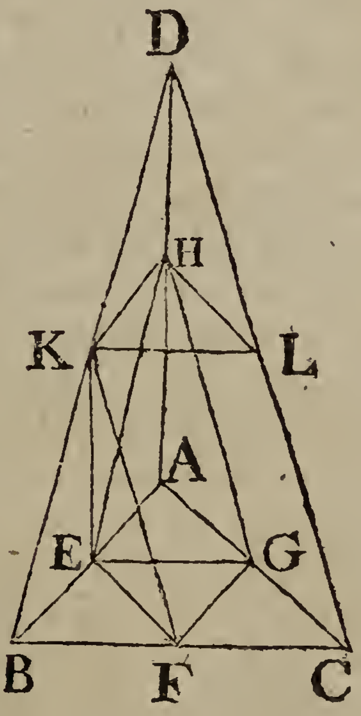
the ſquare of BD, ſo is the circle EFGH to a ſpace leſs than the circle ABCD, which has been demonſtrated to be impoſſible: Therefore the ſquare of BD is not to the ſquare of FH as the circle ABCD is to any ſpace greater than the circle EFGH: And it has been demonſtrated, that neither is the ſquare of BD to the ſquare of FH, as the circle ABCD to any ſpace leſs than the circle EFGH: Wherefore, as the ſquare of BD to the ſquare of FH, ſo is the circle ABCD to the circle EFGH †. Circles therefore are, &c. Q. E. D.

PROP. III. THEOR.

See N.

EVERY pyramid having a triangular baſe, may be divided into two equal and ſimilar pyramids having triangular baſes, and which are ſimilar to the whole pyramid; and into two equal priſms which together are greater than half of the whole pyramid.

Let there be a pyramid of which the baſe is the triangle ABC and its vertex the point D: The pyramid ABCD may be divided into two equal and ſimilar pyramids having triangular baſes, and ſimilar to the whole; and into two equal priſms which together are greater than half of the whole pyramid.



Divide AB, BC, CA, AD, DB, DC, each into two equal parts in the points E, F, G, H, K, L, and join EH, EG, GH, HK, KL, LH, EK, KF, FG. Becauſe AE is equal to EB, and AH to HD, HE is parallel [a] to DB: For the ſame reaſon, HK is parallel to AB: Therefore HEBK is a parallelogram, and HK equal [b] to EB: But EB is equal to AE; therefore alſo AE is equal to HK: And AH is equal to HD; wherefore EA, AH are equal to KH, HD, each to each; and the angle EAH is equal [c] to the angle KHD; therefore the baſe EH is equal to the baſe KD, and the triangle

a 2. 6.

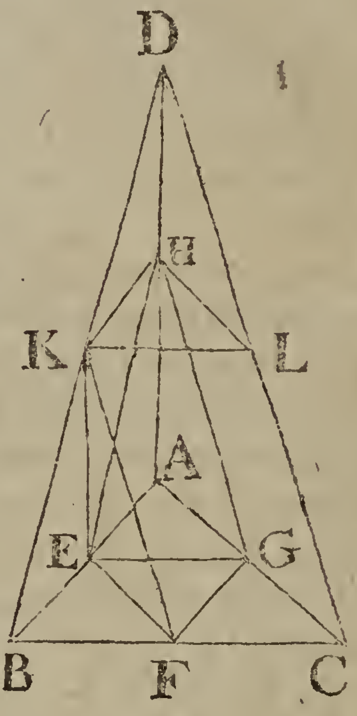
b 34. 1.

c 29. 1.

---

† Becauſe as a fourth proportional to the ſquares of BD, FH and the circle ABCD is poſſible, and that it can neither be leſs nor greater than the circle EFGH, it muſt be equal to it.

AEH equal[d] and ſimilar to the triangle HKD: For the ſame
d 4. 1. reaſon, the triangle AGH is equal and ſimilar to the triangle
HLD: And becauſe the two ſtraight lines EH, HG which
meet one another are parallel to KD, DL that meet one another,
and are not in the ſame plane with them, they contain
e 10. 11. equal[e] angles; therefore the angle EHG is equal to the angle
KDL. Again, becauſe EH, HG are equal to KD, DL, each
to each, and the angle EHG equal to the angle KDL; therefore
the baſe EG is equal to the baſe KL: And the triangle
EHG equal[d] and ſimilar to the triangle KDL: For the ſame
reaſon, the triangle AEG is alſo equal and ſimilar to the triangle
HKL. Therefore the pyramid of which the baſe is the
triangle AEG, and of which the vertex is the point H, is equal[f]
f c. 11. and ſimilar to the pyramid the
baſe of which is the triangle KHL, and
vertex the point D: And becauſe HK
is parallel to AB a ſide of the triangle
ADB, the triangle ADB is equiangular
to the triangle HDK, and their
g 4. 6. ſides are proportionals[g]: Therefore the
triangle ADB is ſimilar to the triangle
HDK: And for the ſame reaſon, the
triangle DBC is ſimilar to the triangle
DKL; and the triangle ADC to the
triangle HDL; and alſo the triangle
ABC to the triangle AEG: But the
triangle AEG is ſimilar to the triangle
HKL, as before was proved; therefore
h 21. 6. the triangle ABC is ſimilar[h] to the
triangle HKL. And the pyramid of
which the baſe is the triangle ABC, and vertex the point D,
i B. 11. is therefore ſimilar[i] to the pyramid of which the baſe is the triangle
& 11. HKL, and vertex the ſame point D: But the pyramid of
Def. 11. which the baſe is the triangle HKL, and vertex the point D, is
ſimilar, as has been proved, to the pyramid the baſe of which is
the triangle AEG, and vertex the point H: Wherefore the pyramid
the baſe of which is the triangle ABC, and vertex the
point D, is ſimilar to the pyramid of which the baſe is the triangle
AEG and vertex H: Therefore each of the pyramids
AEGH, HKLD is ſimilar to the whole pyramid ABCD: And
k 41. 1. becauſe BF is equal to FC, the parallelogram EBFG is double[k]
of the triangle GFC: But when there are two priſms of the
ſame



ſame altitude, of which one has a parallelogram for its baſe, and the other a triangle that is half of the parallelogram, theſe priſms are equal [a] to one another; therefore the priſm having the parallelogram EBFG for its baſe, and the ſtraight line KH oppoſite to it, is equal to the priſm having the triangle GFC for its baſe, and the triangle HKL oppoſite to it; for they are of the ſame altitude, becauſe they are between the parallel [b] planes ABC, HKL: And it is manifeſt that each of theſe priſms is greater than either of the pyramids of which the triangles AEG, HKL are the baſes, and the vertices the points H, D; becauſe, if EF be joined, the priſm having the parallelogram EBFG for its baſe, and KH the ſtraight line oppoſite to it, is greater than the pyramid of which the baſe is the triangle EBF, and vertex the point K; but this pyramid is equal [c] to the pyramid the baſe of which is the triangle AEG, and vertex the point H; becauſe they are contained by equal and ſimilar planes: Wherefore the priſm having the parallelogram EBFG for its baſe, and oppoſite ſide KH, is greater than the pramid of which the baſe is the triangle AEG, and vertex the point H: And the priſm of which the baſe is the parallelogram EBFG, and oppoſite ſide KH is equal to the priſm having the triangle GFC for its baſe, and HKL the triangle oppoſite to it; and the pyramid of which the baſe is the triangle AEG, and vertex H, is equal to the pyramid of which the baſe is the triangle HKL, and vertex D: Therefore the two priſms before mentioned are greater than the two pyramids of which the baſes are the triangles AEG, HKL, and vertices the points H, D. Therefore the whole pyramid of which the baſe is the triangle ABC, and vertex the point D, is divided into two equal pyramids ſimilar to one another, and to the whole pyramid; and into two equal priſms; and the two priſms are together greater than half of the whole pyramid. Q. E. D.

[a] 40. 11.

[b] 15. 11.

[c] C. 11.

PROP. IV. THEOR.

See N. IF there be two pyramids of the ſame altitude, upon triangular baſes, and each of them be divided into two equal pyramids ſimilar to the whole pyramid, and alſo into two equal priſms; and if each of theſe pyramids be divided in the ſame manner as the firſt two, and ſo on: As the baſe of one of the firſt two pyramids is to the baſe of the other, ſo ſhall all the priſms in one of them be to all the priſms in the other that are produced by the ſame number of diviſions.

Let there be two pyramids of the ſame altitude upon the triangular baſes ABC, DEF, and having their vertices in the points G, H; and let each of them be divided into two equal pyramids ſimilar to the whole, and into two equal priſms; and let each of the pyramids thus made be conceived to be divided in the like manner, and ſo on: As the baſe ABC is to the baſe DEF, ſo are all the priſms in the pyramid ABCG to all the priſms in the pyramid DEFH made by the ſame number of diviſions,

Make the ſame conſtruction as in the foregoing propoſition: And becauſe BX is equal to XC, and AL to LC, therefore XL is parallel [a] to AB, and the triangle ABC ſimilar to the triangle LXC: For the ſame reaſon, the triangle DEF is ſimilar to RVF: And becauſe BC is double of CX, and EF double of FV, therefore BC is to CX, as EF to FV: And upon BC, CX are deſcribed the ſimilar and ſimilarly ſituated rectilineal figures ABC, LXC; and upon EF, FV, in like manner, are deſcribed the ſimilar figures DEF, RVF: Therefore, as the triangle ABC is to the triangle LXC, ſo [b] is the triangle DEF to the triangle RVF, and, by permutation, as the triangle ABC to the triangle DEF, ſo is the triangle LXC to the triangle RVF: And becauſe the planes ABC, OMN, as alſo the planes DEF, STY are parallel [c], the perpendiculars drawn from the points G, H to the baſes ABC, DEF, which, by the hypotheſis, are equal to one another, ſhall be cut each into two equal [d] parts by the planes OMN, STY, becauſe the ſtraight lines GC, HF are cut into two equal parts in the points N, Y by the ſame planes: Therefore the priſms LXCOMN, RVFSTY are of the ſame altitude; and therefore, as the baſe LXC to the

a 2. 6.

b 22. 6.

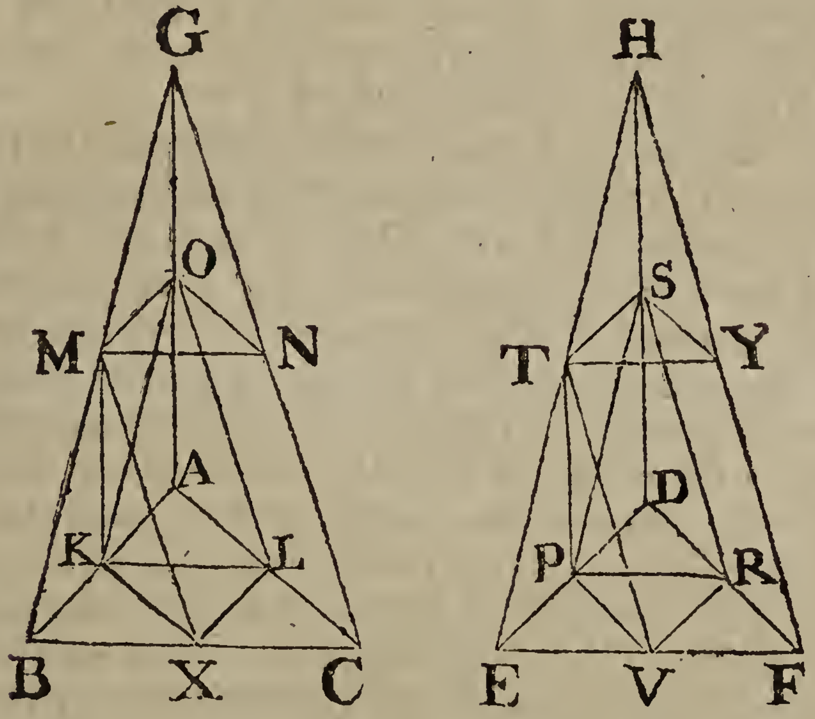
c 5. 11.

d 17. 11.

the baſe RVF; that is, as the triangle ABC to the triangle DEF, ſo [a] is the priſm having the triangle LXC for its baſe, and OMN the triangle oppoſite to it, to the priſm of which the baſe is the triangle RVF, and the oppoſite triangle STY: And becauſe the two priſms in the pyramid ABCG are equal to one another, and alſo the two priſms in the pyramid DEFH equal to one another, as the priſm of which the baſe is the parallelogram KBXL and oppoſite ſide MO, to the priſm having the triangle LXC for its baſe, and OMN the triangle oppoſite to it; ſo is the priſm of which the baſe [b] is the parallelogram, PEVR, and oppoſite ſide TS, to the priſm of which the baſe is the triangle RVF, and oppoſite triangle STY. Therefore, componendo, as the priſms KBXLMO LXCOMN together are unto the priſm LXCOMN; ſo are the priſms PEVRTS, RVFSTY to the priſm RVFSTY: And permutando, as the priſms KBXLMO, LXCOMN are to the priſms PEVRTS, RVFSTY; ſo is the priſm LXCOMN to the priſm RVFSTY: But as the priſm LXCOMN to the priſm RVFSTY, ſo is, as has been proved, the baſe ABC to the baſe DEF: Therefore, as the baſe ABC to the baſe DEF, ſo are the two priſms in the pyramid ABCG to the two priſms in the pyramid DEFH: And likewiſe if the pyramids now made, for example, the two OMNG, STYH be divided in the ſame manner; as the baſe OMN is to the baſe STY, ſo ſhall the two priſms in the pyramid OMNG be to the two priſms in the pyramid STYH: But the baſe OMN is to the baſe STY, as the baſe ABC to the baſe DEF; therefore, as the baſe ABC to the baſe DEF, ſo are the

[a] Cor. 32. 11.

[b] 7. 5.



Book XII. the two priſms in the pyramid ABCG to the two priſms in the pyramid DEFH; and ſo are the two priſms in the pyramid OMNG to the two priſms in the pyramid STYH; and ſo are all four to all four: And the ſame thing may be ſhewn of the priſms made by dividing the pyramids AKLO and DPRS, and of all made by the ſame number of diviſions. Q. E. D.

PROP. V. THEOR.

See N. PYRAMIDS of the ſame altitude which have triangular baſes, are to one another as their baſes.

Let the pyramids of which the triangles ABC, DEF are the baſes, and of which the vertices are the points G, H, be of the ſame altitude: As the baſe ABC to the baſe DEF, ſo is the pyramid ABCG to the pyramid DEFH.

For, if it be not ſo, the baſe ABC muſt be to the baſe DEF, as the pyramid ABCG to a ſolid either leſs than the pyramid DEFH, or greater than it *. Firſt, let it be to a ſolid leſs than it, viz. to the ſolid Q: And divide the pyramid DEFH into two equal pyramids, ſimilar to the whole, and into two equal priſms: Therefore theſe two priſms are greater [a] than the half of the whole pyramid. And again, let the pyramids made by this diviſion be in like manner divided, and ſo on, until the pyramids which remain undivided in the pyramid DEFH be, all of them together, leſs than the exceſs of the pyramid DEFH above the ſolid Q: Let theſe, for example, be the pyramids DPRS, STYH: Therefore the priſms, which make the reſt of the pyramid DEFH, are greater than the ſolid Q: Divide likewiſe the pyramid ABCG in the ſame manner, and into as many parts, as the pyramid DEFH: Therefore, as the baſe ABC to the baſe DEF, ſo [b] are the priſms in the pyramid ABCG to the priſms in the pyramid DEFH: But as the baſe ABC to the baſe DEF, ſo, by hypotheſis, is the pyramid ABCG to the ſolid Q; and therefore, as the pyramid ABCG to the ſolid Q, ſo are the priſms in the pyramid ABCG to the priſms in the pyramid DEFH: But the pyramid ABCG is greater than the priſms contained in it; wherefore [c] alſo the ſolid Q is greater than the priſms in the pyramid DEFH. But is it alſo leſs, which is impoſſible. Therefore the baſe ABC is not to the

a 3. 12.

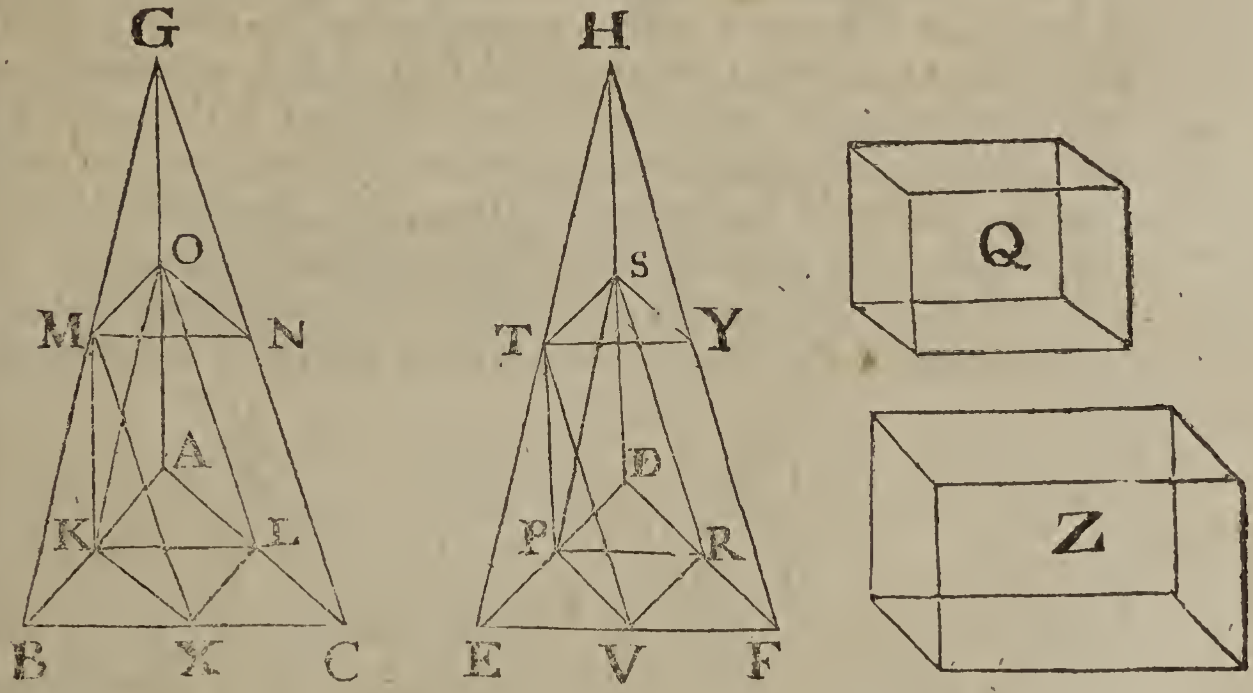
b 4. 12.

c 14. 5.

---

* This may be explained the ſame way as at the note † in propoſition 2. in the like caſe.

the baſe DEF, as the pyramid ABCG to any ſolid which is leſs than the pyramid DEFH. In the ſame manner it may be demonſtrated, that the baſe DEF is not to the baſe ABC, as the pyramid DEFH to any ſolid which is leſs than the pyramid ABCG. Nor can the baſe ABC be to the baſe DEF, as the pyramid ABCG to any ſolid which is greater than the pyramid DEFH. For if it be poſſible, let it be ſo to a greater, viz. the ſolid Z. And becauſe the baſe ABC is to the baſe DEF as the pyramid ABCG to the ſolid Z; by inverſion, as the baſe DEF to the baſe ABC, ſo is the ſolid Z to the pyramid ABCG. But as the ſolid Z is to the pyramid ABCG, ſo is the pyramid



DEFH to ſome ſolid *, which muſt be leſs [a] than the pyramid ABCG, becauſe the ſolid Z is greater than the pyramid DEFH. And therefore, as the baſe DEF to the baſe ABC, ſo is the pyramid DEFH to a ſolid leſs than the pyramid ABCG; the contrary to which has been proved. Therefore the baſe ABC is not to the baſe DEF, as the pyramid ABCG to any ſolid which is greater than the pyramid DEFH. And it has been proved, that neither is the baſe ABC to the baſe DEF, as the pyramid ABCG to any ſolid which is leſs than the pyramid DEFH. Therefore, as the baſe ABC is to the baſe DEF, ſo is the pyramid ABCG to the pyramid DEFH. Wherefore pyramids, &c. Q. E. D.

a 14. 5.

---

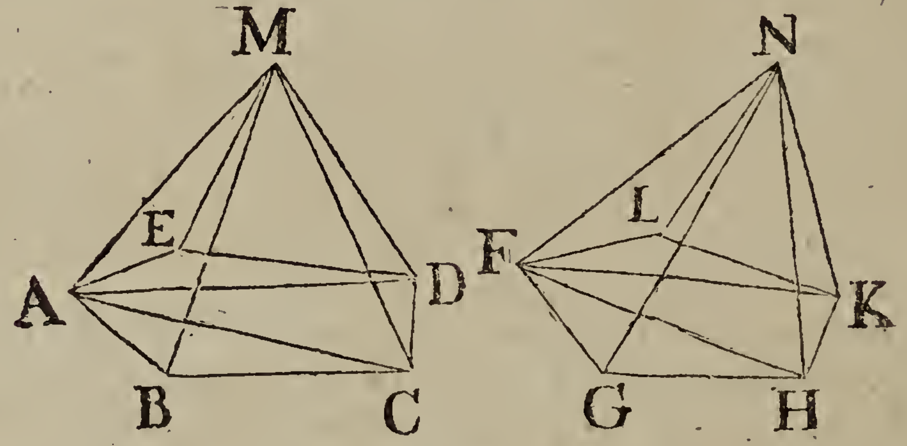
* This may be explained the ſame way as the like at the mark † in prop. 2.

PROP. VI. THEOR.

See N. PYRAMIDS of the ſame altitude which have polygons for their baſes, are to one another as their baſes.

Let the pyramids which have the polygons ABCDE, FGHKL for their baſes, and their vertices in the points M, N, be of the ſame altitude: As the baſe ABCDE to the baſe FGHKL, ſo is the pyramid ABCDEM to the pyramid FGHKLN.

Divide the baſe ABCDE into the triangles ABC, ACD, ADE; and the baſe FGHKL into the triangles FGH, FHK, FKL: And upon the baſes ABC, ACD, ADE let there be as many pyramids of which the common vertex is the point M, and upon the remaining baſes as many pyramids having their common vertex in the point N: Therefore, ſince the triangle ABC is to the triangle FGH, as [a] the pyramid ABCM to the pyramid FGHN; and the triangle ACD to the triangle FGH, as the pyramid ACDM to the pyramid FGHN; and alſo the triangle ADE to the triangle FGH, as the pyramid ADEM to the pyramid FGHN; as all the firſt antecedents to their common conſequent; ſo [b] are all the other antecedents to their common conſequent; that is, as the baſe ABCDE to the baſe FGH, ſo is the pyramid ABCDEM to the pyramid FGHN: And, for the ſame reaſon, as the baſe FGHKL to the baſe FGH, ſo is the pyramid FGHKLN to the pyramid FGHN: And, by inverſion, as the baſe FGH to the baſe FGHKL, ſo is the pyramid FGHN to the pyramid FGHKLN: Then, becauſe as the baſe ABCDE to the baſe FGH, ſo is the pyramid ABCDEM to the pyramid FGHN; and as the baſe FGH to the baſe FGHKL, ſo is the pyramid FGHN to the pyramid FGHKLN; therefore,



a 5. 12.

b 2. Cor. 24. 5.

therefore, *ex æquali* [c], as the baſe ABCDE to the baſe FGHKL, ſo the pyramid ABCDEM to the pyramid FGHKLN. Therefore pyramids, &c. Q. E. D. c 22. 5.

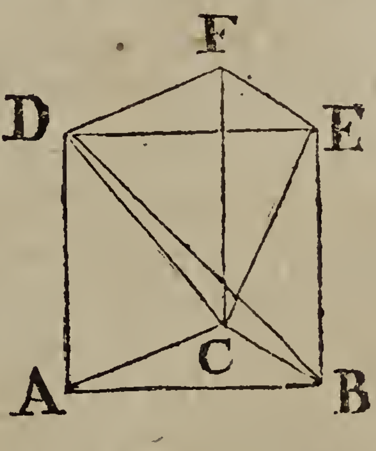
PROP. VII. THEOR.

EVERY priſm having a triangular baſe may be divided into three pyramids that have triangular baſes, and are equal to one another.

Let there be a priſm of which the baſe is the triangle ABC, and let DEF be the triangle oppoſite to it: The priſm ABCDEF may be divided into three equal pyramids having triangular baſes.

Join BD, EC, CD; and becauſe ABED is a parallelogram of which BD is the diameter, the triangle ABD is equal [a] to the triangle EBD; therefore the pyramid of which the baſe is the triangle ABD, and vertex the point C, is equal [b] to the pyramid of which the baſe is the triangle EBD, and vertex the point C: But this pyramid is the ſame with the pyramid the baſe of which is the triangle EBC, and vertex the point D; for they are contained by the ſame planes: Therefore the pyramid of which the baſe is the triangle ABD, and vertex the point C, is equal to the pyramid, the baſe of which is the triangle EBC, and vertex the point D: Again, becauſe FCBE is a parallelogram of which the diameter is CE, the triangle ECF is equal [a] to the triangle ECB; therefore the pyramid of which the baſe is the triangle ECB, and vertex the point D, is equal to the pyramid, the baſe of which is the triangle ECF, and vertex the point D: But the pyramid of which the baſe is the triangle ECB, and vertex the point D has been proved equal to the pyramid of which the baſe is the triangle ABD, and vertex the point C. Therefore the priſm ABCDEF is divided into three equal pyramids having triangular baſes, viz. into the pyramids ABDC, EBDC, ECFD: And becauſe the pyramid of which the baſe is the triangle ABD, and vertex the point C, is the ſame with the pyramid of which the baſe is the triangle ABC, and vertex the point D, for they are contained by the ſame planes; and that the pyramid of which the baſe is the triangle ABD, and vertex the point C, has been demonſtrated

a 34. 1.
b 5. 12.



demonſtrated to be a third part of the priſm the baſe of which is the triangle ABC, and to which DEF is the oppoſite triangle; therefore the pyramid of which the baſe is the triangle ABC, and vertex the point D, is the third part of the priſm which has the ſame baſe, viz. the triangle ABC, and DEF is the oppoſite triangle. Q. E. D.

Cor. 1. From this it is manifeſt, that every pyramid is the third part of a priſm which has the ſame baſe, and is of an equal altitude with it; for if the baſe of the priſm be any other figure than a triangle, it may be divided into priſms having triangular baſes.

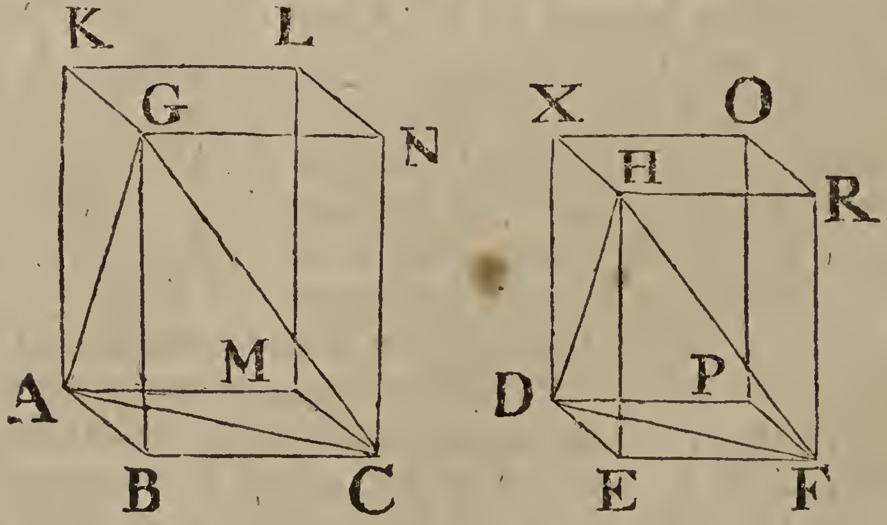
Cor. 2. Priſms of equal altitudes are to one another as their baſes; becauſe the pyramids upon the ſame baſes, and of the ſame altitude, are [c] to one another as their baſes. c 6. 12.

PROP. VIII. THEOR.

SIMILAR pyramids having triangular baſes are one to another in the triplicate ratio of that of their homologous ſides.

Let the pyramids having the triangles ABC, DEF for their baſes, and the points G, H for their vertices, be ſimilar, and ſimilarly ſituated; the pyramid ABCG has to the pyramid DEFH, the triplicate ratio of that which the ſide BC has to the homologous ſide EF.

Complete the parallelograms ABCM, GBCN, ABGK, and the ſolid parallelepiped BGML contained by theſe planes and



thoſe oppoſite to them: And, in like manner, complete the ſolid parallelepiped EHPO contained by the three parallelograms DEFP, HEFR, DEHX, and thoſe oppoſite to them: And be-

cauſe

cauſe the pyramid ABCG is ſimilar to the pyramid DEFH,
the angle ABC is equal [a] to the angle DEF, and the angle GBC a 11. def. 11.
to the angle HEF, and ABG to DEH: And AB is [b] to BC,
as DE to EF; that is, the ſides about the equal angles are proportionals; b 1. def. 6.
wherefore the parallelogram BM is ſimilar to EP:
For the ſame reaſon, the parallelogram BN is ſimilar to ER,
and BK to EX: Therefore the three parallelograms BM, BN,
BK are ſimilar to the three EP, ER, EX: But the three BM,
BN, BK, are equal and ſimilar [c] to the three which are oppoſite c 24. 11.
to them, and the three EP, ER, EX equal and ſimilar to
the three oppoſite to them: Wherefore the ſolids BGML,
EHPO are contained by the ſame number of ſimilar planes;
and their ſolid angles are equal [d]; and therefore the ſolid d B. 11.
BGML, is ſimilar [a] to the ſolid EHPO: But ſimilar ſolid parallelepipeds
have the triplicate [e] ratio of that which their homologous e 33. 11.
ſides have: Therefore the ſolid BGML has to the
ſolid EHPO the triplicate ratio of that which the ſide BC has
to the homologous ſide EF: But as the ſolid BGML is to the
ſolid EHPO, ſo is [f] the pyramid ABCG to the pyramid DEFH; f 15. 5.
becauſe the pyramids are the ſixth part of the ſolids, ſince the
priſm, which is the half [g] of the ſolid parallelepiped, is triple [h] g 28. 11.
of the pyramid. Wherefore likewiſe the pyramid ABCG has h 7. 12.
to the pyramid DEFH, the triplicate ratio of that which BC
has to the homologous ſide EF. Q. E. D.

Cor. From this it is evident, that ſimilar pyramids which See N.
have multangular baſes, are likewiſe to one another in the triplicate
ratio of their homologous ſides: For they may be divided
into ſimilar pyramids having triangular baſes, becauſe the
ſimilar polygons, which are their baſes, may be divided into
the ſame number of ſimilar triangles homologous to the whole
polygons; therefore as one of the triangular pyramids in the
firſt multangular pyramid is to one of the triangular pyramids
in the other, ſo are all the triangular pyramids in the firſt to all
the triangular pyramids in the other; that is, ſo is the firſt
multangular pyramid to the other: But one triangular pyramid
is to its ſimilar triangular pyramid, in the triplicate ratio of
their homologous ſides; and therefore the firſt multangular pyramid
has to the other, the triplicate ratio of that which one of
the ſides of the firſt has to the homologous ſide of the other.

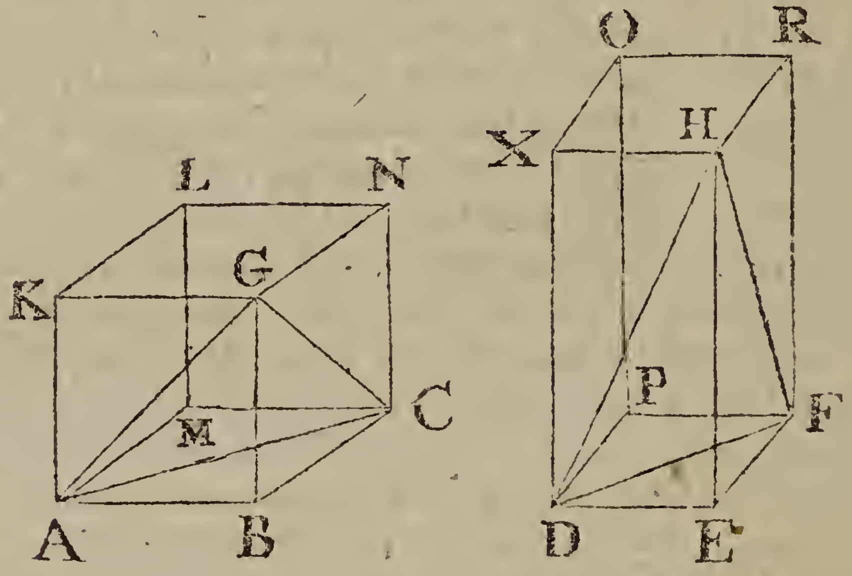
PROP.

PROP. IX. THEOR.

THE baſes and altitudes of equal pyramids having triangular baſes are reciprocally proportional: And triangular pyramids of which the baſes and altitudes are reciprocally proportional, are equal to one another.

Let the pyramids of which the triangles ABC, DEF are the baſes, and which have their vertices in the points G, H, be equal to one another: The baſes and altitudes of the pyramids ABCG, DEFH are reciprocally proportional, viz. the baſe ABC is to the baſe DEF, as the altitude of the pyramid DEFH to the altitude of the pyramid ABCG.

Complete the parallelograms AC, AG, GC, DF, DH, HF; and the ſolid parallelepipeds BGML, EHPO contained by these planes and thoſe oppoſite to them: And becauſe the pyramid ABCG is equal to the pyramid DEFH, and that the ſolid BGML is ſextuple of the pyramid ABCG, and the ſolid EHPO ſextuple of the pyramid DEFH; therefore the ſolid
a 1. Ax. 5. BGML is equal [a] to the ſolid EHPO: But the baſes and altitudes of equal ſolid parallelepipeds are reciprocally propor-
b 34. 11. tional [b]; therefore as the baſe BM to the baſe EP, ſo is the altitude of the ſolid EHPO to the altitude of the ſolid BGML:
c 15. 5. But as the baſe BM to the baſe EP, ſo is [c] the triangle ABC to the triangle DEF; therefore as the triangle ABC to the triangle DEF, ſo is the altitude of the ſolid EHPO to the altitude of the ſolid BGML: But the altitude of the ſolid EHPO is the ſame with the altitude of the pyramid DEFH; and the altitude of the ſolid BGML is the ſame with the altitude of the pyramid



pyramid ABCG: Therefore, as the baſe ABC to the baſe DEF, ſo is the altitude of the pyramid DEFH to the altitude of the pyramid ABCG: Wherefore the baſes and altitudes of the pyramids ABCG, DEFH are reciprocally proportional.

Again, let the baſes and altitudes of the pyramids ABCG, DEFH be reciprocally proportional, viz. the baſe ABC to the baſe DEF, as the altitude of the pyramid DEFH to the altitude of the pyramid ABCG: The pyramid ABCG is equal to the pyramid DEFH.

The ſame conſtruction being made, becauſe as the baſe ABC to the baſe DEF, ſo is the altitude of the pyramid DEFH to the altitude of the pyramid ABCG: And as the baſe ABC to the baſe DEF, ſo is the parallelogram BM to the parallelogram EP; therefore the parallelogram BM is to EP, as the altitude of the pyramid DEFH to the altitude of the pyramid ABCG: But the altitude of the pyramid DEFH is the ſame with the altitude of the ſolid parallelepiped EHPO; and the altitude of the pyramid ABCG is the ſame with the altitude of the ſolid parallelepiped BGML: As, therefore, the baſe BM to the baſe EP, ſo is the altitude of the ſolid parallelepiped EHPO to the altitude of the ſolid parallelepiped BGML. But ſolid parallelepipeds having their baſes and altitudes reciprocally proportional, are equal [b] to one another. Therefore the ſolid parallelepiped BGML is equal to the ſolid parallelepiped EHPO. And the pyramid ABCG is the ſixth part of the ſolid BGML, and the pyramid DEFH is the ſixth part of the ſolid EHPO. Therefore the pyramid ABCG is equal to the pyramid DEFH. Therefore the baſes, &c. Q. E. D. b 34. 11.

PROP. X. THEOR.

EVERY cone is the third part of a cylinder which has the ſame baſe, and is of an equal altitude with it.

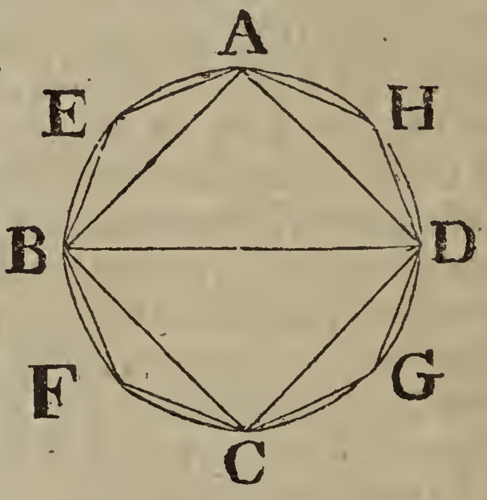
Let a cone have the ſame baſe with a cylinder, viz. the circle ABCD, and the ſame altitude. The cone is the third part of the cylinder; that is, the cylinder is triple of the cone.

If the cylinder be not triple of the cone, it muſt either be greater than the triple, or leſs than it. Firſt, Let it be greater than the triple; and deſcribe the ſquare ABCD in the circle; this ſquare is greater than the half of the circle ABCD *. Upon

---

* As was ſhewn in prop. 2. of this book.

Book XII. Upon the ſquare ABCD erect a priſm of the ſame altitude with the cylinder; this priſm is greater than half of the cylinder; becauſe if a ſquare be deſcribed about the circle, and a priſm erected upon the ſquare, of the ſame altitude with the cylinder, the inſcribed ſquare is half of that circumſcribed; and upon theſe ſquare baſes are erected ſolid parallelepipeds, viz. the priſms of the ſame altitude; therefore the priſm upon the ſquare ABCD is the half of the priſm upon the ſquare deſcribed about the circle: Becauſe they are to one another as their baſes [a]: And the cylinder is leſs than the priſm upon the ſquare deſcribed about the circle ABCD: Therefore the priſm upon the ſquare ABCD of the ſame altitude with the cylinder, is greater than half of the cylinder. Biſect the circumferences AB, BC, CD, DA in the points E, F, G, H; and join AE, EB, BF, FC, CG, GD, DH, HA: Then, each of the triangles AEB, BFC, CGD, DHA is greater than the half of the ſegment of the circle in which it ſtands, as was ſhewn in prop. 2. of this book. Erect priſms upon each of theſe triangles of the ſame altitude with the cylinder; each of theſe priſms is greater than half of the ſegment of the cylinder in which it is; becauſe if, through the points E, F, G, H, parallels be drawn to AB, BC, CD, DA, and parallelograms be completed upon the ſame AB, BC, CD, DA, and ſolid parallelepipeds be erected upon the parallelograms; the priſms upon the triangles AEB, BFC, CGD, DHA are the halves of the ſolid parallelepipeds [b]. And the ſegments of the cylinder which are upon the ſegments of the circle cut off by AB, BC, CD, DA, are leſs than the ſolid parallelepipeds which contain them. Therefore the priſms upon the triangles AEB, BFC, CGD, DHA, are greater than half of the ſegments of the cylinder in which they are; therefore, if each of the circumferences be divided into two equal parts, and ſtraight lines be drawn from the points of diviſion to the extremities of the circumferences, and upon the triangles thus made, priſms be erected of the ſame altitude with the cylinder, and ſo on, there muſt at length remain ſome ſegments of the cylinder which together are leſs [c] than the exceſs of the cylinder above the triple of the cone. Let them be thoſe upon the ſegments of the circle AE, EB, BF, FC,



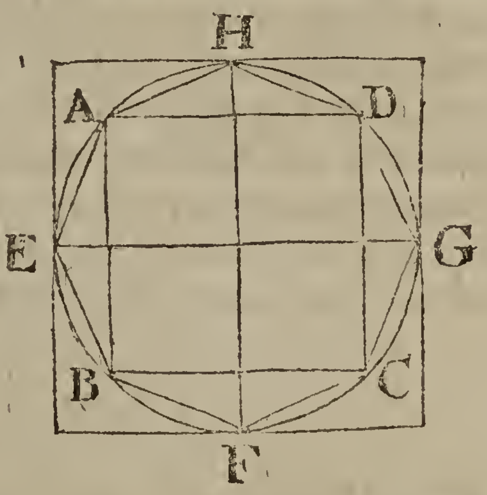
a 32. 11.

b 2. Cor. 7. 12.

c Lemma.

FC, CG, GD, DH, HA. Therefore the reſt of the cylinder, that is, the priſm of which the baſe is the polygon AEBFCGDH, and of which the altitude is the ſame with that of the cylinder, is greater than the triple of the cone: But this priſm is triple[d] of the pyramid upon the ſame baſe, of which the vertex is the ſame with the vertex of the cone; therefore the pyramid upon the baſe AEBFCGDH, having the ſame vertex with the cone, is greater than the cone, of which the baſe is the circle ABCD: But it is alſo leſs, for the pyramid is contained within the cone; which is impoſſible. Nor can the cylinder be leſs than the triple of the cone. Let it be leſs, if poſſible: Therefore, inverſely, the cone is greater than the third part of the cylinder. In the circle ABCD deſcribe a ſquare; this ſquare is greater than the half of the circle: And upon the ſquare ABCD erect a pyramid having the ſame vertex with the cone; this pyramid is greater than the half of the cone; becauſe as was before demonſtrated, if a ſquare be deſcribed about the circle, the ſquare ABCD is the half of it; and if upon theſe ſquares there be erected ſolid parallelepipeds of the ſame altitude with the cone, which are alſo priſms, the priſm upon the ſquare ABCD ſhall be the half of that which is upon the ſquare deſcribed about the circle; for they are to one another as their baſes[e]; as are alſo the third parts of them: Therefore the pyramid, the baſe of which is the ſquare ABCD, is half of the pyramid upon the ſquare deſcribed about the circle: But this laſt pyramid is greater than the cone which it contains; therefore the pyramid upon the ſquare ABCD, having the ſame vertex with the cone, is greater than the half of the cone. Biſect the circumferences AB, BC, CD, DA in the points E, F, G, H, and join AE, EB, BF, FC, CG, GD, DH, HA: Therefore each of the triangles AEB, BFC, CGD, DHA is greater than half of the ſegment of the circle in which it is: Upon each of theſe triangles erect pyramids having the ſame vertex with the cone. Therefore each of theſe pyramids is greater than the half of the ſegment of the cone in which it is, as before was demonſtrated of the priſms and ſegments of the cylinder; and thus dividing each of the circumferences into two equal parts, and joining the

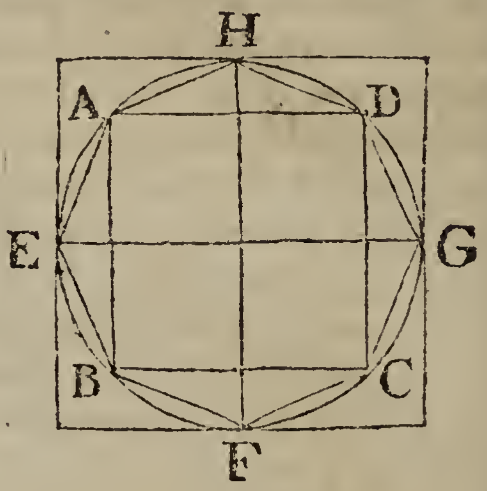
points



[d] 1. Cor. 7. 12.

[e] 32. 11.

points of diviſion and their extremities by ſtraight lines, and upon the triangles erecting pyramids having their vertices the ſame with that of the cone, and ſo on, there muſt at length remain ſome ſegments of the cone, which together ſhall be leſs than the exceſs of the cone, above the third part of the cylinder. Let theſe be the ſegments upon AE, EB, BF, FC, CG, GD, DH, HA. Therefore the reſt of the cone, that is, the pyramid, of which the baſe is the polygon AEBFCGDH, and of which the vertex is the ſame with that of the cone, is greater than the third part of the cylinder. But this pyramid is the third part of the priſm upon the ſame baſe AEBFCGDH, and of the ſame altitude with the cylinder. Therefore this priſm is greater than the cylinder of which the baſe is the circle ABCD. But it is alſo leſs, for it is contained within the cylinder; which is impoſſible. Therefore the cylinder is not leſs than the triple of the cone. And it has been demonſtrated that neither is it greater than the triple. Therefore the cylinder is triple of the cone, or, the cone is the third part of the cylinder. Wherefore every cone, &c. Q. E. D.



PROP. XI. THEOR.

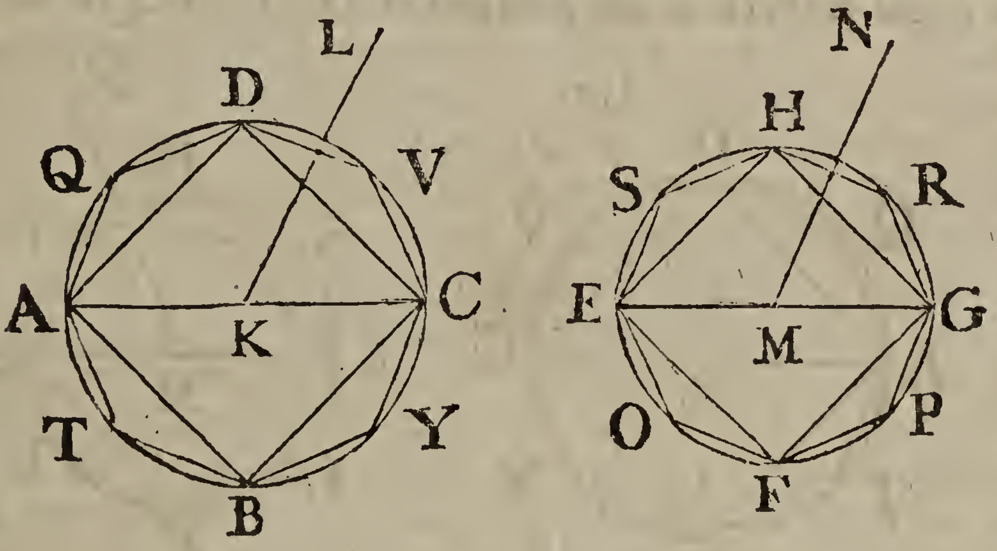
See N.

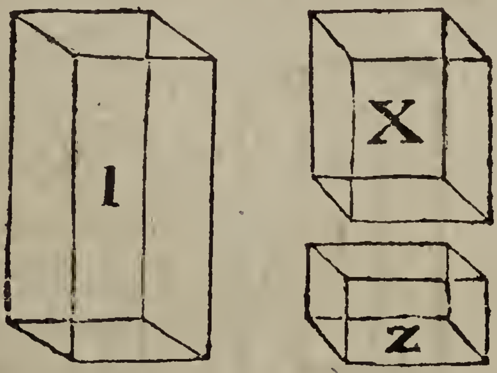
CONES and cylinders of the ſame altitude, are to one another as their baſes.

Let the cones and cylinders, of which the baſes are the circles ABCD, EFGH, and the axes KL, MN, and AC, EG the diameters of their baſes, be of the ſame altitude. As the circle ABCD to the circle EFGH, ſo is the cone AL to the cone EN.

If it be not ſo, let the circle ABCD be to the circle EFGH, as the cone AL to ſome ſolid either leſs than the cone EN, or greater than it. Firſt, let it be to a ſolid leſs than EN, viz. to the ſolid X; and let Z be the ſolid which is equal to the exceſs of the cone EN above the ſolid X; therefore the cone EN is equal to the ſolids X, Z together. In the circle EFGH deſcribe the ſquare EFGH, therefore this ſquare is greater than the half of the circle: Upon the ſquare EFGH erect a pyramid of the ſame altitude with the cone; this pyramid is greater than half of the cone. For, if a ſquare be deſcribed about the circle, and a pyramid be erected upon it, having the ſame vertex with the cone *, the pyramid inſcribed in the cone is half of the pyramid circumſcribed about it, becauſe they are to one another as their baſes [a]: But the cone is leſs than the circumſcribed pyramid; therefore the pyramid of which the baſe is the ſquare EFGH, and its vertex the ſame with that of the cone, is greater than half of the cone: Divide the circumferences EF, FG, GH, HE, each into two equal parts in the points O, P, R, S, and join EO, OF, FP, PG, GR, RH, HS, SE: Therefore each of the triangles EOF, FPG, GRH, HSE is greater than half of the ſegment of the

a 6. 12.





circle in which it is: Upon each of theſe triangles erect a pyramid having the ſame vertex with the cone; each of theſe pyramids is greater than the half of the ſegment of the cone in which it is: And thus dividing each of theſe circumferences into two equal parts, and from the points of diviſion drawing ſtraight lines to the extremities of the circumferences, and upon each of the triangles thus made erecting pyramids having the ſame vertex with the cone, and ſo on, there muſt at length remain ſome ſegments of the cone which are together leſs [b] than the ſolid Z: Let theſe be the ſegments upon EO, OF, FP,

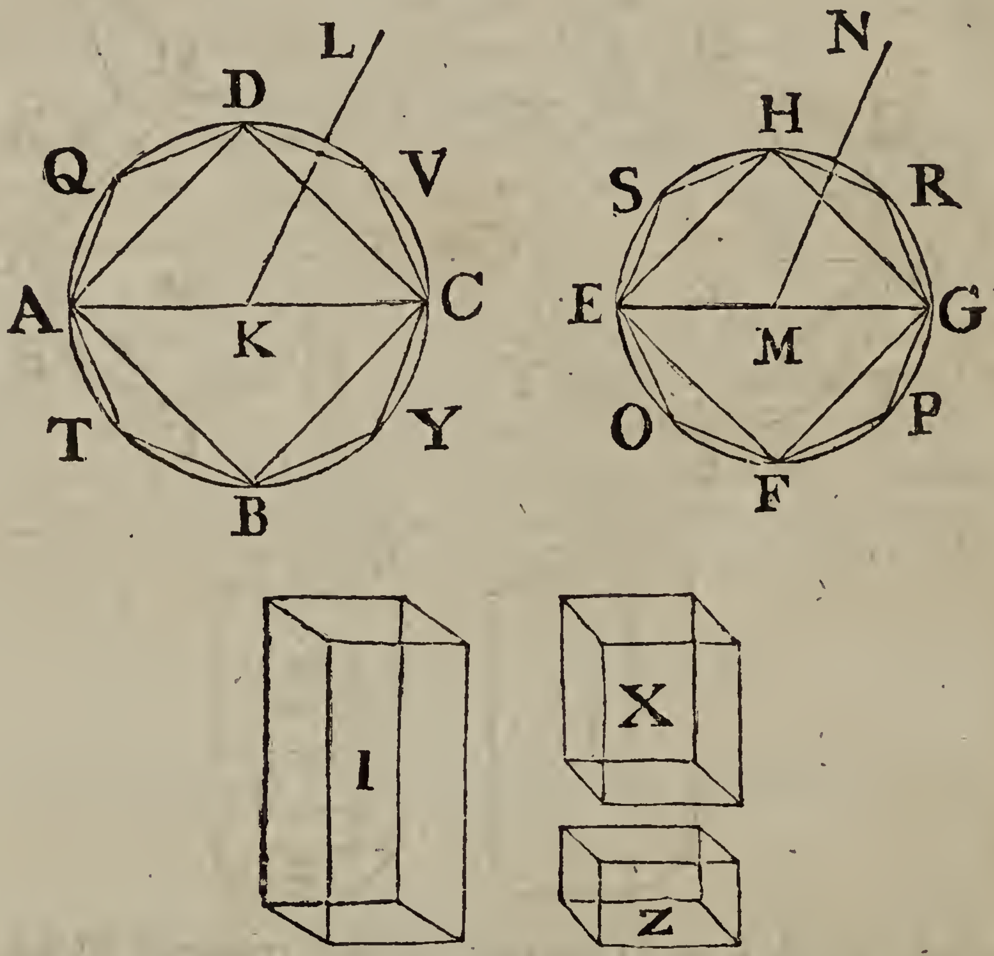
b Lem. 1.

PG,

---

* Vertex is put in place of altitude which is in the Greek, becauſe the pyramid, in what follows, is ſuppoſed to be circumſcribed about the cone, and ſo muſt have the ſame vertex. And the ſame change is made in ſome places following.

Book XII. PG, GR, RH, HS, SE: Therefore the remainder of the cone,
viz. the pyramid of which the baſe is the polygon EOFPGRHS,
and its vertex the ſame with that of the cone, is greater than
the ſolid X: In the circle ABCD deſcribe the polygon
ATBYCVDQ ſimilar to the polygon EOFPGRHS, and upon
it erect a pyramid having the ſame vertex with the cone AL:
a 1. 12. And becauſe as the ſquare of AC is to the ſquare of EG, ſo [a] is
the polygon ATBYCVDQ to the polygon EOFPGRHS; and
b 2. 12. as the ſquare of AC to the ſquare of EG, ſo is [b] the circle
11. 5. ABCD to the circle EFGH; therefore the circle ABCD [c] is to
the circle EFGH, as the polygon ATBYCVDQ to the poly-



gon EOFPGRHS: But as the circle ABCD to the circle EFGH,
d 6. 12. ſo is the cone AL to the ſolid X; and as the polygon
ATBYCVDQ to the polygon EOFPGRHS, ſo is [d] the pyramid of which the baſe is the firſt of theſe polygons, and vertex L, to the pyramid of which the baſe is the other polygon,
and its vertex N: Therefore, as the cone AL to the ſolid X, ſo
is the pyramid of which the baſe is the polygon ATBYCVDQ,
and vertex L, to the pyramid the baſe of which is the polygon
e 14. 5. EOFPGRHS, and vertex N: But the cone AL is greater than
the pyramid contained in it; therefore the ſolid X is greater [e]
than the pyramid in the cone EN. But it is leſs, as was ſhown,
which

which is abſurd: Therefore the circle ABCD is not to the circle EFGH, as the cone AL to any ſolid which is leſs than the cone EN. In the ſame manner it may be demonſtrated that the circle EFGH is not to the circle ABCD, as the cone EN to any ſolid leſs than the cone AL. Nor can the circle ABCD be to the circle EFGH, as the cone AL to any ſolid greater than the cone EN: For, if it be poſſible, let it be ſo to the ſolid I, which is greater than the cone EN: Therefore, by inverſion, as the circle EFGH to the circle ABCD, ſo is the ſolid I to the cone AL: But as the ſolid I to the cone AL, ſo is the cone EN to ſome ſolid, which muſt be leſs[a] than the cone AL, becauſe the ſolid I is greater than the cone EN: Therefore, as the circle EFGH is to the circle ABCD, ſo is the cone EN to a ſolid leſs than the cone AL, which was ſhewn to be impoſſible: Therefore the circle ABCD is not to the circle EFGH, as the cone AL is to any ſolid greater than the cone EN: And it has been demonſtrated that neither is the circle ABCD to the circle EFGH, as the cone AL to any ſolid leſs than the cone EN: Therefore the circle ABCD is to the circle EFGH, as the cone AL to the cone EN: But as the cone is to the cone, ſo[b] is the cylinder to the cylinder, becauſe the cylinders are triple[c] of the cone each to each. Therefore, as the circle ABCD to the circle EFGH, ſo are the cylinders upon them of the ſame altitude. Wherefore cones and cylinders of the ſame altitude are to one another as their baſes. Q. E. D.

a 14. 5.
b 15. 5.
c 10. 12.

PROP. XII. THEOR.

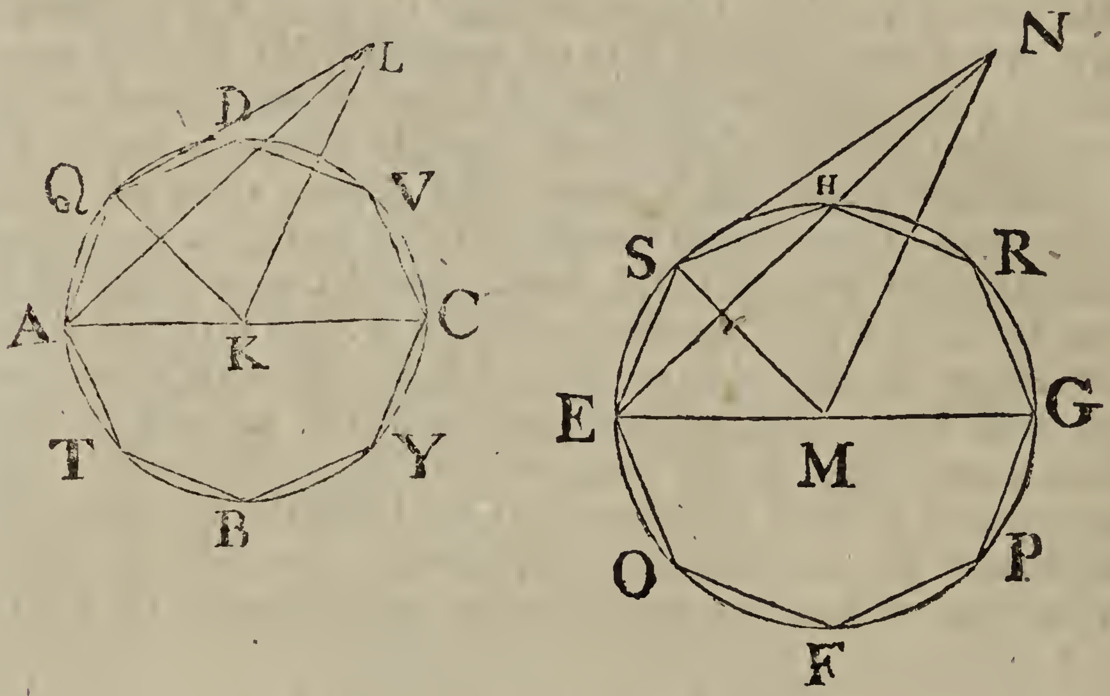
SIMILAR cones and cylinders have to one another the triplicate ratio of that which the diameters of their baſes have. See N.

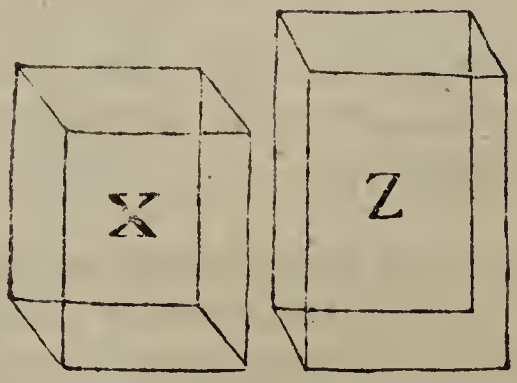
Let the cones and cylinders of which the baſes are the circles ABCD, EFGH, and the diameters of the baſes AC, EG, and KL, MN, the axis of the cones or cylinders, be ſimilar: The cone, of which the baſe is the circle ABCD, and vertex the point L, has to the cone of which the baſe is the circle EFGH, and vertex N, the triplicate ratio of that which AC has to EG.

For if the cone ABCDL has not to the cone EFGHN the triplicate ratio of that which AC has to EG, the cone ABCDL ſhall have the triplicate of that ratio to ſome ſolid which is leſs

or

Book XII. or greater than the cone EFGHN. Firſt, let it have it to a leſs, viz. to the ſolid X. Make the ſame conſtruction as in the preceding propoſition, and it may be demonſtrated the very ſame way as in that propoſition, that the pyramid of which the baſe is the polygon EOFPGRHS, and vertex N, is greater than the ſolid X. Deſcribe alſo in the circle ABCD the polygon ATBYCVDQ ſimilar to the polygon EOFPGRHS, upon which erect a pyramid having the ſame vertex with the cone; and let LAQ be one of the triangles containing the pyramid upon the polygon ATBYCVDQ the vertex of which is L; and let NES be one of the triangles containing the pyramid upon the





polygon EOFPGRHS of which the vertex is N; and join KQ, MS: Becauſe then the cone ABCDL is ſimilar to the cone EFGHN, AC is [a] to EG, as the axis KL to the axis MN; and as AC to EG, ſo [b] is AK to EM; therefore as AK to EM, ſo is KL to MN; and, alternately, AK to KL, as EM to MN: And the right angles AKL, EMN are equal; therefore the ſides about theſe equal angles being proportionals the triangle AKL is ſimilar [c] to the triangle EMN. Again, becauſe AK is to KQ, as EM to MS, and that theſe ſides are about

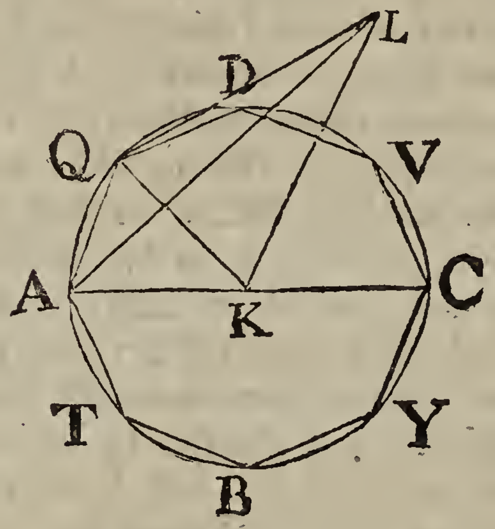
[a] 24. def. 11.

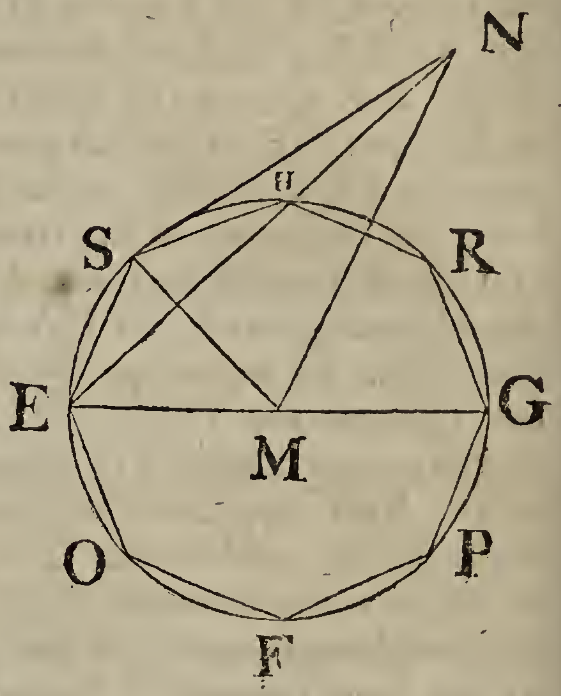
[b] 15. 5.

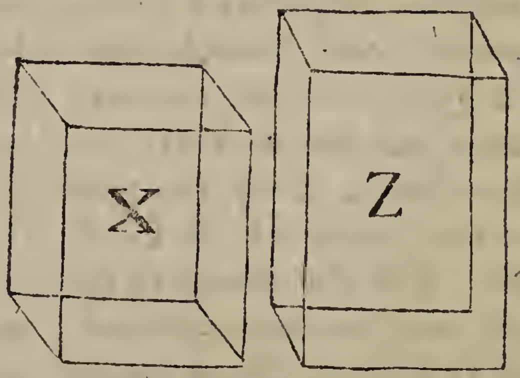
[c] 6. 6.

about equal angles AKQ, EMS, becauſe theſe angles are,
each of them, the ſame part of four right angles at the centres
K, M; therefore the triangle AKQ is ſimilar [a] to the triangle a 6. 6.
EMS: And becauſe it has been ſhown that as AK to KL, ſo
is EM to MN, and that AK is equal to KQ; and EM to MS,
as QK to KL, ſo is SM to MN; and therefore the ſides about
the right angles QKL, SMN being proportionals, the triangle
LKQ is ſimilar to the triangle NMS: And becauſe of the ſimilarity
of the triangles AKL, EMN, as LA is to AK, ſo is
NE to EM; and by the ſimilarity of the triangles AKQ,
EMS, as KA to AQ, ſo ME to ES; ex æquali [b], LA is b 22. 5.
to AQ, as NE to ES. Again, becauſe of the ſimilarity of the
triangles LQK, NSM, as LQ to QK, ſo NS to SM; and
from the ſimilarity of the triangles KAQ, MES, as KQ to
QA, ſo MS to SE; ex æquali [b], LQ is to QA, as NS to SE:
And it was proved that QA is to AL, as SE to EN; therefore,
again, ex æquali, as QL to LA, ſo is SN to NE: Wherefore
the triangles LQA, NSE, having the ſides about all their
angles proportionals, are equiangular [c] and ſimilar to one an- c 5. 6.
other: And therefore the pyramid of which the baſe is the triangle
AKQ, and vertex L, is ſimilar to the pyramid the baſe
of which is the triangle EMS, and vertex N, becauſe their
ſolid angles are equal [d] to one another, and they are contained d B. 11.
by the ſame number of ſimilar planes: But ſimilar pyramids
which have triangular baſes have to one another the triplicate
[e] ratio of that which their homologous ſides have; therefore e 8. 12.
the pyramid AKQL has to the pyramid EMSN the triplicate
ratio of that which AK has to EM. In the ſame manner, if
ſtraight lines be drawn from the points D, V, C, Y, B, T,
to K, and from the points H, R, G, P, F, O to M, and pyramids
be erected upon the triangles having the ſame vertices
with the cones, it may be demonſtrated that each pyramid in
the firſt cone has to each in the other, taking them in the ſame
order, the triplicate ratio of that which the ſide AK has to the
ſide EM; that is, which AC has to EG: But as one antecedent
to its conſequent, ſo are all the antecedents to all the
conſequents [f]; therefore as the pyramid AKQL to the pyra- f 12. 5.
mid EMSN, ſo is the whole pyramid the baſe of which is the
polygon DQATBYCV, and vertex L, to the whole pyramid
of which the baſe is the polygon HSEOFPGR, and vertex N.
Wherefore alſo the firſt of theſe two laſt named pyramids has
to the other the triplicate ratio of that which AC has to EG.
But, by the hypotheſis, the cone of which the baſe is the circle
ABCD, and vertex L, has to the ſolid X, the triplicate
ratio of that which AC has to EG; therefore, as the cone of

Book XII. which the baſe is the circle ABCD, and vertex L, is to the ſolid X, ſo is the pyramid the baſe of which is the polygon DQATBYCV, and vertex L, to the pyramid the baſe of which is the polygon HSEOFPGR and vertex N: But the ſaid cone is greater than the pyramid contained in it, therefore the ſolid
a 14. 5. X is greater [a] than the pyramid, the baſe of which is the polygon HSEOFPGR, and vertex N; but it is alſo leſs; which is impoſſible: Therefore the cone of which the baſe is the circle







ABCD and vertex L, has not to any ſolid which is leſs than the cone of which the baſe is the circle EFGH and vertex N, the triplicate ratio of that which AC has to EG. In the ſame manner it may be demonſtrated that neither has the cone EFGHN to any ſolid which is leſs than the cone ABCDL, the triplicate ratio of that which EG has to AC. Nor can the cone ABCDL have to any ſolid which is greater than the cone EFGHN, the triplicate ratio of that which AC has to EG: For, if it be poſſible, let it have it to a greater, viz. to the ſolid Z: Therefore, inverſely, the ſolid Z has to the cone ABCDL, the triplicate ratio of that which EG has to AC: But as the ſolid Z is to the

a 14. 5.

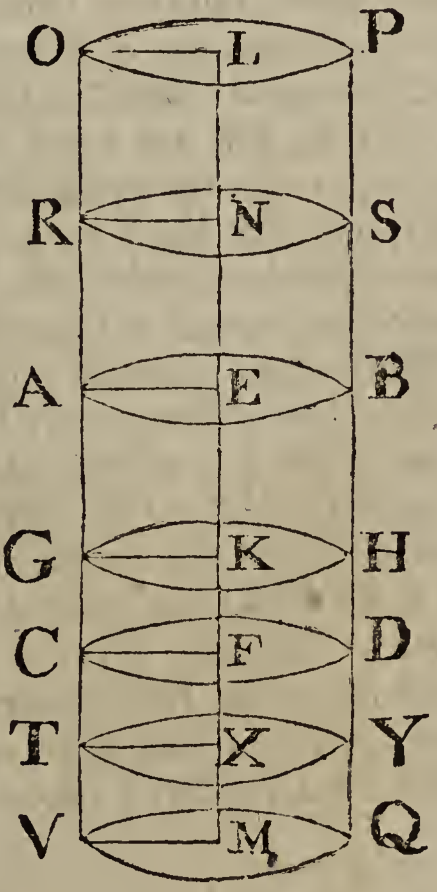
the cone ABCDL, ſo is the cone EFGHN to ſome ſolid, which muſt be leſs [a] than the cone ABCDL, becauſe the ſolid Z is greater than the cone EFGHN: Therefore the cone EFGHN has to a ſolid which is leſs than the cone ABCDL, the triplicate ratio of that which EG has to AC, which was demonſtrated to be impoſſible: Therefore the cone ABCDL has not to any ſolid greater than the cone EFGHN, the triplicate ratio of that which AC has to EG; and it was demonſtrated that it could not have that ratio to any ſolid leſs than the cone EFGHN: Therefore the cone ABCDL has to the cone EFGHN, the triplicate ratio of that which AC has to EG: But as the cone is to the cone, ſo [b] the cylinder to the cylinder; for every cone is the third part of the cylinder upon the ſame baſe, and of the ſame altitude: Therefore alſo the cylinder has to the cylinder, the triplicate ratio of that which AC has to EG: Wherefore ſimilar cones, &c. Q. E. D.

b 15. 5.

PROP. XIII. THEOR.

See N.

IF a cylinder be cut by a plane parallel to its oppoſite planes, or baſes; it divides the cylinder into two cylinders, one of which is to the other as the axis of the firſt to the axis of the other.



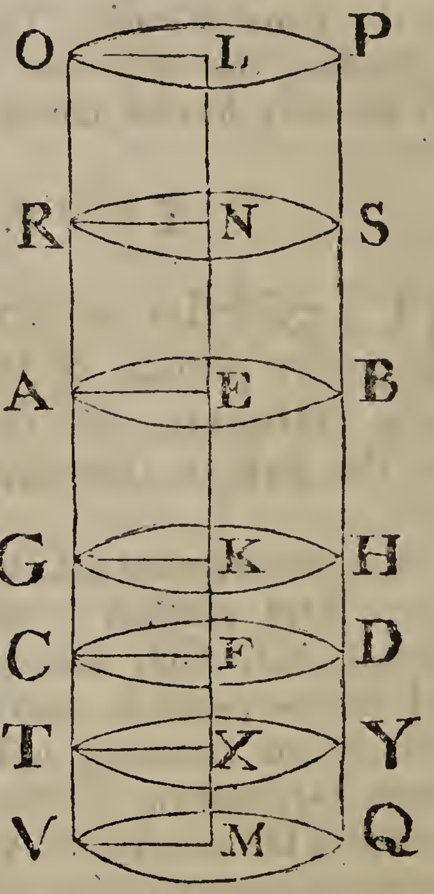
a 16. 11.

Let the cylinder AD be cut by the plane GH parallel to the oppoſite planes AB, CD, meeting the axis EF in the point K, and let the line GH be the common ſection of the plane GH and the ſurface of the cylinder AD: Let AEFC be the parallelogram in any poſition of it, by the revolution of which about the ſtraight line EF the cylinder AD is deſcribed; and let GK be the common ſection of the plane GH, and the plane AEFC: And becauſe the parallel planes AB, GH are cut by the plane AEKG, AE, KG, their common ſections with it are parallel [a]; wherefore AK is a parallelogram, and GK equal to EA the ſtraight line from the centre of the circle AB: For the ſame reaſon, each of the ſtraight lines drawn from the point K to the line GH may be proved to be equal to thoſe which are drawn from the centre of the circle AB to its circumference, and are therefore all equal to one another. Therefore the line GH is the circumference of a circle[a] of which the centre is the point K: Therefore the plane GH divides the cylinder AD into the cylinders AH, GD; for they are the ſame which would be deſcribed by the revolution of the parallelograms AK, GF, about the ſtraight lines EK, KF: And it is to be ſhown that the cylinder AH is to the cylinder HC, as the axis EK to the axis KF.

a 15. def. 1.

Produce the axis EF both ways; and take any number of ſtraight lines EN, NL, each equal to EK; and any number FX, XM, each equal to FK; and let planes parallel to AB, CD paſs through the points L, N, X, M: Therefore the common ſections of theſe planes with the cylinder produced are circles the centres of which are the points L, N, X, M, as was proved of the plane GH; and theſe planes cut off the cylinders, PR, RB, DT, TQ: And becauſe the axes LN, NE, EK are all equal: therefore the cylinders PR, RB, BG are[b] to one another as their baſes; but their baſes are equal, and therefore the cylinders PR, RB, BG are equal: And becauſe the axes LN, NE, EK are equal to one another, as alſo the cylinders PR, RB, BG, and that there are as many axes as cylinders; therefore, whatever multiple the axis KL is of the axis KE, the ſame multiple is the cylinder PG of the cylinder GB: For the ſame reaſon whatever multiple the axis MK is of the axis KF, the ſame multiple is the cylinder QG of the cylinder GD: And if the axis KL be equal to the axis KM the cylinder PG is equal to the cylinder GQ; and if the axis KL be greater than the axis KM the cylinder PG is greater than the cylinder QG; and if leſs, leſs: Since therefore there are four magnitudes, viz. the axes EK, KF, and the cylinders BG, GD, and that of the axis EK and cylinder BG there has been taken any equimultiples whatever, viz. the

b 11. 12.



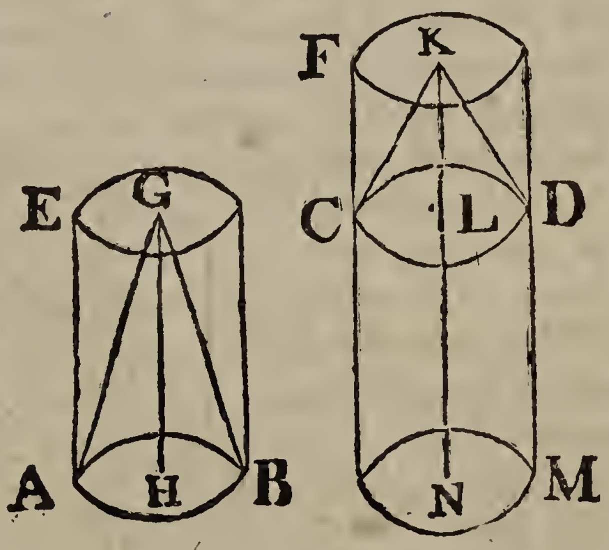
axis KL and cylinder PG; and of the axis KF and cylinder GD, any equimultiples whatever, viz. the axis KM and cylinder GQ; and it has been demonſtrated, if the axis KL be greater than the axis KM, the cylinder PG is greater than the cylinder GQ; and if equal, equal; and if leſs, leſs: Therefore [d] the axis EK is to the axis KF, as the cylinder BG to the cylinder GD. Wherefore, if a cylinder, &c. Q. E. D. d 5. def. 5.

PROP. XIV. THEOR.

CONES and cylinders upon equal baſes are to one another as their altitudes.

Let the cylinders EB, FD be upon the equal baſes AB, CD: As the cylinder EB to the cylinder FD, ſo is the axis GH to the axis KL.

Produce the axis KL to the point N, and make LN equal to the axis GH, and let CM be a cylinder of which the baſe is CD, and axis LN, and becauſe the cylinders EB, CM have the ſame altitude, they are to one another as their baſes [a]: But a 11. 12. their baſes are equal, therefore alſo the cylinders EB, CM are equal. And becauſe the cylinder FM is cut by the plane CD parallel to its oppoſite planes, as the cylinder CM to the cylinder FD, ſo is [b] the axis LN to the axis KL. But b 13. 12. the cylinder CM is equal to the cylinder EB, and the axis LN to the axis GH: Therefore as the cylinder EB to the cylinder FD, ſo is the axis GH to the axis KL: And as the cylinder EB to the cylinder FD, ſo is [c] the cone ABG to c 15. 5. the cone CDK, becauſe the cylinders are triple [d] of the cones: d 10. 12. Therefore alſo the axis GH is to the axis KL, as the cone ABG to the cone CDK, and the cylinder EB to the cylinder FD. Wherefore cones, &c. Q. E. D.

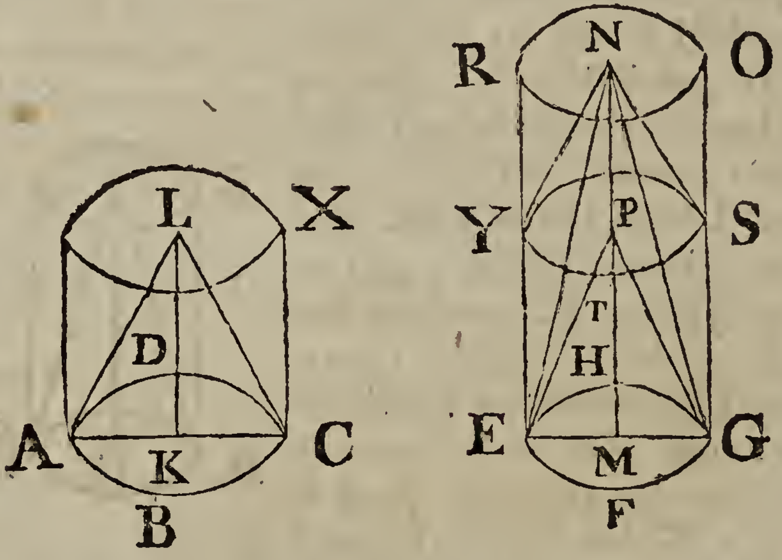


PROP. XV. THEOR.

See N. THE baſes and altitudes of equal cones and cylinders, are reciprocally proportional; and if the baſes and altitudes be reciprocally proportional, the cones and cylinders are equal to one another.

Let the circles ABCD, EFGH, the diameters of which are AC, EG be the baſes, and KL, MN the axis, as alſo the altitudes, of equal cones and cylinders; and let ALC, ENG be the cones, and AX, EO the cylinders: The baſes and altitudes of the cylinders AX, EO are reciprocally proportional; that is, as the baſe ABCD to the baſe EFGH, ſo is the altitude MN to the altitude KL.

Either the altitude MN is equal to the altitude KL, or theſe
altitudes are not equal. Firſt, let them be equal; and the
cylinders AX, EO being alſo equal, and cones and cylinders
a 11. 12. of the ſame altitude being to one another as their baſes[a], there-
b A. 5. fore the baſe ABCD is equal[b] to the baſe EFGH; and as the
baſe ABCD is to the baſe EFGH, ſo is the altitude MN to
the altitude KL.



But let the altitudes KL, MN, be unequal, and MN the greater of the two, and from MN take MP equal to KL, and through the point P cut the cylinder EO by the plane TYS parallel to the oppoſite planes of the circles EFGH, RO; therefore the common ſection of the plane TYS and the cylinder EO is a circle, and conſequently ES is a cylinder, the baſe of which is the
circle EFGH, and altitude MP: And becauſe the cylinder AX
is equal to the cylinder EO, as AX is to the cylinder ES, ſo
c 7. 5. [c] is the cylinder EO to the ſame ES. But as the cylinder AX
to the cylinder ES, ſo[a] is the baſe ABCD to the baſe EFGH;
for the cylinders AX, ES are of the ſame altitude; and as the
d 13. 12. cylinder EO to the cylinder ES, ſo[d] is the altitude MN to
the altitude MP, becauſe the cylinder EO is cut by the plane

TYS parallel to its oppoſite planes. Therefore as the baſe ABCD to the baſe EFGH, ſo is the altitude MN to the altitude MP: But MP is equal to the altitude KL; wherefore as the baſe ABCD to the baſe EFGH, ſo is the altitude MN to the altitude KL; that is, the baſes and altitudes of the equal cylinders AX, EO are reciprocally proportional.

But let the baſes and altitudes of the cylinders AX, EO, be reciprocally proportional, viz. the baſe ABCD to the baſe EFGH, as the altitude MN to the altitude KL: The cylinder AX is equal to the cylinder EO.

Firſt, let the baſe ABCD, be equal to the baſe EFGH; then becauſe as the baſe ABCD is to the baſe EFGH, ſo is the altitude MN to the altitude KL; MN is equal [b] to KL, and therefore the cylinder AX is equal [a] to the cylinder EO. [b] A. 5. [a] 11. 12.

But let the baſes ABCD, EFGH be unequal, and let ABCD be the greater; and becauſe, as ABCD is to the baſe EFGH, ſo is the altitude MN to the altitude KL; therefore MN is greater [b] than KL. Then, the ſame conſtruction being made as before, becauſe as the baſe ABCD to the baſe EFGH, ſo is the altitude MN to the altitude KL; and becauſe the altitude KL is equal to the altitude MP; therefore the baſe ABCD is [a] to the baſe EFGH, as the cylinder AX to the cylinder ES; and as the altitude MN to the altitude MP or KL, ſo is the cylinder EO to the cylinder ES: Therefore the cylinder AX is to the cylinder ES, as the cylinder EO is to the ſame ES: Whence the cylinder AX is equal to the cylinder EO: and the ſame reaſoning holds in cones. Q. E. D.

PROP. XVI. PROB.

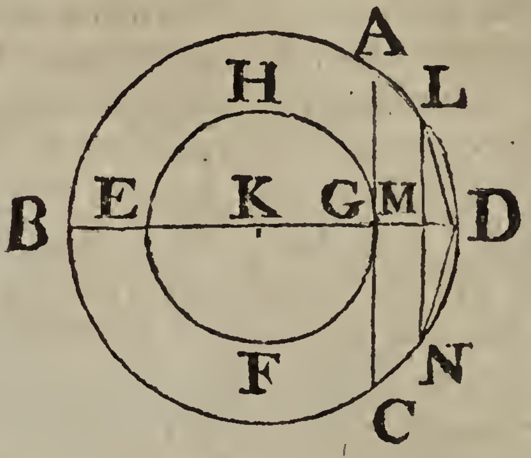
TO deſcribe in the greater of two circles that have the ſame centre, a polygon of an even number of equal ſides, that ſhall not meet the leſſer circle.

Let ABCD, EFGH be two given circles having the ſame centre K: It is required to inſcribe in the greater circle ABCD a polygon of an even number of equal ſides, that ſhall not meet the leſſer circle.

Through the centre K draw the ſtraight line BD, and from the point G, where it meets the circumferences of the leſſer circle,

a 16. 3.
b Lemma.

circle, draw GA at right angles to BD, and produce it to C; therefore AC touches [a] the circle EFGH: Then, if the circumference BAD be biſected, and the half of it be again biſected, and ſo on, there muſt at length remain a circumference leſs than AD: Let this be LD: and from the point L draw LM perpendicular to BD, and produce it to N; and join LD, DN. Therefore LD is equal to DN; and becauſe LN is parallel to AC and that AC touches the circle EFGH; therefore LN does not meet the circle EFGH: And much leſs ſhall the ſtraight lines LD, DN meet the circle EFGH:



So that if ſtraight lines equal to LD be applied in the circle ABCD from the point L around to N, there ſhall be deſcribed in the circle a polygon of an even number of equal ſides not meeting the leſſer circle. Which was to be done.

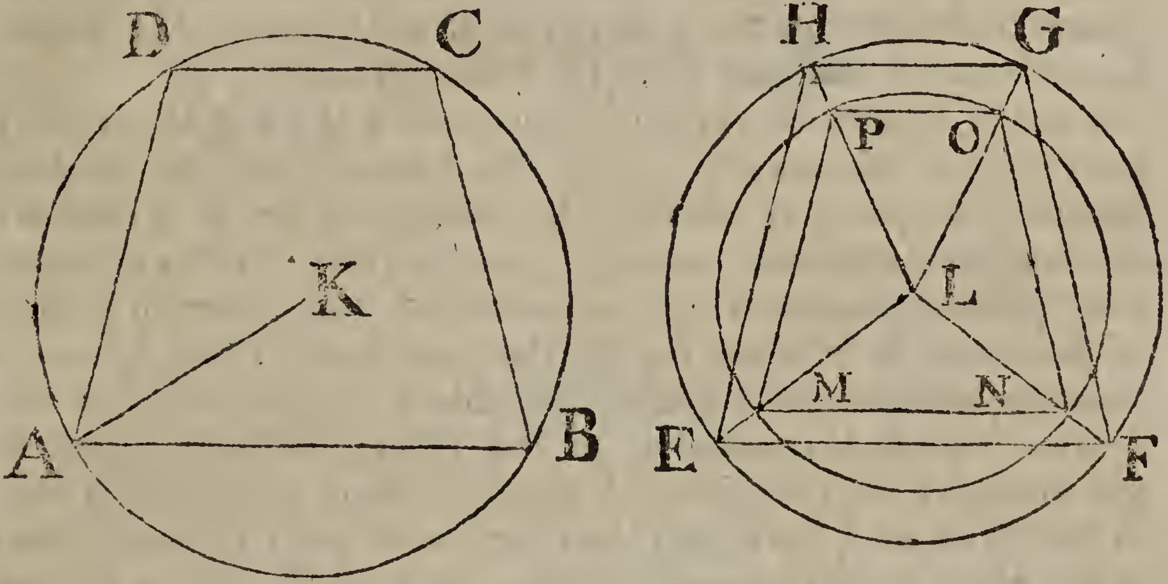
LEMMA II.

If two trapeziums ABCD, EFGH be inſcribed in the circles, the centres of which are the points K, L; and if the ſides AB, DC be parallel, as alſo EF, HG; and the other four ſides AD, BC, EH, FG, be all equal to one another; but the ſide AB greater than EF, and DC greater than HG. The ſtraight line KA from the centre of the circle in which the greater ſides are, is greater than the ſtraight line LE drawn from the centre to the circumference of the other circle.

a 28. 3.

If it be poſſible, let KA be not greater than LE; then KA muſt be either equal to it, or leſs. Firſt, let KA be equal to LE: Therefore, becauſe in two equal circles AD, BC in the one are equal to EH, FG in the other, the circumferences AD, BC are equal [a] to the circumferences EH, FG; but becauſe the ſtraight lines AB, DC are reſpectively greater than EF, GH, the circumferences AB, DC are greater than EF, HG: Therefore the whole circumference ABCD is greater than the whole EFGH: but it is alſo equal to it, which is impoſſible: Therefore the ſtraight line KA is not equal to LE.

But let KA be leſs than LE, and make LM equal to KA, and from the centre L, and diſtance LM deſcribe the circle MNOP, meeting the ſtraight lines LE, LF, LG, LH, in M, N, O, P; and join MN, NO, OP, PM which are reſpectively parallel [a] to, and leſs than EF, FG, GH, HE: Then becauſe EH is greater than MP, AD is greater than MP; and a 2. 6.



the circles ABCD, MNOP are equal; therefore the circumference AD is greater than MP; for the ſame reaſon, the circumference BC is greater than NO; and becauſe the ſtraight line AB is greater than EF which is greater than MN, much more is AB greater than MN: Therefore the circumference AB is greater than MN; and, for the ſame reaſon, the circumference DC is greater than PO: Therefore the whole circumference ABCD is greater than the whole MNOP; but it is likewiſe equal to it, which is impoſſible: Therefore KA is not leſs than LE; nor is it equal to it; the ſtraight line KA muſt therefore be greater than LE. Q. E. D.

Cor. And if there be an iſoſceles triangle the ſides of which are equal to AD, BC, but its baſe leſs than AB the greater of the two ſides AB, DC; the ſtraight line KA may, in the ſame manner, be demonſtrated to be greater than the ſtraight line drawn from the centre to the circumference of the circle deſcribed about the triangle.

PROP.

PROP. XVII. PROB.

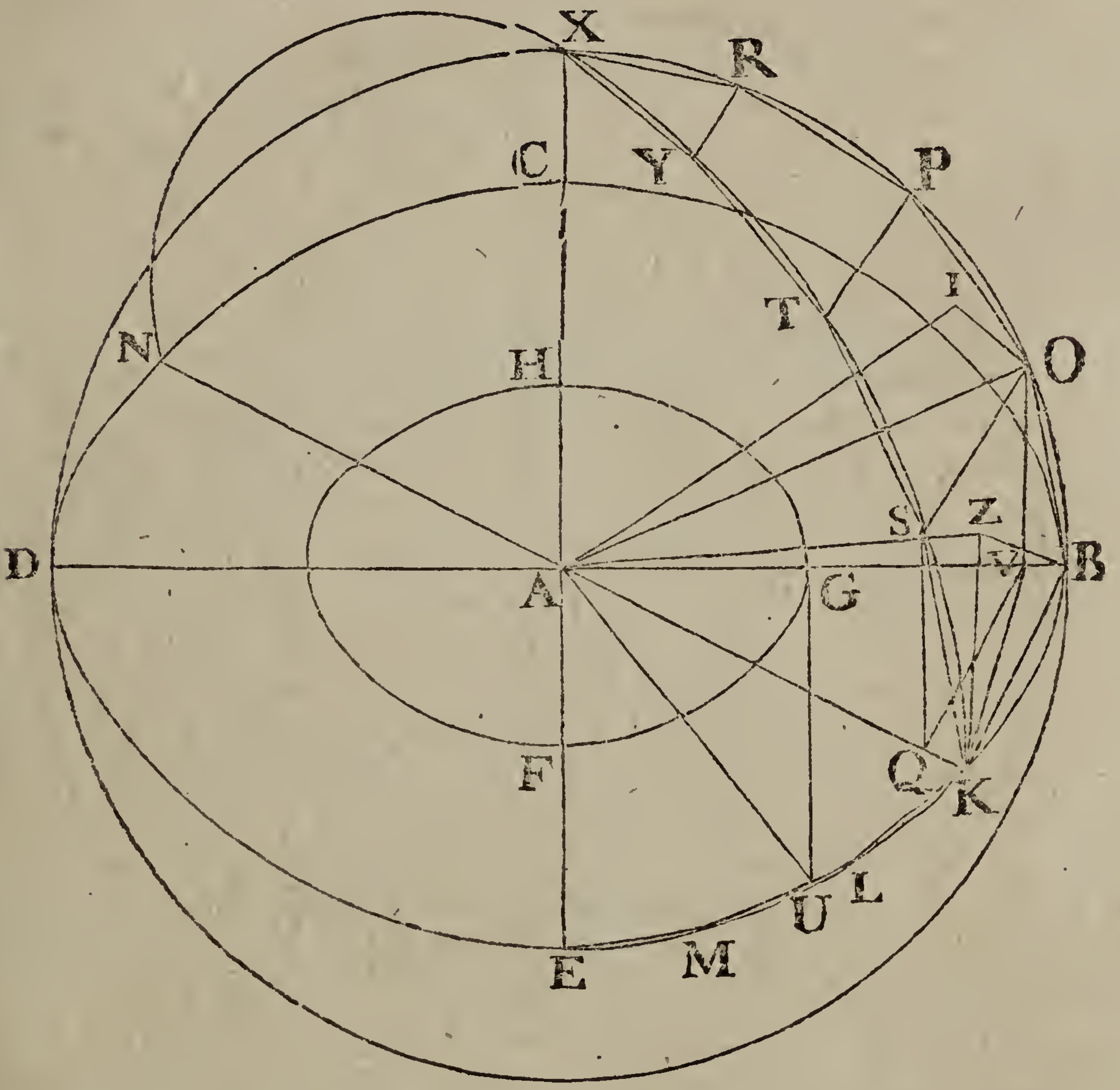
See N. TO deſcribe in the greater of two ſpheres which have the ſame centre, a ſolid polyhedron, the ſuperficies of which ſhall not meet the leſſer ſphere.

Let there be two ſpheres about the ſame centre A; it is required to deſcribe in the greater a ſolid polyhedron, the ſuperficies of which ſhall not meet the leſſer ſphere.

Let the ſpheres be cut by a plane paſſing through the centre; the common ſections of it with the ſpheres ſhall be circles; becauſe the ſphere is deſcribed by the revolution of a ſemicircle about the diameter remaining unmoveable; ſo that in whatever poſition the ſemicircle be conceived, the common ſection of the plane in which it is with the ſuperficies of the ſphere is the circumference of a circle: and this is a great circle of the ſphere, becauſe the diameter of the ſphere, which is likewiſe
a 15. 3. the diameter of the circle, is greater [a] than any ſtraight line in the circle or ſphere: Let then the circle made by the ſection of the plane with the greater ſphere be BCDE, and with the leſſer ſphere be FGH; and draw the two diameters BD, CE, at right angles to one another; and in BCDE, the greater of
b 16. 12. the two circles, deſcribe [b] a polygon of an even number of equal ſides not meeting the leſſer circle FGH; and let its ſides, in BE the fourth part of the circle, be BK, KL, LM, ME; join KA and produce it to N; and from A draw AX at right angles to the plane of the circle BCDE meeting the ſuperficies of the ſphere in the point X; and let planes paſs through AX and each of the ſtraight lines BD, KN, which, from what has been ſaid, ſhall produce great circles on the ſuperficies of the ſphere, and let BXD, KXN be the ſemicircles thus made upon the diameters BD, KN: Therefore, becauſe XA is at right angles to the plane of the circle BCDE, every plane which
c 18. 11. paſſes through XA is at right [c] angles to the plane of the circle BCDE; wherefore the ſemicircles BXD, KXN are at right angles to that plane: And becauſe the ſemicircles BED, BXD, KXN, upon the equal diameters BD, KN, are equal to one another, their halves BE, BX, KX, are equal to one another: Therefore, as many ſides of the polygon as are in BE, ſo many there are in BX, KX equal to the ſides BK, KL, LM, ME: Let theſe polygons be deſcribed, and their ſides be BO, OP, PR, RX; KS, ST, TY, YX, and join OS,

OS, PT, RY; and from the points O, S draw OV, SQ perpendiculars to AB, AK: And becauſe the plane BOXD is at right angles to the plane BCDE, and in one of them BOXD, OV is drawn perpendicular to AB the common ſection of the planes, therefore OV is perpendicular [a] to the plane BCDE: For the ſame reaſon SQ is perpendicular to the ſame plane, becauſe the plane KSXN is at right angles to the plane BCDE. Join VQ; and becauſe in the equal ſemicircles BXD, KXN the

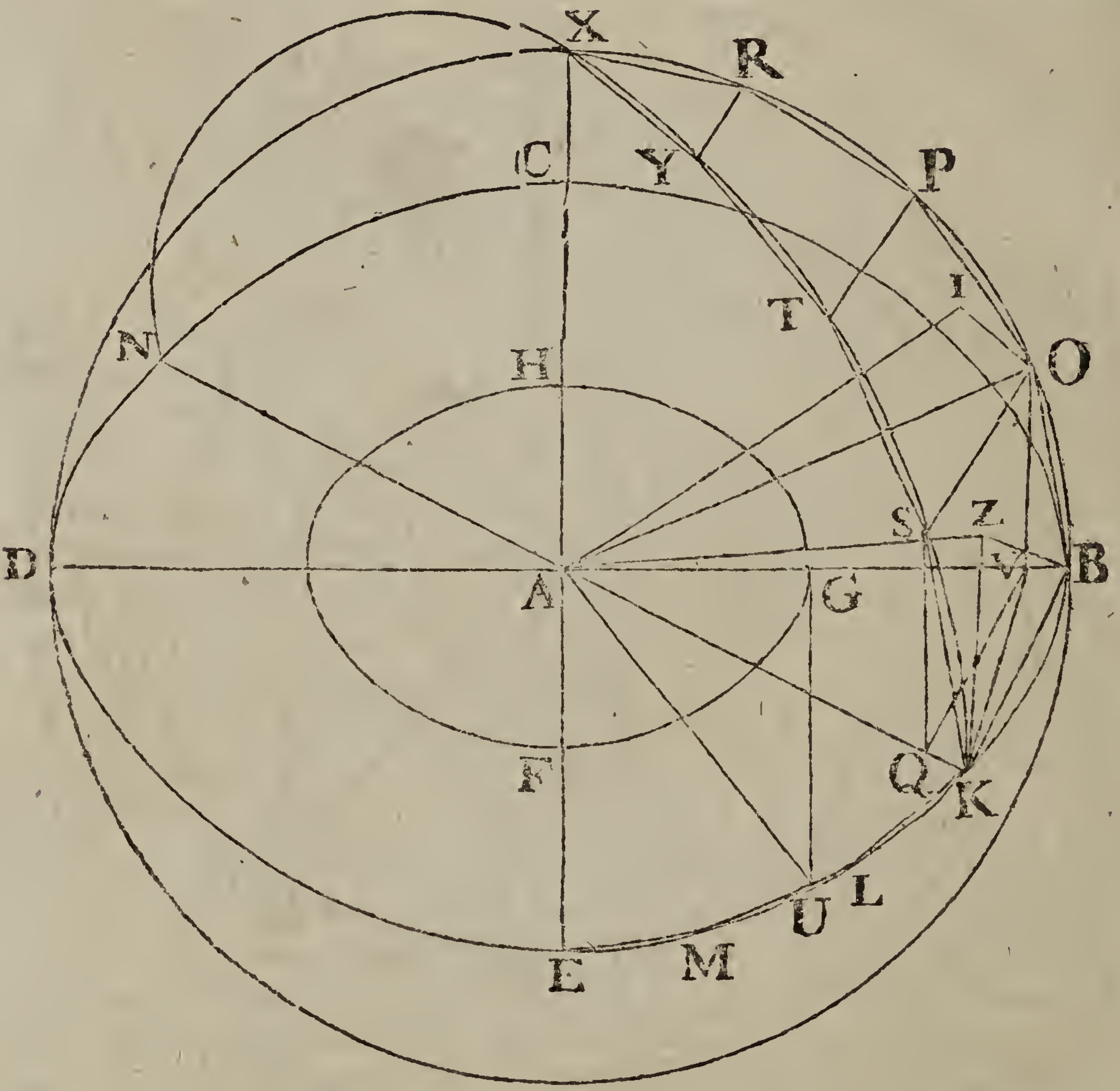
a 4. def. 11.



circumferences BO, KS are equal, and OV, SQ are perpendicular to their diameters, therefore [d] OV is equal to SQ, and BV equal to KQ. But the whole BA is equal to the whole KA, therefore the remainder VA is equal to the remainder QA: As therefore BV is to VA, ſo is KQ to QA, wherefore VQ is parallel [e] to BK: And becauſe OV, SQ are each of them at right angles to the plane of the circle BCDE, OV is parallel [f] to SQ; and it has been proved that it is alſo equal to it; therefore QV, SO are equal and parallel [g]: And becauſe QV is parallel to SO, and alſo to KB; OS is parallel [h] to BK; and therefore BO, KS which join them are in the ſame plane in

d 26. 1.
e 2. 6.
f 6. 11.
g 33. 1.
h 9. 11.

Book XII. in which theſe parallels are, and the quadrilateral figure KBOS is in one plane: And if PB, TK be joined, and perpendiculars be drawn from the points P, T to the ſtraight lines AB, AK it may be demonſtrated that TP is parallel to KB in the very ſame way that SO was ſhown to be parallel to the ſame KB; a 9. 11. wherefore [a] TP is parallel to SO, and the quadrilateral figure SOPT is in one plane: For the ſame reaſon the quadrilateral b 2. 11. TPRY is in one plane: And the figure YRX is alſo in one plane [b].



Therefore, if from the points, O, S, P, T, R, Y there be drawn ſtraight lines to the point A, there ſhall be formed a ſolid polyhedron between the circumferences BX, KX compoſed of pyramids the baſes of which are the quadrilaterals KBOS, SOPT, TPRY, and the triangle YRX, and of which the common vertex is the point A: And if the ſame conſtruction be made upon each of the ſides KL, LM, ME, as has been done upon BK, and the like be done alſo in the other three quadrants, and in the other hemiſphere; there ſhall be formed a ſolid polyhedron deſcribed in the ſphere, compoſed of pyramids, the baſes of which are the aforeſaid quadrilateral figures, and the triangle YRX, and thoſe formed in the like manner in the reſt of the ſphere, the common vertex of them all being the point A: And the ſuperficies of this ſolid polyhedron does not meet the leſſer ſphere in which is the circle FGH: For, from the point A draw[a] AZ perpendicular to the plane of the quadrilateral KBOS meeting it in Z, and join BZ, ZK: And becauſe AZ is perpendicular to the plane KBOS, it makes right angles with every ſtraight line meeting it in that plane; therefore AZ is perpendicular to BZ and ZK: And becauſe AB is equal to AK, and that the ſquares of AZ, ZB, are equal to the ſquare of AB; and the ſquares of AZ, ZK to the ſquare of AK[b]: therefore the ſquares of AZ, ZB are equal to the ſquares of AZ, ZK: Take from theſe equals the ſquare of AZ, the remaining ſquare of BZ is equal to the remaining ſquare of ZK; and therefore the ſtraight line BZ is equal to ZK: In the like manner it may be demonſtrated, that the ſtraight lines drawn from the point Z to the points O, S are equal to BZ or ZK: Therefore the circle deſcribed from the centre Z, and diſtance ZB ſhall paſs through the points K, O, S, and KBOS ſhall be a quadrilateral figure in the circle: And becauſe KB is greater than QV, and QV equal to SO, therefore KB is greater than SO: But KB is equal to each of the ſtraight lines BO, KS; wherefore each of the circumferences cut off by KB, BO, KS is greater than that cut off by OS; and theſe three circumferences, together with a fourth equal to one of them, are greater than the ſame three together with that cut off by OS; that is, than the whole circumference of the circle; therefore the circumference ſubtended by KB is greater than the fourth part of the whole circumference of the circle KBOS, and conſequently the angle BZK at the centre is greater than a right angle: And becauſe the angle BZK is obtuſe, the ſquare of BK is greater[c] than the ſquares of BZ, ZK; that is, greater than twice the ſquare of BZ. Join KV, and becauſe in the triangles KBV, OBV, KB, BV are equal to OB, BV, and that they contain equal angles; the angle KVB is equal[d] to the angle OVB: And OVB is a right angle; therefore alſo KVB is a right angle: And becauſe BD is leſs than twice DV, the rectangle contained by DB, BV is leſs than twice the rectangle DVB; that is[e], the ſquare of KB is leſs than twice the ſquare of KV: But the ſquare of KB is greater than twice the ſquare of BZ; therefore the ſquare of KV is greater

[a] 11. 11.

[b] 47. 1.

[c] 12. 2.

[d] 4. 1.

[e] 8. 6.

Book XII. greater than the ſquare of BZ: And becauſe BA is equal to AK, and that the ſquares of BZ, ZA are equal together to the ſquare of BA, and the ſquares of KV, VA to the ſquare of AK; therefore the ſquares of BZ, ZA are equal to the ſquares of KV, VA; and of theſe the ſquare of KV is greater than the ſquare of BZ; therefore the ſquare of VA is leſs than the ſquare of ZA, and the ſtraight line AZ greater than VA: Much more then is AZ greater than AG; becauſe, in the preceding propoſition, it was ſhown that KV falls without the circle FGH: And AZ is perpendicular to the plane KBOS, and is therefore the ſhorteſt of all the ſtraight lines that can be drawn from A, the centre of the ſphere to that plane. Therefore the plane KBOS, does not meet the leſſer ſphere.

And that the other planes between the quadrants BX, KX fall without the leſſer ſphere, is thus demonſtrated: From the point A draw AI perpendicular to the plane of the quadrilateral SOPT, and join IO; and, as was demonſtrated of the plane KBOS and the point Z, in the ſame way it may be ſhown that the point I is the centre of a circle deſcribed about SOPT: and that OS is greater than PT; and PT was ſhown to be parallel to OS: Therefore, becauſe the two trapeziums KBOS, SOPT inſcribed in circles have their ſides BK, OS parallel, as alſo OS, PT; and their other ſides BO, KS, OP, ST all equal to one another, and that BK is greater than OS, and OS greater than PT, therefore the ſtraight line ZB is greater [a] than IO. Join AO which will be equal to AB; and becauſe AIO, AZB are right angles, the ſquares of AI, IO are equal to the ſquare of AO or of AB; that is, to the ſquares of AZ, ZB; and the ſquare of ZB is greater than the ſquare of IO, therefore the ſquare of AZ is leſs than the ſquare of AI; and the ſtraight line AZ leſs than the ſtraight line AI: And it was proved that AZ is greater than AG; much more then is AI greater than AG: Therefore the plane SOPT falls wholly without the leſſer ſphere: In the ſame manner it may be demonſtrated that the plane TPRY falls without the ſame ſphere, as alſo the triangle YRX, viz. by the Cor. of 2d Lemma. And after the ſame way it may be demonſtrated that all the planes which contain the ſolid polyhedron, fall without the leſſer ſphere. Therefore in the greater of two ſpheres which have the ſame centre, a ſolid polyhedron is deſcribed, the ſuperficies of which does not meet the leſſer ſphere. Which was to be done.

[a] 2. Lem 12.

But the ſtraight line AZ may be demonſtrated to be greater than AG otherwiſe, and in a ſhorter manner, without the help of Prop. 16. as follows. From the point G draw GU at right angles to AG and join AU. If then the circumference BE be biſected, and its half again biſected, and ſo on, there will at length be left a circumference leſs than the circumference which is ſubtended by a ſtraight line equal to GU inſcribed in the circle BCDE: Let this be the circumference KB: Therefore the ſtraight line KB is leſs than GU: And becauſe the angle BZK is obtuſe, as was proved in the preceding, therefore BK is greater than BZ: But GU is greater than BK; much more then is GU greater than BZ, and the ſquare of GU than the ſquare of BZ; and AU is equal to AB; therefore the ſquare of AU, that is, the ſquares of AG, GU are equal to the ſquare of AB, that is, to the ſquares of AZ, ZB; but the ſquare of BZ is leſs than the ſquare of GU; therefore the ſquare of AZ is greater than the ſquare of AG, and the ſtraight line AZ conſequently greater than the ſtraight line AG.

Cor. And if in the leſſer ſphere there be deſcribed a ſolid polyhedron by drawing ſtraight lines betwixt the points in which the ſtraight lines from the centre of the ſphere drawn to all the angles of the ſolid polyhedron in the greater ſphere meet the ſuperficies of the leſſer; in the ſame order in which are joined the points in which the ſame lines from the centre meet the ſuperficies of the greater ſphere; the ſolid polyhedron in the ſphere BCDE has to this other ſolid polyhedron the triplicate ratio of that which the diameter of the ſphere BCDE has to the diameter of the other ſphere: For if theſe two ſolids be divided into the ſame number of pyramids, and in the ſame order; the pyramids ſhall be ſimilar to one another, each to each: Becauſe they have the ſolid angles at their common vertex, the centre of the ſphere, the ſame in each pyramid, and their other ſolid angle at the baſes equal to one another, each to each [a], becauſe they are contained by three plane angles equal each to each; and the pyramids are contained by the ſame number of ſimilar planes; and are therefore ſimilar [b] to one another, each to each: But ſimilar pyramids have to one another the triplicate [c] ratio of their homologous ſides. Therefore the pyramid of which the baſe is the quadrilateral KBOS, and vertex A, has to the pyramid in the other ſphere of the ſame order, the triplicate ratio of their homologous ſides; that is, of that ratio, which AB from the centre of the greater ſphere has to the ſtraight line from the ſame centre to the

a B. 11.

b 11. Def. 11.

c Cor. 8. 12.

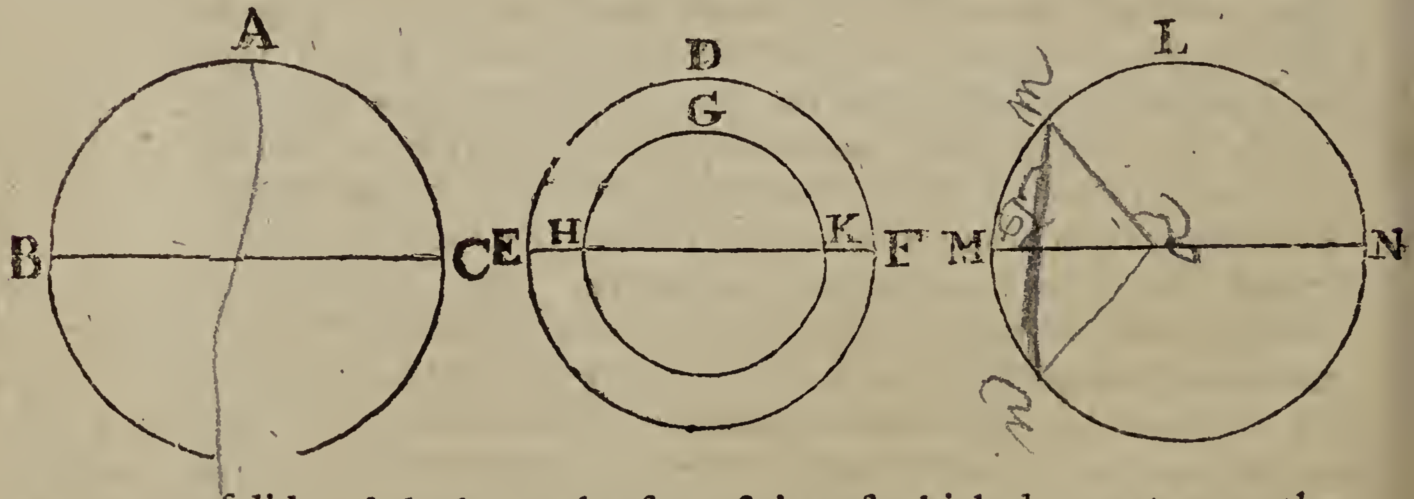
Book XII the ſuperficies of the leſſer ſphere. And in like manner, each pyramid in the greater ſphere has to each of the ſame order in the leſſer, the triplicate ratio of that which AB has to the ſemidiameter of the leſſer ſphere. And as one antecedent is to its conſequent, ſo are all the antecedents to all the conſequents. Wherefore the whole ſolid polyhedron in the greater ſphere has to the whole ſolid polyhedron in the other, the triplicate ratio of that which AB the ſemidiameter of the firſt has to the ſemidiameter of the other; that is, which the diameter BD of the greater has to the diameter of the other ſphere.

PROP. XVIII. THEOR.

SPHERES have to one another the triplicate ratio of that which their diameters have.

Let ABC, DEF be two ſpheres of which the diameters are BC, EF. The ſphere ABC has to the ſphere DEF the triplicate ratio of that which BC has to EF.

For, if it has not, the ſphere ABC ſhall have to a ſphere either leſs or greater than DEF, the triplicate ratio of that which BC has to EF. Firſt, let it have that ratio to a leſs, viz. to the ſphere GHK; and let the ſphere DEF have the ſame centre with GHK; and in the greater ſphere DEF deſcribe [a] a 17. 12.



a ſolid polyhedron, the ſuperficies of which does not meet the leſſer ſphere GHK; and in the ſphere ABC deſcribe another ſimilar to that in the ſphere DEF: Therefore the ſolid polyhedron in the ſphere ABC has to the ſolid polyhedron in the b Cor. 17. 12. ſphere DEF, the triplicate ratio [b] of that which BC has to EF. But the ſphere ABC has to the ſphere GHK, the triplicate ratio of that which BC has to EF; therefore, as the ſphere ABC to the ſphere GHK, ſo is the ſaid polyhedron in the ſphere ABC to the ſolid polyhedron in the ſphere DEF: But the ſphere ABC is greater than the ſolid polyhedron in it; therefore [c] alſo the ſphere GHK is greater than the ſolid polyhedron in the ſphere DEF: But it is alſo leſs, becauſe it is contained within it, which is impoſſible: Therefore the ſphere ABC has not to any ſphere leſs than DEF, the triplicate ratio of that which BC has to EF. In the ſame manner, it may be demonſtrated, that the ſphere DEF has not to any ſphere leſs than ABC, the triplicate ratio of that which EF has to BC. Nor can the ſphere ABC have to any ſphere greater than DEF, the triplicate ratio of that which BC has to EF: For, if it can, let it have that ratio to a greater ſphere LMN: Therefore, by inverſion, the ſphere LMN has to the ſphere ABC, the triplicate ratio of that which the diameter EF has to the diameter BC. But as the ſphere LMN to ABC, ſo is the ſphere DEF to ſome ſphere, which muſt be leſs [c] than the ſphere ABC, becauſe the ſphere LMN is greater than the ſphere DEF: Therefore the ſphere DEF has to a ſphere leſs than ABC the triplicate ratio of that which EF has to BC; which was ſhewn to be impoſſible: Therefore the ſphere ABC has not to any ſphere greater than DEF the triplicate ratio of that which BC has to EF: And it was demonſtrated, that neither has it that ratio to any ſphere leſs than DEF. Therefore the ſphere ABC has to the ſphere DEF, the triplicate ratio of that which BC has to EF. Q. E. D.

[c] 14. 5.

FINIS.

NOTES

CRITICAL AND GEOMETRICAL

CONTAINING

An Account of thoſe things in which this Edition
differs from the Greek text; and the Reaſons of
the Alterations which have been made. As alſo
Obſervations on ſome of the Propoſitions.

BY ROBERT SIMSON, M. D.

Emeritus Profeſſor of Mathematics in the Univerſity of Glaſgow.

EDINBURGH:

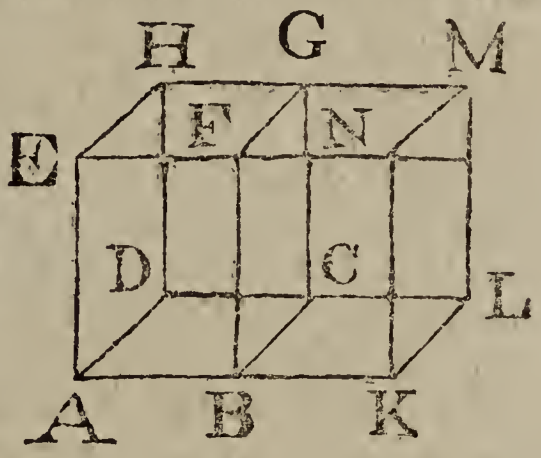
Printed for F. Wingrave, London; and E. Balfour, Edinburgh.

M,DCC,XCIV.

NOTES, &c.

DEFINITION I. BOOK I.

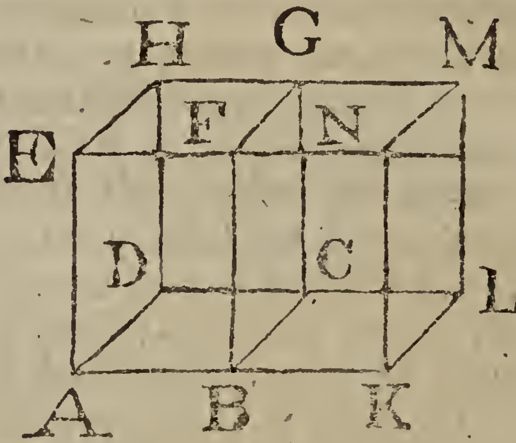
IT is neceſſary to conſider a ſolid, that is, a magnitude which has length, breadth, and thickneſs, in order to underſtand aright the definitions of a point, line, and ſuperficies; for theſe all ariſe from a ſolid, and exiſt in it: The boundary, or boundaries which contain a ſolid are called ſuperficies, or the boundary which is common to two ſolids which are contiguous, or which divides one ſolid into two contiguous parts, is called a ſuperficies: Thus, if BCGF be one of the boundaries which contain the ſolid ABCDEFGH, or which is the common boundary of this ſolid, and the ſolid BKLCFNMG, and is therefore in the one as well as the other ſolid, is called a ſuperficies, and has no thickneſs: For, if it have any, this thickneſs muſt either be a part of the thickneſs of the ſolid AG, or the ſolid BM, or a part of the thickneſs of each of them. It cannot be a part of the thickneſs of the ſolid BM; becauſe if this ſolid be removed from the ſolid AG, the ſuperficies BCGF, the boundary of the ſolid AG, remains ſtill the ſame as it was. Nor can it be a part of the thickneſs of the ſolid AG; becauſe, if this be removed from the ſolid BM, the ſuperficies BCGF, the boundary of the ſolid BM does nevertheleſs remain, therefore the ſuperficies BCGF has no thickneſs, but only length and breadth.



The boundary of a ſuperficies is called a line, or a line is the common boundary of two ſuperficies that are contiguous, or which divides one ſuperficies into two contiguous parts: Thus, if BC be one of the boundaries which contain the ſuperficies ABCD, or which is the common boundary of this ſuperficies, and of the ſuperficies KBCL which is contiguous to it, this boundary BC is called a line, and has no breadth: For if it have any, this muſt be part either of the breadth of the ſuperficies ABCD, or of the ſuperficies KBCL or part of each of them. It is not part of the breadth of the ſuperficies KBCL; for, if this ſuperficies be removed from the ſuperficies ABCD, the

the line BC which is the boundary of the ſuperficies ABCD remains the ſame as it was: Nor can the breadth that BC is ſuppoſed to have, be a part of the breadth of the ſuperficies ABCD; becauſe, if this be removed from the ſuperficies, KBCL, the line BC which is the boundary of the ſuperficies KBCL does nevertherleſs remain: Therefore the line BC has no breadth: And becauſe the line BC is in a ſuperficies, and that a ſuperficies has no thickneſs, as was ſhewn; therefore a line has neither breadth nor thickneſs, but only length.

The boundary of a line is called a point, or a point is the common boundary or extremity of two lines that are contiguous: Thus, if B be the extremity of the line AB, or the common extremity of the two lines AB, KB, this extremity is called a point, and has no length: For if it have any, this length muſt either be part of the length of the line AB, or of the line KB. It is not part of the length of KB; for if the line KB be removed from AB, the point B which is the extremity of the line AB remains the ſame as it was: Nor is it part of the length of the line AB; for, if AB be removed from the line KB, the point B, which is the extremity of the line KB, does nevertheleſs remain: Therefore the point B has no length: And becauſe a point is in a line, and a line has neither breadth nor thickneſs, therefore a point has no length, breadth, nor thickneſs. And in this manner the definitions of a point, line, and ſuperficies, are to be underſtood.



DEF. VII. B. I.

Inſtead of this definition as it is in the Greek copies, a more diſtinct one is given from a property of a plane ſuperficies, which is manifeſtly ſuppoſed in the elements, viz. that a ſtraight line drawn from any point in a plane to any other in it, is wholly in that plane.

DEF. VIII. B. I.

It ſeems that he who made this definition deſigned that it ſhould comprehend not only a plane angle contained by two ſtraight lines, but likewiſe the angle which ſome conceive to be made by a ſtraight line and a curve, or by two curve lines, which meet one another in a plane: But, tho' the meaning of the words επ' ευϑειας, that is, in a ſtraight line, or in the ſame direction, be plain, when two ſtraight lines are ſaid to be in a ſtraight line, it does not appear what ought to be underſtood by theſe words, when a ſtraight line and a curve, or two curve lines, are ſaid to be in the ſame direction; at leaſt it cannot be explained in this place; which makes it probable that this definition, and that of the angle of a ſegment, and what is ſaid of the angle of a ſemicircle, and the angles of ſegments, in the 16. and 31. propoſitions of Book 3. are the additions of ſome leſs ſkilful editor: On which account, eſpecially ſince they are quite uſeleſs, theſe definitions are diſtinguiſhed from the reſt by inverted double commas.

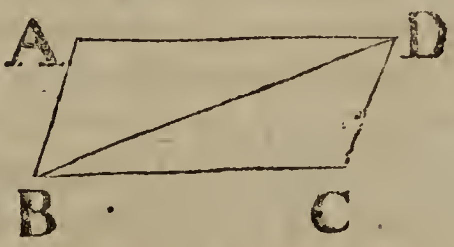
DEF. XVII. B. I.

The words, "which alſo divides the circle into two equal "parts," are added at the end of this definition in all the copies, but are now left out as not belonging to the definition, being only a corollary from it. Proclus demonſtrates it by conceiving one of the parts into which the diameter divides the circle, to be applied to the other; for it is plain they muſt coincide, elſe the ſtraight lines from the centre to the circumference would not be all equal: The ſame thing is eaſily deduced from the 31. prop. of Book 3. and the 24. of the ſame; from the firſt of which it follows that ſemicircles are ſimilar ſegments of a circle: And from the other, that they are equal to one another.

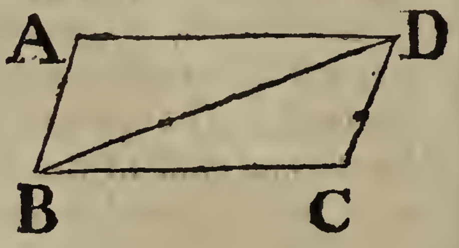
DEF. XXXIII. B. I.

This definition has one condition more than is neceſſary; becauſe every quadrilateral figure which has its oppoſite ſides equal to one another, has likewiſe its oppoſite angles equal; and on the contrary.

Let ABCD be a quadrilateral figure of which the oppoſite ſides AB, CD are equal to one another; as alſo AD and BC: Join BD; the two ſides AD, DB are equal to the two CB, BD, and the baſe AB is equal to the baſe CD; therefore by prop. 8. of Book 1. the angle ADB is equal to the angle CBD; and by prop. 4. B. 1. the angle BAD is equal to the angle DCB, and ABD to BDC; and therefore alſo the angle ADC is equal to the angle ABC.



And if the angle BAD be equal to the oppoſite angle BCD, and the angle ABC to ADC; the oppoſite ſides are equal: Becauſe, by prop. 32. B. 1. all the angles of the quadrilateral figure ABCD are together equal to four right angles, and the two angles BAD, ADC are together equal to the two angles BCD, ABC: Wherefore BAD, ADC are the half of all the four angles; that is, BAD and ADC are equal to two right angles: And therefore AB, CD are parallels by prop. 28. B. 1. In the ſame manner AD, BC are parallels: Therefore ABCD is a parallelogram, and its oppoſite ſides are equal by 34. prop. B. 1.



PROP. VII. B. I.

There are two caſes of this propoſition, one of which is not in the Greek text, but is as neceſſary as the other: And that the caſe left out has been formerly in the text, appears plainly from this, that the ſecond part of prop. 5. which is neceſſary to the demonſtration of this caſe, can be of no uſe at all in the elements, or any where elſe, but in this demonſtration; becauſe the ſecond part of prop. 5. clearly follows from the firſt part, and prop. 13. B. 1. This part muſt therefore have been added to prop. 5. upon account of ſome propoſition betwixt the 5. and 13. but none of theſe ſtand in need of it except the 7. propoſition, on account of which it has been added: Beſides, the tranſlation from the Arabic has this caſe explicitly demonſtrated. And Proclus acknowledges that the ſecond part of prop. 5. was added upon account of prop. 7. but gives a ridiculous reaſon for it, "that it might afford an anſwer to objections made "againſt the 7." as if the caſe of the 7. which is left out, were, as he expreſsly makes it, an objection againſt the propoſition itſelf. Whoever is curious may read what Proclus ſays of this in his commentary on the 5. and 7. propoſitions; for it is not worth while to relate his trifles at full length.

It was thought proper to change the enunciation of this 7. prop. ſo as to preſerve the very ſame meaning; the literal tranſlation from the Greek being extremely harſh, and difficult to be underſtood by beginners.

PROP. XI. B. I.

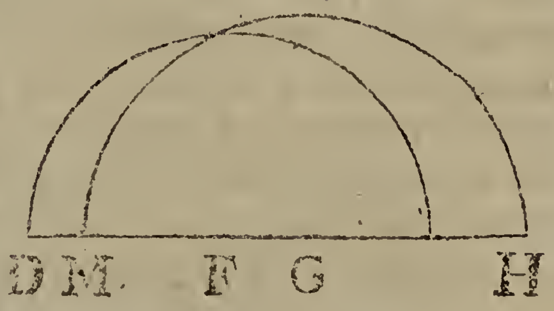
A corollary is added to this propoſition, which is neceſſary to Prop. 1. B. 11. and otherwiſe.

PROP. XX. and XXI. B. I.

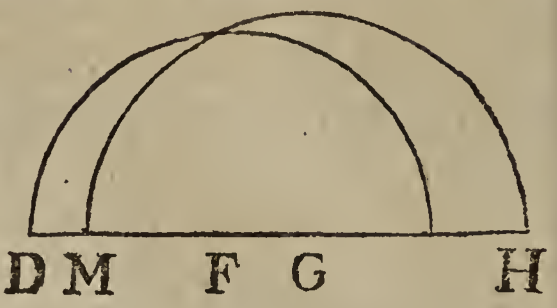
Proclus, in his commentary relates, that the Epicureans derided this propoſition, as being manifeſt even to aſſes, and needing no demonſtration; and his anſwer is, that though the truth of it be manifeſt to our ſenſes, yet it is ſcience which muſt give the reaſon why two ſides of a triangle are greater than the third: But the right anſwer to this objection againſt this and the 21ſt, and ſome other plain propoſitions, is, that the number of axioms ought not to be encreaſed without neceſſity, as it muſt be if theſe propoſitions be not demonſtrated. Monſ. Clairault, in the preface to his elements of geometry, publiſhed in French at Paris, anno 1741, ſays, That Euclid has been at the pains to prove, that the two ſides of a triangle which is included within another are together leſs than the two ſides of the triangle which includes it; but he has forgot to add this condition, viz. that the triangles muſt be upon the ſame baſe; becauſe, unleſs this be added, the ſides of the included triangle may be greater than the ſides of the triangle which includes it, in any ratio which is leſs than that of two to one, as Pappus Alexandrinus has demonſtrated in Prop. 3. B. 3. of his mathematical collections.

PROP. XXII. B. I.

Some authors blame Euclid becauſe he does not demonſtrate, that the two circles made uſe of in the conſtruction of this problem muſt cut one another: But this is very plain from the determination he has given, viz. that any two of the ſtraight lines DF, FG, GH muſt be greater than the third: For who is ſo dull, tho' only beginning to learn the elements, as not to perceive that the circle deſcribed from the centre F, at the diſtance FD, muſt meet FH betwixt F and H, becauſe FD is leſs than FH; and that for the like reaſon, the circle deſcribed from the centre G, at the diſtance GH or GM, muſt meet DG betwixt D and

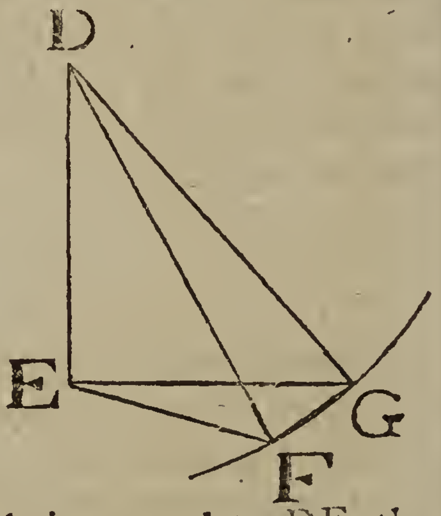


and G; and that theſe circles muſt meet one another, becauſe FD and GH are together greater than FG? And this determination is eaſier to be underſtood than that which Mr Thomas Simpſon derives from it, and puts inſtead of Euclid's, in the 49th page of his elements of geometry, that he may ſupply the omiſſion he blames Euclid for, which determination is, that any of the three ſtraight lines muſt be leſs than the ſum, but greater than the difference of the other two: From this he ſhews the circles muſt meet one another, in one caſe: and ſays, that it may be proved after the ſame manner in any other caſe: But the ſtraight line GM which he bids take from GF may be greater than it, as in the figure here annexed; in which caſe his demonſtration muſt be changed into another.



PROP. XXIV. B. I.

To this is added, "of the two ſides DE, DF, let DE be "that which is not greater than the other;" that is, take that ſide of the two DE, DF which is not greater than the other, in order to make with it the angle EDG equal to BAC; becauſe without this reſtriction, there might be three different caſes of the propoſition, as Campanus and others make.



Mr Thomas Simpſon, in p. 262. of the ſecond edition of his elements of geometry, printed anno 1760, obſerves in his notes, that it ought to have been ſhown, that the point F falls below the line EG; this probably Euclid omitted, as it is very eaſy to perceive that DG being equal to DF, the point G is in the circumference of a circle deſcribed from the centre D at the diſtance DF, and muſt be in that part of it which is above the ſtraight line EF, becauſe DG falls above DF, the angle EDG being greater than the angle EDF.

PROP. XXIX. B. I.

The propoſition which is uſually called the 5th poſtulate, or 11th axiom, by ſome the 12th, on which this 29th depends, has given a great deal to do, both to ancient and modern geometers: It ſeems not to be properly placed among the Axioms, as indeed, it is not ſelf-evident; but it may be demonſtrated thus:

DEFINITION I.

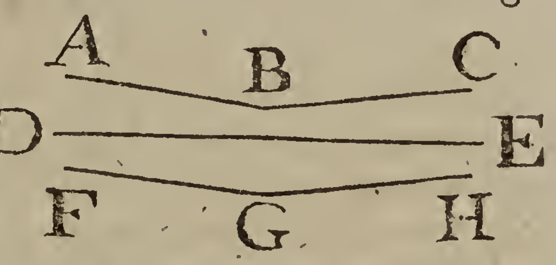
The diſtance of a point from a ſtraight line, is the perpendicular drawn to it from the point.

DEF. 2.

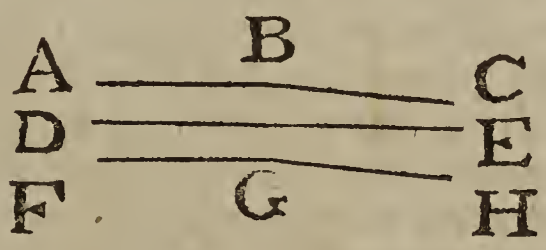
One ſtraight line is ſaid to go nearer to, or further from, another ſtraight line, when the diſtances of the points of the firſt from the other ſtraight line become leſs or greater than they were; and two ſtraight lines, are ſaid to keep the ſame diſtance from one another, when the diſtance of the points of one of them from the other is always the ſame.

AXIOM.

A ſtraight line cannot firſt come nearer to another ſtraight line, and then go further from it, before it cuts it; and, in like manner, a ſtraight line cannot go further from another ſtraight line, and then come nearer to it; nor can a ſtraight line keep the ſame diſtance from another ſtraight line, and then come nearer to it, or go further from it; for a ſtraight line keeps always the ſame direction.



For example, the ſtraight line ABC cannot firſt come nearer to the ſtraight line DE, as from the point A to the point B, and then, from the point B to the point C, go further from the ſame DE: And in like manner, the ſtraight line FGH cannot go further from DE, as from F to G, and then from G to H, come nearer to the ſame DE: And ſo in the laſt caſe, as in fig. 2.

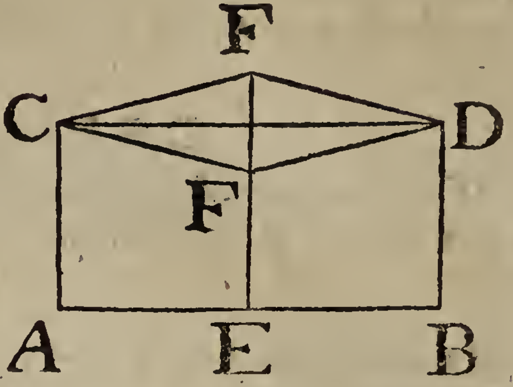


See the figure above.

PROP. I.

If two equal ſtraight lines AC, BD, be each at right angles to the ſame ſtraight line AB: If the points C, D be joined by the ſtraight line CD, the ſtraight line EF drawn from any point E in AB unto CD, at right angles to AB, ſhall be equal to AC, or BD.

If EF be not equal to AC, one of them muſt be greater than the other; let AC be the greater; then, becauſe FE is leſs than CA, the ſtraight line CFD is nearer to the ſtraight line AB at the point F than at the point C, that is, CF comes nearer to AB from the point C to F: But becauſe DB is greater than FE the ſtraight line CFD is further from AB at the point D than at F, that is, FD goes further from AB from F to D: Therefore the ſtraight line CFD firſt comes nearer to the ſtraight line AB, and then goes further from it, before it cuts it; which is impoſſible. If FE be ſaid to be greater than CA, or DB, the ſtraight line CFD firſt goes further from the ſtraight line AB, and then comes nearer to it; which is alſo impoſſible. Therefore FE is not unequal to AC, that is, it is equal to it.

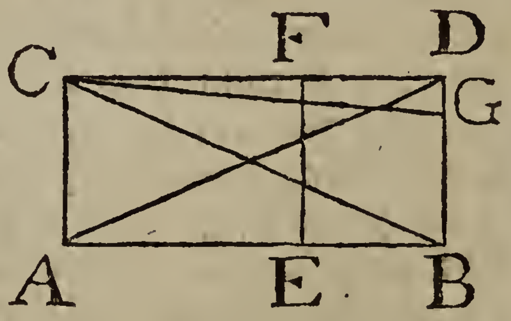


PROP. 2.

If two equal ſtraight lines AC, BD be each at right angles to the ſame ſtraight line AB; the ſtraight line CD which joins their extremities makes right angles with AC and BD.

Join AD, BC; and becauſe, in the triangles CAB, DBA, CA, AB are equal to DB, BA, and the angle CAB equal to the angle DBA; the baſe BC is equal[a] to the baſe AD: And in the triangles ACD, BDC, AC, CD are equal to BD, DC, and the baſe AD is equal to the baſe BC: Therefore the angle ACD is equal[b] to the angle BDC: From any point E in AB draw EF unto CD, at right angles to AB; therefore by Prop. 1. EF is equal to AC, or BD; wherefore, as has been juſt now ſhown, the angle ACF is equal to the angle EFC: In the ſame manner, the angle BDF is equal to the angle EFD; but the angles ACD, BDC are equal; therefore the angles EFC and EFD are equal, and right angles[c]; wherefore alſo the angles ACD, BDC are right angles.

[a] 4. 1.
[b] 8. 1.
[c] 10. def. 1.



Cor. Hence, if two ſtraight lines AB, CD be at right angles to the ſame ſtraight line AC, and if betwixt them a ſtraight line BD be drawn at right angles to either of them, as to AB; then BD is equal to AC, and BDC is a right angle.

If AC be not equal to BD, take BG equal to AC, and join CG: Therefore, by this Propoſition, the angle ACG is a right angle; but ACD is alſo a right angle; wherefore the angles ACD, ACG are equal to one another, which is impoſſible. Therefore BD is equal to AC; and by this propoſition BDC is a right angle.

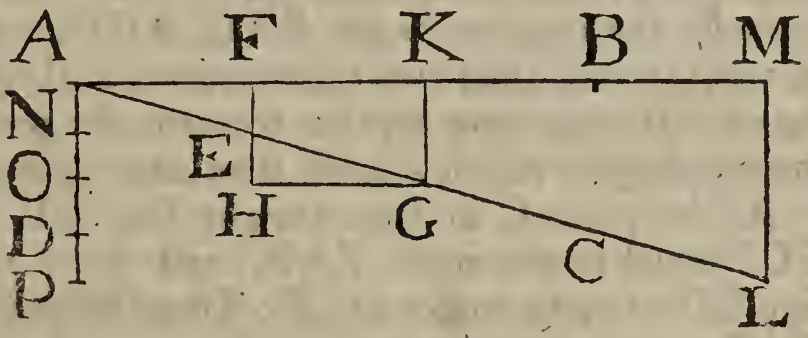
PROP. 3.

If two ſtraight lines which contain an angle be produced, there may be found in either of them a point from which the perpendicular drawn to the other ſhall be greater than any given ſtraight line.

Let AB, AC be two ſtraight lines which make an angle with one another, and let AD be the given ſtraight line; a point may be found either in AB or AC, as in AC, from which the perpendicular drawn to the other AB ſhall be greater than AD.

In AC take any point E, and draw EF perpendicular to AB; produce AE to G, ſo that EG be equal to AE; and produce FE to H, and make EH equal to FE, and join HG. Becauſe, in the triangles AEF, GEH, AE, EF are equal to GE, EH, each to each, and contain equal [a] angles, the angle GHE is therefore equal [b] to the angle AFE which is a right angle: Draw GK perpendicular to AB; and becauſe the ſtraight lines FK, HG are at right angles to FH, and KG at right angles to FK, KG is equal to FH, by Cor. Pr. 2. that is, to the double of FE.

a 15. 1.
b 4. 1.



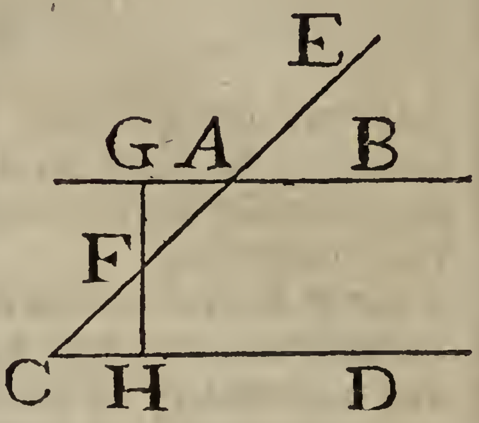
In the ſame manner, if AG be produced to L, ſo that GL be equal to AG, and LM be drawn perpendicular to AB, then LM is double of GK, and ſo on. In AD take AN equal to FE, and AO equal to KG, that is, to the double of FE, or AN; alſo, take AP, equal to LM, that is, to the double of KG, or AO; and let this be done till the ſtraight line taken be greater than AD: Let this ſtraight line ſo taken be AP, and becauſe AP is equal to LM, therefore LM is greater than AD. Which was to be done.

PROP. 4.

If two ſtraight lines AB, CD make equal angles EAB, ECD with another ſtraight line EAC towards the ſame parts of it; AB and CD are at right angles to ſome ſtraight line.

Book I.

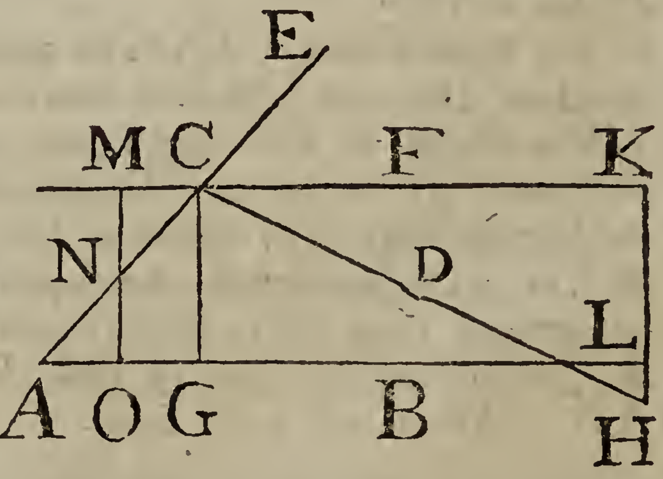
Biſect AC in F, and draw FG perpendicular to AB; take CH in the ſtraight line CD equal to AG, and on the contrary ſide of AC to that on which AG is, and join FH: Therefore, in the triangles AFG, CFH, the ſides FA, AG are equal to FC, CH, each to each, and the angle
a 15. 1. FAG, that [a] is EAB is equal to the
b 4. 1. angle FCH; wherefore [b] the angle AGF is equal to CHF, and AFG to the angle CFH: To theſe laſt add the common angle AFH; therefore the two angles AFG, AFH are equal to the two angles CFH, HFA, which two laſt are equal together to two
c 13. 1. right angles [c], therefore alſo AFG,
d 14. 1. AFH are equal to two right angles, and conſequently [d] GF and FH are in one ſtraight line. And becauſe AGF is a right angle, CHF which is equal to it is alſo a right angle; Therefore the ſtraight lines AB, CD are at right angles to GH.



PROP. 5.

If two ſtraight lines AB, CD be cut by a third ACE ſo as to make the interior angles BAC, ACD, on the ſame ſide of it, together leſs than two right angles; AB and CD being produced ſhall meet one another towards the parts on which are the two angles which are leſs than two right angles.

a 23. 1. At the point C in the ſtraight line CE make [a] the angle ECF equal to the angle EAB, and draw to AB the ſtraight line CG at right angles to CF: Then, becauſe the angles ECF, EAB are equal to one another, and that the angles ECF, FCA are together
b 13. 1. equal [b] to two right angles, the angles EAB, FCA are equal to two right angles. But by the hypotheſis, the angles EAB, ACD are together leſs than two right angles; therefore the angle FCA is greater than ACD, and CD falls between CF and AB: And becauſe CF and CD make an angle with one another, by Prop. 3. a point may be found in either of them CD, from which the perpendicular drawn to CF ſhall be greater than the ſtraight line CG.



Let this point be H, and draw HK perpendicular to CF meeting AB in L: And becauſe AB, CF contain equal angles with AC on the ſame ſide of it, by Prop. 4. AB and CF are at right angles to the ſtraight line MNO which biſects AC in N and is perpendicular to CF: Therefore by Cor. Prop. 2. CG and KL which are at right angles to CF are equal to one another: And HK is greater than CG, and therefore is greater than KL, and conſequently the point H is in KL produced. Wherefore the ſtraight line CDH drawn betwixt the points C, H which are on contrary ſides of AL, muſt neceſſarily cut the ſtraight line AB.

PROP. XXXV. B. I.

The demonſtration of this Propoſition is changed, becauſe, if the method which is uſed in it was followed, there would be three caſes to be ſeparately demonſtrated, as is done in the tranſlation from the Arabic; for, in the Elements, no caſe of a Propoſition that requires a different demonſtration, ought to be omitted. On this account, we have choſen the method which Monſ. Clairault has given, the firſt of any, as far as I know, in his Elements, page 21, and which afterwards Mr Simpſon gives in his page 32. But whereas Mr Simpſon makes uſe of prop. 26. B. 1. from which the equality of the two triangles does not immediately follow, becauſe, to prove that, the 4 of B. 1. muſt likewiſe be made uſe of, as may be ſeen in the very ſame caſe in the 34. Prop. B. 1. it was thought better to make uſe only of the 4. of B. 1.

PROP. XLV. B. I.

The ſtraight line KM is proved to be parallel to FL from the 33. Prop.; whereas KH is parallel to FG by conſtruction, and KHM, FGL have been demonſtrated to be ſtraight lines. A corollary is added from Commandine, as being often uſed.

PROP. XIII. B. II.

IN this Propoſition only acute angled triangles are mentioned, whereas it holds true of every triangle: And the demonſtrations of the caſes omitted are added; Commandine and Clavius have likewiſe given their demonſtrations of theſe caſes.

PROP. XIV. B. II.

In the demonſtration of this, ſome Greek editor has ignorantly inſerted the words, " but if not, one of the two BE, Book II. "ED is the greater: let BE be the greater, and produce it to "F," as if it was of any conſequence whether the greater or leſſer be produced: Therefore, inſtead of theſe words, there ought to be read only, "but if not, produce BE to F."

PROP. I. B. III.

Book III. SEVERAL authors, eſpecially among the modern mathematicians and logicians, inveigh too ſeverely againſt indirect or Apagogic demonſtrations, and ſometimes ignorantly enough; not being aware that there are ſome things that cannot be demonſtrated any other way: Of this the preſent propoſition is a very clear inſtance, as no direct demonſtration can be given of it: Becauſe, beſides the definition of a circle, there is no principle or property relating to a circle antecedent to this problem, from which either a direct or indirect demonſtration can be deduced: Wherefore it is neceſſary that the point found by the conſtruction of the problem be proved to be the centre of the circle, by the help of this definition, and ſome of the preceding propoſitions: And becauſe, in the demonſtration, this propoſition muſt be brought in, viz. ſtraight lines from the centre of a circle to the circumference are equal, and that the point found by the conſtruction cannot be aſſumed as the centre, for this is the thing to be demonſtrated: it is manifeſt ſome other point muſt be aſſumed as the centre: and if from this aſſumption an abſurdity follows, as Euclid demonſtrates there muſt, then it is not true that the point aſſumed is the centre; and as any point whatever was aſſumed, it follows that no point, except that found by the conſtruction, can be the centre, from which the neceſſity of an indirect demonſtration in this caſe is evident.

PROP. XIII. B. III.

As it is much eaſier to imagine that two circles may touch one another within in more points than one, upon the ſame ſide, than upon oppoſite ſides; the figure of that caſe ought not to have been omitted; but the conſtruction in the Greek text would not have ſuited with this figure ſo well, becauſe the centres of the circles muſt have been placed near to the circumferences: On which account another conſtruction and demonſtration is given, which is the ſame with the ſecond part of that which Campanus has tranſlated from the Arabic, where, without any reaſon, the demonſtration is divided into two parts.

PROP. XV. B. III.

The converſe of the ſecond part of this propoſition is wanting, though in the preceding, the converſe is added, in a like caſe, both in the enunciation and demonſtration; and it is now added in this. Beſides, in the demonſtration of the firſt part of this 15th, the diameter AD (ſee Commandine's figure, is proved to be greater than the ſtraight line BC by means of another ſtraight line MN; whereas it may be better done without it: On which accounts we have given a different demonſtration, like to that which Euclid gives in the preceding 14th, and to that which Theodoſius gives in prop. 6. B. 1. of his Spherics, in this very affair.

PROP. XVI. B. III.

In this we have not followed the Greek nor the Latin tranſlation literally, but have given what is plainly the meaning of the propoſition, without mentioning the angle of the ſemicircle, or that which ſome call the cornicular angle which they conceive to be made by the circumference and the ſtraight line which is at right angles to the diameter, at its extremity; which angles have furniſhed matter of great debate between ſome of the modern geometers, and given occaſion of deducing ſtrange conſequences from them, which are quite avoided by the manner in which we have expreſſed the propoſition. And in like manner, we have given the true meaning of prop. 31. B. 3, without mentioning the angles of the greater or leſſer ſegments: Theſe paſſages, Vieta, with good reaſon, ſuſpects to be adulterated in the 386th page of his Oper. Math.

PROP. XX. B. III.

The firſt words of the ſecond part of this demonſtration, "*κεκλασθω δη παλιν*," are wrong tranſlated by Mr Briggs and Dr Gregory "Rurſus inclinetur;" for the tranſlation ought to be "Rurſus inflectatur," as Commandine has it: A ſtraight line is ſaid to be inflected either to a ſtraight, or curve line, when a ſtraight line is drawn to this line from a point, and from the point in which it meets it, a ſtraight line making an angle with the former is drawn to another point, as is evident from the 90th prop. of Euclid's Data: For this the whole line betwixt the firſt and laſt points, is inflected or broken at

Book III. the point of inflection, where the two ſtraight lines meet. And in the like ſenſe two ſtraight lines are ſaid to be inflected from two points to a third point, when they make an angle at this point; as may be ſeen in the deſcription given by Pappus Alexandrinus of Apollonius's Books de Locis planis, in the preface to his 7th book: We have made the expreſſion fuller from the 90th Prop. of the Data.

PROP. XXI. B. III.

There are two caſes of this propoſition, the ſecond of which, viz. when the angles are in a ſegment not greater than a ſemicircle, is wanting in the Greek: And of this a more ſimple demonſtration is given than that which is in Commandine, as being derived only from the firſt caſe, without the help of triangles.

PROP. XXIII. and XXIV. B. III.

In propoſition 24. it is demonſtrated, that the ſegment AEB muſt coincide with the ſegment CFD, (ſee Commandine's figure), and that it cannot fall otherwiſe, as CGD, ſo as to cut the other circle in a third point G, from this, that, if it did, a circle could cut another in more points than two: But this ought to have been proved to be impoſſible in the 23d Prop, as well as that one of the ſegments cannot fall within the other: This part then is left out in the 24th, and put in its proper place, the 23d Propoſition.

PROP. XXV. B. III.

This propoſition is divided into three caſes, of which two have the ſame conſtruction and demonſtration; therefore it is now divided only into two caſes.

PROP. XXXIII. B. III.

This alſo in the Greek is divided into three caſes, of which two, viz. one, in which the given angle is acute, and the other in which it is obtuſe, have exactly the ſame conſtruction and demonſtration; on which account, the demonſtration of the laſt caſe is left out as quite ſuperfluous, and the addition of ſome unſkilful editor; beſides the demonſtration of the caſe when the angle given is a right angle, is done a round about way, and is therefore changed to a more ſimple one, as was done by Clavius.

PROP. XXXV. B. III.

As the 25th and 33d propoſitions are divided into more caſes, ſo this 35th is divided into fewer caſes than are neceſſary. Nor can it be ſuppoſed that Euclid omitted them becauſe they are eaſy; as he has given the caſe, which by far is the eaſieſt of them all, viz. that in which both the ſtraight lines paſs through the centre: And in the following propoſition he ſeparately demonſtrates the caſe in which the ſtraight line paſſes through the centre, and that in which it does not paſs through the centre: So that it ſeems Theon, or ſome other, has thought them too long to inſert: But caſes that require different demonſtrations, ſhould not be left out in the Elements, as was before taken notice of: Theſe caſes are in the tranſlation from the Arabic, and are now put into the text.

PROP. XXXVII. B. III.

At the end of this, the words, "in the ſame manner it may "be demonſtrated, if the centre be in AC," are left out as the addition of ſome ignorant editor.

DEFINITIONS of BOOK IV.

Book IV.

WHEN a point is in a ſtraight line, or any other line, this point is by the Greek geometers ſaid αϖτεσϑαι, to be upon, or in that line, and when a ſtraight line or circle meets a circle any way, the one is ſaid αϖτεσϑαι to meet the other: But when a ſtraight line or circle meets a circle ſo as not to cut it, it is ſaid εφαπτεσϑαι, to touch the circle; and theſe two terms are never promiſcuouſly uſed by them: Therefore, in the 5th definition of B. 4. the compound εφαπτηταϊ muſt be read, inſtead of the ſimple απτηται: And in the 1ſt, 2d, 3d, and 6th definitions in Commandine's tranſlation, "tangit," muſt be read inſtead of "contingit:". And in the 2d and 3d definitions of Book 3. the ſame change muſt be made: But in the Greek text of propoſitions 11th, 12th, 13th, 18th, 19th, Book 3. the compound verb is to be put for the ſimple.

PROP. IV. B. IV.

In this, as alſo in the 8th and 13th propoſitions of this book, it is demonſtrated indirectly, that the circle touches a ſtraight line; whereas in the 17th, 33d, and 37th propoſitions of book 3. the ſame thing is directly demonſtrated: And this way we

Book IV. have choſen to uſe in the propoſitions of this book, as it is ſhorter.

PROP. V. B. IV.

The demonſtration of this has been ſpoiled by ſome unſkilful hand: For he does not demonſtrate, as is neceſſary, that the two ſtraight lines which biſect the ſides of the triangle at right angles muſt meet one another; and, without any reaſon, he divides the propoſition into three caſes; whereas, one and the ſame conſtruction and demonſtration ſerves for them all, as Campanus has obſerved; which uſeleſs repetitions are now left out: The Greek text alſo in the corollary is manifeſtly vitiated, where mention is made of a given angle, though there neither is, nor can be any thing in the propoſition relating to a given angle.

PROP. XV. and XVI. B. IV.

In the corollary of the firſt of theſe, the words equilateral and equiangular are wanting in the Greek: And in prop. 16. inſtead of the circle ABCD, ought to be read the circumference ABCD: Where mention is made of its containing fifteen equal parts.

DEF. III. B. V.

Book V. MANY of the modern mathematicians reject this definition: The very learned Dr Barrow has explained it at large at the end of his third lecture of the year 1666, in which alſo he anſwers the objections made againſt it as well as the ſubject would allow: And at the end gives his opinion upon the whole, as follows:

" I ſhall only add, that the author had, perhaps, no other " deſign in making this definition, than (that he might more " fully explain and embelliſh his ſubject) to give a general " and ſummary idea of ratio to beginners, by premiſing " this metaphyſical definition, to the more accurate defini- " tions of ratios that are the ſame to one another, or one of " which is greater, or leſs than the other: I call it a meta- " phyſical, for it is not properly a mathematical definition, " ſince nothing in mathematics depends on it, or is deduced, " nor, as I judge, can be deduced from it: And the defini- " tion of analogy, which follows, viz. Analogy is the ſimi-

" litude of ratios, is of the ſame kind, and can ſerve for no " purpoſe in mathematics, but only to give beginners ſome " general, tho' groſs and confuſed notion of analogy: But the " whole of the doctrine of ratios, and the whole of mathema- " tics, depend upon the accurate mathematical definitions which " follows this: To theſe we ought principally to attend, as the " doctrine of ratios is more perfectly explained by them; this " third, and others like it, may be entirely ſpared without any " loſs to geometry; as we ſee in the 7th book of the elements, " where the proportion of numbers to one another is defined, " and treated of, yet without giving any definition of the ratio " of numbers; tho' ſuch a definition was as neceſſary and uſe- " ful to be given in that book, as in this: But indeed there is " ſcarce any need of it in either of them: Though I think that " a thing of ſo general and abſtracted a nature, and thereby " the more difficult to be conceived and explained, cannot be " more commodiouſly defined than as the author has done: " Upon which account I thought fit to explain it at large, and " defend it againſt the captious objections of thoſe who attack " it." To this citation from Dr Barrow I have nothing to add, except that I fully believe the 3d and 8th definitions are not Euclid's, but added by ſome unſkilful editor.

DEF. XI. B. V.

It was neceſſary to add the word "continual" before "pro- " portionals" in this definition; and thus it is cited in the 33d prop. of Book 11.

After this definition ought to have followed the definition of compound ratio, as this was the proper place for it; duplicate and triplicate ratio being ſpecies of compound ratio. But Theon has made it the 5th def. of B. 6. where he gives an abſurd and entirely uſeleſs definition of compound ratio: For this reaſon we have placed another definition of it betwixt the 11th and 12th of this book, which, no doubt, Euclid gave; for he cites it expreſsly in prop. 23. B. 6. and which Clavius, Herigon, and Barrow, have likewiſe given, but they retain alſo Theon's, which they ought to have left out of the elements.

DEF. XIII. B. V.

This, and the reſt of the definitions following, contain the explication of ſome terms which are uſed in the 5th and following books; which, except a few, are eaſily enough underſtood from the propoſitions of this book where they are firſt mentioned: They ſeem to have been added by Theon, or ſome other. However it be, they are explained ſomething more diſtinctly for the ſake of learners.

PROP. IV. B. V.

In the conſtruction preceding the demonſtration of this, the words ἁ ετυχε, any whatever, are twice wanting in the Greek, as alſo in the Latin tranſlations; and are now added, as being wholly neceſſary.

Ibid. in the demonſtration; in the Greek, and in the Latin tranſlation of Commandine, and in that of Mr Henry Briggs, which was publiſhed at London in 1620, together with the Greek text of the firſt ſix books, which tranſlation in this place is followed by Dr Gregory in his edition of Euclid, there is this ſentence following, viz. "and of A and C have been taken equimultiples K, L; and of B and D, any equimultiples "whatever (ἁ ετυχε) M, N;" which is not true, the words "any whatever;" ought to be left out: And it is ſtrange that neither Mr Briggs, who did right to leave out theſe words in one place of prop. 13. of this book, nor Dr Gregory, who changed them into the word "ſome" in three places, and left them out in a fourth of that ſame prop. 13. did not alſo leave them out in this place of prop. 4. and in the ſecond of the two places where they occur in prop. 17. of this book, in neither of which they can ſtand conſiſtent with truth: And in none of all theſe places, even in thoſe which they corrected in their Latin tranſlation have they cancelled the words ἁ ετυχε in the Greek text, as they ought to have done.

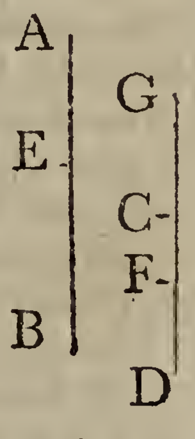
The ſame words ἁ ετυχε are found in four places of prop. 11. of this book, in the firſt and laſt of which they are neceſſary, but in the ſecond and third, though they are true, they are quite ſuperfluous; as they likewiſe are in the ſecond of the two places in which they are found in the 12th prop. and in the like places of Prop. 22. 23. of this book; but are wanting in the laſt place of prop. 23. as alſo in prop. 25. Book 11.

COR. IV. PROP. B. V.

This corollary has been unſkilfully annexed to this propoſition, and has been made inſtead of the legitimate demonſtration, which, without doubt, Theon, or ſome other editor, has taken away, not from this, but from its proper place in this book: The author of it deſigned to demonſtrate, that if four magnitudes E, G, F, H, be proportionals, they are alſo proportionals inverſely; that is, G is to E, as H to F; which is true; but the demonſtration of it does not in the leaſt depend upon this 4th prop. or its demonſtration: For, when he ſays, "becauſe it is demonſtrated that if K be greater than M, L is "greater than N," &c. This indeed is ſhewn in the demonſtration of the 4th prop. but not from this, that E, G, F, H are proportionals; for this laſt is the concluſion of the propoſition. Wherefore theſe words, "becauſe it is demonſtrated," &c. are wholly foreign to his deſign: And he ſhould have proved, that if K be greater than M, L is greater than N, from this, that E, G, F, H are proportionals, and from the 5th def. of this book, which he has not; but is done in propoſition B, which we have given in its proper place, inſtead of this corollary; and another corollary is placed after the 4th prop. which is often of uſe; and is neceſſary to the demonſtration of prop. 18. of this book.

PROP. V. B. V.

In the conſtruction which precedes the demonſtration of this propoſition, it is required that EB may be the ſame multiple of CG, that AE is of CF; that is, that EB be divided into as many equal parts, as there are parts in AE equal to CF: From which it is evident, that this conſtruction is not Euclid's; for he does not ſhow the way of dividing ſtraight lines, and far leſs other magnitudes, into any number of equal parts, until the 9th propoſition of B. 6; and he never requires any thing to be done in the conſtruction of which he had not before given the method of doing: For this reaſon, we have changed the conſtruction to one, which, without doubt, is Euclid's, in which nothing is required but to add a magnitude to itſelf a certain number of times; and this is to be found in the tranſlation from the Arabic, though the enunciation of the propoſition and the demonſtration are there very much ſpoiled. Jacobus Peletarius, who was the firſt, as far as I know, who took notice of this error, gives alſo the right conſtruction in his edition of Euclid, after he had given the other which he blames: He ſays, he would not leave it out, becauſe it was fine, and might ſharpen one's genius to invent others like it;



whereas there is not the leaſt difference between the two demonſtrations, except a ſingle word in the conſtruction, which very probably has been owing to an unſkilful librarian. Clavius likewiſe gives both the ways; but neither he nor Peletarius takes notice of the reaſon why the one is preferable to the other.

PROP. VI. B. V.

There are two caſes of this propoſition, of which only the firſt and ſimpleſt is demonſtrated in the Greek: And it is probable Theon thought it was ſufficient to give this one, ſince he was to make uſe of neither of them in his mutilated edition of the 5th book; and he might as well have left out the other, as alſo the 5th propoſition, for the ſame reaſon; The demonſtration of the other caſe is now added, becauſe both of them, as alſo the fifth propoſition, are neceſſary to the demonſtration of the 18th propoſition of this Book. The tranſlation from the Arabic gives both caſes briefly.

PROP. A. B. V.

This propoſition is frequently uſed by geometers, and it is neceſſary in the 25th prop. of this book, 31ſt of the 6th, and 34th of the 11th, and 15th of the 12th book: It ſeems to have been taken out of the elements by Theon, becauſe it appeared evident enough to him, and others, who ſubſtitute the confuſed and indiſtinct idea the vulgar have of proportionals, in place of that accurate idea which is to be got from the 5th def. of this book. Nor can there be any doubt that Eudoxus or Euclid gave it a place in the elements, when we ſee the 7th and 9th of the ſame book demonſtrated, tho' they are quite as eaſy and evident as this. Alphonſus Borellus takes occaſion from this propoſition to cenſure the 5th definition of this book very ſeverely, but moſt unjuſtly: In p. 126. of his Euclid reſtored, printed at Piſa in 1658, he ſays, "Nor can even this leaſt de-"gree of knowledge be obtained from the foreſaid property," viz. that which is contained in 5th def. 5. "That, if four "magnitudes be proportionals, the third muſt neceſſarily be "greater than the fourth, when the firſt is greater than the "ſecond: as Clavius acknowledges in the 16th prop. of the "5th book of the elements." But though Clavius makes no ſuch acknowledgment expreſsly, he has given Borellus a handle to ſay this of him; becauſe when Clavius, in the above cited place, cenſures Commandine, and that very juſtly, for demonſtrating this propoſition by help of the 16th of the 5th; yet he himſelf gives no demonſtration of it, but thinks it plain from the nature of proportionals, as he writes in the end of the 14th and 16th prop. B. 5. of his edition, and is followed by Herigon in Schol. 1. prop. 14th B. 5. as if there was any nature of proportionals antecedent to that which is to be derived and underſtood from the definition of them: And indeed, though it is very eaſy to give a right demonſtration of it, no body, as far as I know, has given one, except the learned Dr Barrow, who, in anſwer to Borrellus's objection, demonſtrates it indirectly, but very briefly and clearly, from the 5th definition, in the 322 page of his Lect. Mathem. from which definition it may alſo be eaſily demonſtrated directly: On which account we have placed it next to the propoſitions concerning equimultiples.

PROP. B. BOOK V.

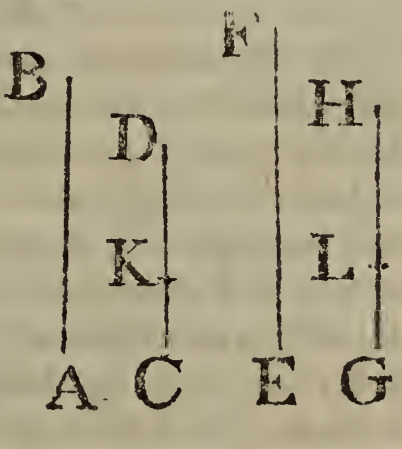
This alſo is eaſily deduced from the 5th def. B. 5. and therefore is placed next to the other; for it was very ignorantly made a corollary from the 4th prop. of this Book. See the note on that corollary.

PROP. C. B. V.

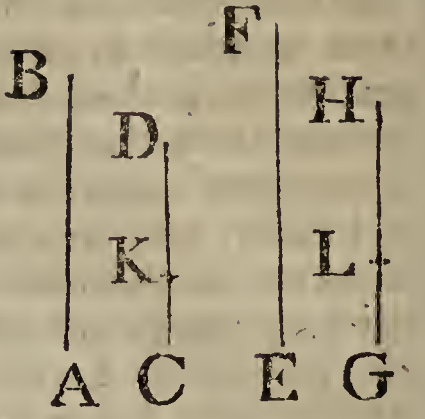
This is frequently made uſe of by geometers, and is neceſſary to the 5th and 6th propoſitions of the 10th book. Clavius, in his notes ſubjoined to the 8th def. of book 5. demonſtrates it only in numbers, by help of ſome of the propoſitions of the 7th book: in order to demonſtrate the property contained in the 5th definition of the 5th book, when applied to numbers, from the property of proportionals contained in the 20th def. of the 7th book: And moſt of the commentators judge it difficult to prove that four magnitudes which are proportionals according to the 20th def. of 7th book, are alſo proportionals according to the 5th def. of 5th book. But this is eaſily made out, as follows:

Firſt, if A, B, C, D be four magnitudes, ſuch that A is the ſame multiple, or the ſame part of B, which C is of D; A, B, C, D are proportionals: This is demonſtrated in propoſition C.

Secondly, if AB contain the ſame parts of CD that EF does of GH; in this caſe likewiſe AB is to CD, as EF to GH.



Let CK be a part of CD, and GL the ſame part of GH; and let AB be the ſame multiple of CK, that EF is of GL: Therefore, by prop. C. of 5th book, AB is to CK, as EF to GL: And CD, GH are equimultiples of CK, GL the ſecond and fourth; wherefore, by Cor. prop. 4. book 5. AB is to CD, as EF to GH.



And if four magnitudes be proportionals according to the 5th def. of book 5. they are alſo proportionals according to the 20th def. of book 7.

Firſt, if A be to B, as C to D; then if A be any multiple or part of B, C is the ſame multiple or part of D, by prop. D. of B. 5.

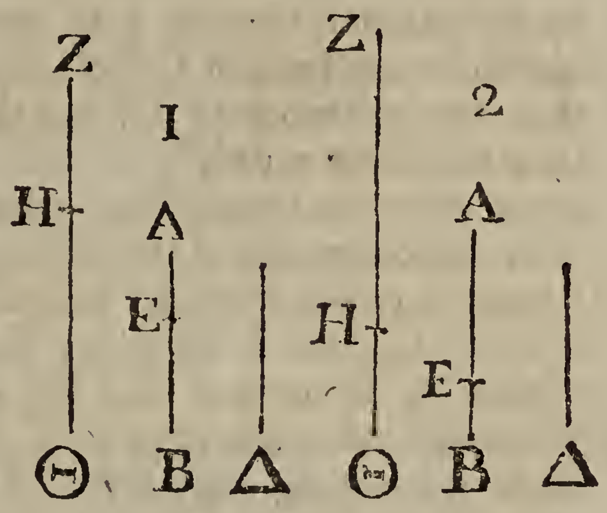
Next, if AB be to CD, as EF to GH; then if AB contains any parts of CD, EF contains the ſame parts of GH: For let CK be a part of CD, and GL the ſame part of GH, and let AB be a multiple of CK: EF is the ſame multiple of GL; Take M the ſame multiple of GL that AB is of CK; therefore by prop. C. of B. 5. AB is to CK, as M to GL; and CD, GH are equimultiples of CK, GL; wherefore by Cor. prop. 4. B. 5. AB is to CD, as M to GH. And, by the hypotheſis, AB is to CD, as EF to GH; therefore M is equal to EF by prop. 9. book 5. and conſequently EF is the ſame multiple of GL that AB is of CK.

PROP. D. B. V.

This is not unfrequently uſed in the demonſtration of other propoſitions, and is neceſſary in that of prop. 9. B. 6. It ſeems Theon has left it out for the reaſons mentioned in the notes at prop. A.

PROP. VIII. B. V.

In the demonſtration of this, as it is now in the Greek, there are two caſes, (ſee the demonſtration in Hergavius, or Dr Gregory's edition), of which the firſt is that in which AE is leſs than EB; and in this, it neceſſarily follows that HΘ the multiple EB is greater than ZH the ſame multiple of AE, which laſt multiple, by the conſtruction, is greater than Δ; whence alſo HΘ muſt be greater than Δ: But in the ſecond caſe, viz. that in which EB is leſs than AE, tho' ZH be greater than Δ, yet HΘ may be leſs than the ſame Δ; ſo that there cannot be taken a multiple of Δ which is the firſt that is greater than K or HΘ, becauſe Δ itſelf is greater than it: Upon this account, the author of this demonſtration found it neceſſary to change one part of the conſtruction that was made uſe of in the firſt caſe: But he has, without any neceſſity, changed alſo another part of it, viz. when he orders to take N that multiple of Δ which is the firſt that is greater than ZH; for he might have taken that multiple of Δ which is the firſt that is greater than HΘ, or K, as was done in the firſt caſe: He likewiſe brings in this K into the demonſtration of both caſes, without any reaſon; for it ſerves to no purpoſe but to lengthen the demonſtration. There is alſo a third caſe, which is not mentioned in this demonſtration, viz. that in which AE in the firſt, or EB in the ſecond of the two other caſes, is greater than D; and in this any equimultiples, as the doubles, of AE, KB are to be taken, as is done in this edition, where all the caſes are at once demonſtrated: And from this it is plain that Theon, or ſome other unſkilful editor, has vitiated this propoſition.



PROP. IX. B. V.

Of this there is given a more explicit demonſtration than that which is now in the elements.

PROP. X. B. V.

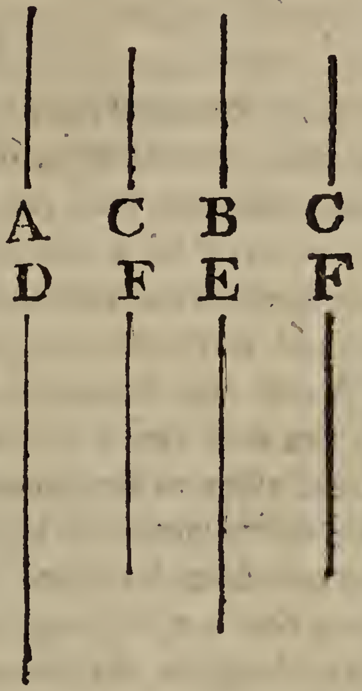
It was neceſſary to give another demonſtration of this propoſition, becauſe that which is in the Greek and Latin, or other editions, is not legitimate: For the words *greater*, *the ſame* or *equal*, *leſſer*, have a quite different meaning when applied to magnitudes and ratios, as is plain from the 5th and 7th definitions of book 5. By the help of theſe let us examine the demonſtration of the 10th prop. which proceeds thus: "Let A "have to C a greater ratio, than B to C: I ſay that A is greater "than B. For if it is not greater, it is either equal, or leſs. "But A cannot be equal to B, becauſe then each of them "would have the ſame ratio to C; but they have not. There"fore A is not equal to B." The force of which reaſoning is this, if A had to C the ſame ratio that B has to C, then if any equimultiples whatever of A and B be taken, and any multiple whatever of C; if the multiple of A be greater than the multiple of C, then, by the 5th def. of book 5. the multiple of B is alſo greater than that of C; but, from the hypotheſis that A has a greater ratio to C, than B has to C, there muſt, by the 7th def. of book 5. be certain equimultiples of A and B, and ſome multiple of C ſuch, that the multiple of A is greater than the multiple of C, but the multiple of B is not greater than the ſame multiple of C: And this propoſition directly contradicts the preceding; wherefore A is not equal to B. The demonſtration of the 10th prop. goes on thus: "But neither is A leſs than B; becauſe then A would have a leſs ratio to C, than B has to it: But it has not a leſs ratio, therefore A is not leſs than B," &c. Here it is ſaid that "A would have a leſs ratio to C, than B has to C," or. which is the ſame thing, that B would have a greater ratio to C, than A to C; that is by 7th def. book 5. there muſt be ſome equimultiples of B and A, and ſome multiple of C, ſuch that the multiple of B is greater than the multiple of C, but the multiple of A is not greater than it: And it ought to have been proved that this can never happen if the ratio of A to C be greater than the ratio of B to C; that is, it ſhould have been proved, that, in this caſe, the multiple of A is always greater than the multiple of C, whenever the multiple of B is greater than the multiple of C; for, when this is demonſtrated, it will be evident that B cannot have a greater ratio to C, than A has to C, or, which is the ſame thing, that A cannot have a leſs ratio to C, than B has to C: But this is not at all proved in the 10th propoſition: but if the 10th were once demonſtrated, it would immediately follow from it, but cannot without it be eaſily demonſtrated, as he that tries to do it will find. Wherefore the 10th propoſition is not ſufficiently demonſtrated. And it ſeems that he who has given the demonſtration of the 10th propoſition as we now have it, inſtead of that which Eudoxus or Euclid had given, has been deceived in applying what is manifeſt, when underſtood of magnitudes, unto ratios, viz. that a magnitude cannot be both greater and leſs than another. That thoſe things which are equal to the ſame are equal to one another, is a moſt evident axiom when underſtood of magnitudes; yet Euclid does not make uſe of it to infer that thoſe ratios which are the ſame to the ſame ratio, are the ſame to one another; but explicitly demonſtrates this in prop. 11. of book 5. The demonſtration we have given of the 10th prop.

is no doubt the ſame with that of Eudoxus or Euclid, as it is immediately and directly derived from the definition of a greater ratio, viz. the 7 of the 5.

The above mentioned propoſition, viz. If A have to C a greater ratio than B to C; and if of A and B there be taken certain equimultiplies, and ſome multiple of C; then if the multiple of B be greater than the multiple of C, the multiple of A is alſo greater than the ſame, is thus demonſtrated.

Let D, E be equimultiples of A, B, and F a multiple of C, ſuch, that E the multiple of B is greater than F; D the multiple of A is alſo greater than F.

Becauſe A has a greater ratio to C, than B to C, A is greater than B, by the 10th prop. B. 5. therefore D the multiple of A is greater than E the ſame multiple of B: And E is greater than F; much more therefore D is greater than F.



PROP. XIII. B. V.

In Commandine's, Briggs's, and Gregory's tranſlations, at the beginning of this demonſtration, it is ſaid, "And the multi-"ple of C is greater than the multiple of D; but the multi-"ple of E is not greater than the multiple of F:" which words are a literal tranſlation from the Greek: But the ſenſe evidently requires that it be read, "ſo that the multiple of C "be greater than the multiple of D; but the multiple of E be "not greater than the multiple of F." And thus this place was reſtored to the true reading in the firſt editions of Commandine's Euclid, printed in 8vo at Oxford; but in the latter editions, at leaſt in that of 1747, the error of the Greek text was kept in.

There is a corollary added to prop. 13. as it is neceſſary to the 20th and 21ſt prop. of this book, and is as uſeful as the propoſition.

PROP. XIV. B. V.

The two caſes of this, which are not in the Greek, are added; the demonſtration of them not being exactly the ſame with that of the firſt caſe.

PROP. XVII. B. V.

The order of the words in a clauſe of this is changed to one more natural: As was alſo done in prop. 1.

PROP. XVIII. B. V.

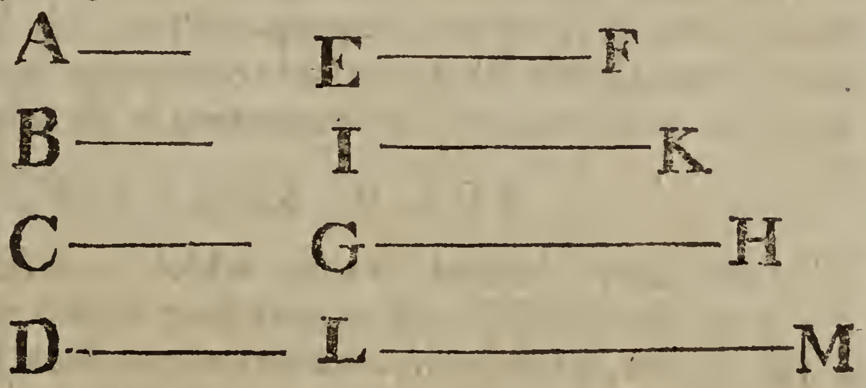
The demonſtration of this is none of Euclid's, nor is it legitimate; for it depends upon this hypotheſis, that to any three magnitudes, two of which, at leaſt, are of the ſame kind, there may be a fourth proportional: which, if not proved, the demonſtration now in the text is of no force: But this is aſſumed without any proof; nor can it, as far as I am able to diſcern, be demonſtrated by the propoſitions preceding this: ſo far is it from deſerving to be reckoned an axiom, as Clavius, after other commentators, would have it, at the end of the definitions of the 5th book. Euclid does not demonſtrate it, nor does he ſhew how to find the fourth proportional, before the 12th prop. of the 6th book: And he never aſſumes any thing in the demonſtration of a propoſition, which he had not before demonſtrated; at leaſt, he aſſumes nothing the exiſtence of which is not evidently poſſible; for a certain concluſion can never be deduced by the means of an uncertain propoſition: Upon this account, we have given a legitimate demonſtration of this propoſition inſtead of that in the Greek and other editions, which very probably Theon, at leaſt ſome other, has put in the place of Euclid's, becauſe he throught it too prolix: And as the 17th prop. of which this 18th is the converſe, is demonſtrated by help of the 1ſt and 2d propoſitions of this book; ſo, in the demonſtration now given of the 18th, the 5th prop. and both caſes of the 6th are neceſſary, and theſe two propoſitions are the converſes of the 1ſt and 2d. Now the 5th and 6th do not enter into the demonſtration of any propoſition in this book as we now have it: Nor can they be of uſe in any propoſition of the Elements, except in this 18th, and this is a manifeſt proof, that Euclid made uſe of them in his demonſtration of it, and that the demonſtration now given, which is exactly the converſe of that of the 17th, as it ought to be, differs nothing from that of Eudoxus or Euclid: For the 5th and 6th have undoubtedly been put into the 5th book for the ſake of ſome propoſitions in it, as all the other propoſitions about equimultiples have been.

Hieronymus Saccherius, in his book named "Euclides ab "omni nævo vindicatus," printed at Milan ann. 1733, in 4to, acknowledges this blemiſh in the demonſtration of the 18th, and that he may remove it, and render the demonſtration we now have of it legitimate, he endeavours to demonſtrate the following propoſition, which is in page 115 of his book, viz.

“Let A, B, C, D be four magnitudes, of which the two firſt are of the one kind, and alſo the two others either of the ſame kind with the two firſt, or of ſome other the ſame kind with one another. I ſay the ratio of the third C to the fourth D, is either equal to, or greater, or leſs than the ratio of the firſt A to the ſecond B.”

And after two propoſitions premiſed as Lemmas, he proceeds thus:

“Either among all the poſſible equimultiples of the firſt A, and of the third C, and, at the ſame time, among all the poſſible equimultiples of the ſecond B, and of the fourth D, there can be found ſome one multiple EF of the firſt A, and one IK of the ſecond B, that are equal to one another; and alſo (in the ſame caſe) ſome one multiple GH of the third C equal to LM the multiple of the fourth D, or ſuch equality is no where to be found. If the firſt caſe happen, [i. e. if ſuch equality is to be found] it is manifeſt from what is before demonſtrated that A is to B, as C to D; but if ſuch ſimultaneous equality be not to be found upon both ſides, it will be found either upon one ſide, as upon the ſide of A [and B;] or it will be found upon neither ſide; if the firſt happen; therefore (from Euclid's definition of greater and leſſer ratio foregoing) A has to B, a greater or *leſs* ratio than C to D; according as GH the multiple of the third C is leſs, or greater than LM the multiple of the fourth D: But if the ſecond caſe happen; therefore upon the one ſide, as upon the ſide of A the firſt and B the ſecond, it may happen that the multiple EF, [viz. of the firſt] may be leſs than IK the multiple of the ſecond, while on the contrary, upon the other ſide, [viz. of C and D] the multiple GH [of the third C] is greater than the other multiple LM [of the fourth D:] And then (from the ſame definition of Euclid) the ra-



" tio of the firſt A to the ſecond B, is leſs than the ratio of the " third C to the fourth D; or on the contrary.

" Therefore the axiom [i. e. the propoſition before ſet down], " remains demonſtrated," &c.

Not in the leaſt; but it remains ſtill undemonſtrated: For what he ſays may happen, may, in innumerable caſes, never happen; and therefore his demonſtration does not hold: For example, if A be the ſide, and B the diameter of a ſquare; and C the ſide, and D the diameter of another ſquare; there can in no caſe be any multiple of A equal to any of B; nor any one of C equal to one of D, as is well known; and yet it can never happen that when any multiple of A is greater than a multiple of B, the multiple of C can be leſs than the multiple of D, nor when the multiple of A is leſs than that of B, the multiple of C can be greater than that of D, viz. taking equimultiples of A and C, and equimultiples of B and D: For A, B, C, D are proportionals; and ſo if the multiple of A be greater, &c. than that of B, ſo muſt that of C be greater, &c. than that of D; by 5th Def. b. 5.

The ſame objection holds good againſt the demonſtration which ſome give of the 1ſt prop. of the 6th book, which we have made againſt this of the 18th prop. becauſe it depends upon the ſame inſufficient foundation with the other.

PROP. XIX. B. V.

A corollary is added to this, which is as frequently uſed as the propoſition itſelf. The corollary which is ſubjoined to it in the Greek, plainly ſhews that the 5th book has been vitiated by editors who were not geometers: For the converſion of ratios does not depend upon this 19th, and the demonſtration which ſeveral of the commentators on Euclid give of converſion is not legitimate, as Clavius has rightly obſerved, who has given a good demonſtration of it which we have put in propoſition E; but he makes it a corollary from the 19th, and begins it with the words, " Hence it eaſily follows," though it does not at all follow from it.

PROP. XX. XXI. XXII. XXIII. XXIV. B. V.

The demonſtrations of the 20th and 21ſt propoſitions, are ſhorter than thoſe Euclid gives of eaſier propoſitions, either in the preceding, or following books: Wherefore it was proper to make them more explicit, and the 22d and 23d propoſitions are, as they ought to be, extended to any number of magnitudes: And, in like manner, may the 24th be, as is taken notice of in a corollary; and another corollary is added, as uſeful as the propoſition, and the words, "any whatever", are ſupplied near the end of prop. 23. which are wanting in the Greek text, and the tranſlations from it.

In a paper writ by Philippus Naudæus, and publiſhed after his death, in the hiſtory of the Royal Academy of Sciences of Berlin, anno 1745, page 50. the 23d prop. of the 5th book, is cenſured as being obſcurely enunciated, and, becauſe of this, prolixly demonſtrated: The enunciation there given is not Euclid's but Tacquet's, as he acknowledges, which, though not ſo well expreſſed, is, upon the matter, the ſame with that which is now in the Elements. Nor is there any thing obſcure in it though the author of the paper has ſet down the proportionals in a diſadvantageous order, by which it appears to be obſcure: But no doubt, Euclid enunciated this 23d, as well as the 22d, ſo as to extend it to any number of magnitudes, which, taken two and two, are proportionals, and not of ſix only; and to this general caſe, the enunciation which Naudæus gives, cannot be well applied.

The demonſtration which is given of this 23d, in that paper, is quite wrong; becauſe, if the proportional magnitudes be plane or ſolid figures, there can no rectangle (which he improperly calls a *product*) be conceived to be made by any two of them: And if it ſhould be ſaid, that in this caſe ſtraight lines are to be taken which are proportional to the figures, the demonſtration would this way become much longer than Euclid's: But, even though his demonſtration had been right, who does not ſee that it could not be made uſe of in the 5th book?

PROP. F, G, H, K. B. V.

Theſe propoſitions are annexed to the 5th book, becauſe they are frequently made uſe of by both ancient and modern geometers: And in many caſes, compound ratios cannot be brought into demonſtration, without making uſe of them.

Whoever deſires to ſee the doctrine of ratios delivered in this 5th book ſolidly defended, and the arguments brought againſt it by And. Tacquet, Alph. Borellus, and others, fully refuted, may read Dr Barrow's mathematical lectures, viz. the 7th and 8th of the year 1666.

The 5th book being thus corrected, I moſt readily agree to what the learned Dr Barrow ſays*, "That there is nothing

---

* Page 336.

Book V. " in the whole body of the elements of a more ſubtile invention, nothing more ſolidly eſtabliſhed, and more accurately " handled than the doctrine of proportionals." And there is ſome ground to hope, that geometers will think that this could not have been ſaid with as good reaſon, ſince Theon's time till the preſent.

DEF. II. and V. of B. VI.

Book VI. THE 2d definition does not ſeem to be Euclid's, but ſome unſkilful editor's: For there is no mention made by Euclid, nor, as far as I know, by any other geometer, of reciprocal figures: It is obſcurely expreſſed, which made it proper to render it more diſtinct: It would be better to put the following definition in place of it, viz.

DEF. II.

Two magnitudes are ſaid to be reciprocally proportional to two others, when one of the firſt is to one of the other magnitudes, as the remaining one of the laſt two is to the remaining one of the firſt.

But the 5th definition, which, ſince Theon's time, has been kept in the elements, to the great detriment of learners, is now juſtly thrown out of them, for the reaſon given in the notes on the 23d prop. of this book.

PROP. I. and II. B. VI.

To the firſt of theſe a corollary is added, which is often uſed: And the enunciation of the ſecond is made more general.

PROP. III. B. VI.

A ſecond caſe of this, as uſeful as the firſt, is given in prop. A; viz. the caſe in which the exterior angle of a triangle is biſected by a ſtraight line: The demonſtration of it is very like to that of the firſt caſe, and upon this account may, probably, have been left out, as alſo the enunciation, by ſome unſkilful editor. At leaſt, it is certain, that Pappus makes uſe of this caſe, as an elementary propoſition, without a demonſtration of it, in prop. 39. of his 7th book of Mathematical Collections.

PROP. VII. B. VI.

To this a caſe is added which occurs not unfrequently in demonſtration.

PROP. VIII. B. VI.

It ſeems plain that ſome editor has changed the demonſtration that Euclid gave of this propoſition: For, after he has demonſtrated, that the triangles are equiangular to one another, he particularly ſhews that their ſides about the equal angles are proportionals, as if this had not been done in the demonſtration of the 4th prop. of this book: this ſuperfluous part is not found in the tranſlation from the Arabic, and is now left out.

PROP. IX. B. VI.

This is demonſtrated in a particular caſe, viz. that in which the third part of a ſtraight line is required to be be cut off; which is not at all like Euclid's manner: Beſides, the author of the demonſtration, from four magnitudes being proportionals, concludes that the third of them is the ſame multiple of the fourth, which the firſt is of the ſecond; now, this is no where demonſtrated in the 5th book, as we now have it: But the editor aſſumes it from the confuſed notion which the vulgar have of proportionals: On this account it was neceſſary to give a general and legitimate demonſtration of this propoſition.

PROP. XVIII. B. VI.

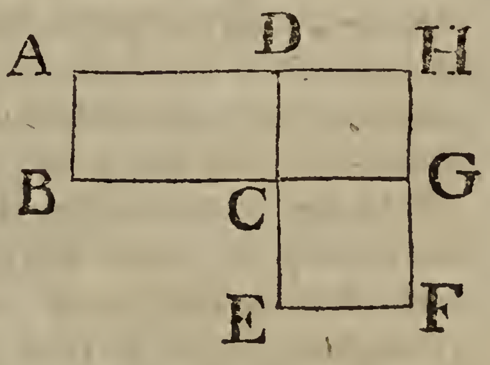
The demonſtration of this ſeems to be vitiated: For the propoſition is demonſtrated only in the caſe of quadrilateral figures, without mentioning how it may be extended to figures of five or more ſides: Beſides, from two triangles being equiangular, it is inferred, that a ſide of the one is to the homologous ſide of the other, as another ſide of the firſt is to the ſide homologous to it of the other, without permutation of the proportionals; which is contrary to Euclid's manner, as is clear from the next propoſition: And the ſame fault occurs again in the concluſion, where the ſides about the equal angles are not ſhewn to be proportionals, by reaſon of again neglecting permutation. On theſe accounts, a demonſtration is given in Euclid's manner, like to that he makes uſe of in the 20th prop. of this book; and it is extended to five-ſided figures, by which it may be ſeen how to extend it to figures of any number of ſides.

PROP. XXIII. B. VI.

Nothing is uſually reckoned more difficult in the elements of geometry by learners, than the doctrine of compound ratio, which Theon has rendered abſurd and ungeometrical, by ſubſtituting the 5th definition of the 6th book in place of the right definition, which without doubt Eudoxus or Euclid gave, in its proper place, after the definition of triplicate ratio, &c. in the 5th book. Theon's definition is this; a ratio is ſaid to be compounded of ratios *οταν ‘αι των λογων πηλικοτητες εφ’ εαυτας πολλαπλασιασθεισαι ϖοιωσι τινα*: Which Commandine thus tranſlates: "quando rationum quantitates inter ſe multiplicatæ aliquam efficiunt rationem;" that is, when the quantities of the ratios being multiplied by one another make a certain ratio. Dr Wallis tranſlates the word *πηλικοτητες* "rationem exponentes," the exponents of the ratios: And Dr Gregory renders the laſt words of the definition by "illius facit quantitatem," makes the quantity of that ratio: But in whatever ſenſe the "quantities," or "exponents of the ratios," and their "multiplication" be taken, the definition will be ungeometrical and uſeleſs: For there can be no multiplication but by a number: Now the quantity or exponent of a ratio (according to Eutochius in his comment. on prop. 4. book 2. of Arch. de Sph. et Cyl. and the moderns explain that term) is the number which multiplied into the conſequent term of a ratio produces the antecedent, or, which is the ſame thing, the number which ariſes by dividing the antecedent by the conſequent; but there are many ratios ſuch, that no number can ariſe from the diviſion of the antecedent by the conſequent; ex. gr. the ratio of which the diameter of a ſquare has to the ſide of it; and the ratio which the circumference of a circle has to its diameter, and ſuch like. Beſides, that there is not the leaſt mention made of this definition in the writings of Euclid, Archimedes, Apollonius, or other ancients, tho' they frequently make uſe of compound ratio: And in this 23d prop. of the 6th book, where compound ratio is firſt mentioned, there is not one word which can relate to this definition, though here, if in any place, it was neceſſary to be brought in; but the right definition is expreſsly cited in theſe words: "But the "ratio of K to M is compounded of the ratio of K to L, “and of the ratio of L to M.” This definition therefore of Theon is quite uſeleſs and abſurd: For that Theon brought it into the elements can ſcarce be doubted; as it is to be found in his commentary upon Ptolemy's Μεγαλη Συνταξις, page 62. where he alſo gives a childiſh explication of it, as agreeing only to ſuch ratios as can be expreſſed by numbers; and from this place the definition and explication have been exactly copied and prefixed to the definitions of the 6th book, as appears from Hervagius's edition: But Zambertus and Commandine, in their Latin tranſlations, ſubjoin the ſame to theſe definitions. Neither Campanus, nor, as it ſeems, the Arabic manuſcripts, from which he made his tranſlation, have this definition. Clavius, in his obſervations upon it, rightly judges that the definition of compound ratio might have been made after the ſame manner in which the definitions of duplicate and triplicate ratio are given, viz. “That as in ſeveral magnitudes that are continual proportionals, Euclid named the ratio of the firſt to the third, the duplicate ratio of the firſt to the ſecond; and the ratio of the firſt to the fourth, the triplicate ratio of the firſt to the ſecond, that is, the ratio compounded of two or three intermediate ratios that are equal to one another, and ſo on; ſo, in like manner, if there be ſeveral magnitudes of the ſame kind, following one another, which are not continual proportionals, the firſt is ſaid to have to the laſt the ratio compounded of all the intermediate ratios,——only for this reaſon, that theſe intermediate ratios are interpoſed betwixt the two extremes, viz. the firſt and laſt magnitudes; even as, in the 10th definition of the 5th book, the ratio of the firſt to the third was called the duplicate ratio, merely upon account of two ratios being interpoſed betwixt the extremes, that are equal to one another: So that there is no difference betwixt this compounding of ratios, and the duplication or triplication of them which are defined in the 5th book, but that in the duplication, triplication, &c. of ratios, all the interpoſed ratios are equal to one another; whereas, in the compounding of ratios, it is not neceſſary that the intermediate ratios ſhould be equal to one another.” Alſo Mr Edmund Scarburgh, in his Engliſh tranſlation of the firſt ſix books, page 238. 266. expreſsly affirms, that the 5th definition of the 6th book, is ſuppoſititious, and that the true definition of compound ratio is contained in the 10th definition of the 5th book, viz. the definition of duplicate ratio, or to be underſtood from it, to wit; in the ſame manner as Clavius has explained it in the preceding citation. Yet theſe, and the reſt of the moderns, do notwithſtanding retain this 5th def. of the 6th book, and illuſtrate and explain it by long commentaries, when they ought rather to have taken it quite away from the elements.

For, by comparing def. 5. book 6. with prop. 5. book 8. it will clearly appear that this definition has been put into the elements in place of the right one which has been taken out of them: becauſe, in prop. 5. book 8. it is demonſtrated that the plane number of which the ſides are C, D has to the plane number of which the ſides are E, Z, ſee Hergavius's or Gregory's edition), the ratio which is compounded of the ratios of their ſides; that is, of the ratios of C to E, and D to Z; and by def. 5. book 6. and the explication given of it by all the commentators, the ratio which is compounded of the ratios of C to E, and D to Z, is the ratio of the product made by the multiplication of the antecedents C, D to the product by the conſequents E, Z, that is, the ratio of the plane number of which the ſides are C, D to the plane number of which the ſides are E, Z. Wherefore the propoſition which is the 5th def. of book 6. is the very ſame with the 5th prop. of book 8. and therefore it ought neceſſarily to be cancelled in one of theſe places; becauſe it is abſurd that the ſame propoſition ſhould ſtand as a definition in one place of the elements, and be demonſtrated in another place of them. Now, there is no doubt that prop. 5. book 8. ſhould have a place in the elements, as the ſame thing is demonſtrated in it concerning plane numbers, which is demonſtrated in prop. 23. book 6. of equiangular parallelograms; wherefore def. 5. book 6. ought not to be in the elements. And from this it is evident that this definition is not Euclid's, but Theon's, or ſome other unſkilful geometer's.

But nobody, as far as I know, has hitherto ſhown the true uſe of compound ratio, or for what purpoſe it has been introduced into geometry: for every propoſition in which compound ratio is made uſe of, may without it be both enunciated and demonſtrated. Now the uſe of compound ratio conſiſts wholly in this, that by means of it, circumlocutions may be avoided, and thereby propoſitions may be more briefly either enunciated or demonſtrated, or both may be done, for inſtance, if this 23d propoſition of the ſixth book were to be enunciated, without mentioning compound ratio, it might be done as follows. If two parallelograms be equiangular, and if as a ſide of the firſt to a ſide of the ſecond, ſo any aſſumed ſtraight line be made to a ſecond ſtraight line; and as the other ſide of the firſt to the other ſide of the ſecond, ſo the ſecond ſtraight line be made a third. The firſt parallelogram is to the ſecond, as the firſt ſtraight line to the third. And the demonſtration would be exactly the ſame as we now have it. But the ancient geometers, when they obſerved this enunciation could be made ſhorter, by giving a name to the ratio which the firſt ſtraight line has to the laſt, by which name the intermediate ratios might likewiſe be ſignified, of the firſt to the ſecond, and of the ſecond to the third, and ſo on, if there were more of them, they called this ratio of the firſt to the laſt, the ratio compounded of the ratios of the firſt to the ſecond, and of the ſecond to the third ſtraight line; that is, in the preſent example, of the ratios which are the ſame with the ratios of the ſides, and by this they expreſſed the propoſition more briefly thus: If there be two equiangular parallelograms, they have to one another the ratio which is the ſame with that which is compounded of ratios that are the ſame with the ratios of the ſides. Which is ſhorter than the preceding enunciation, but has preciſely the ſame meaning. Or yet ſhorter thus: Equiangular parallelograms have to one another the ratio which is the ſame with that which is compounded of the ratios of their ſides. And theſe two enunciations, the firſt eſpecially, agree to the demonſtration which is now in the Greek. The propoſition may be more briefly demonſtrated, as Candalla does, thus: Let ABCD, CEFG be two equiangular parallelograms, and complete the parallelogram CDHG; then, becauſe there are three parallelograms AC, CH, CF, the firſt AC (by the definition of compound ratio) has to the third CF, the ratio which is compounded of the ratio of the firſt AC to the ſecond CH and of the ratio of CH to the third CF; but the parallelogram AC is to the parallelogram CH, as the ſtraight line BC to CG; and the parallelogram CH is to CF, as the ſtraight line CD is to CE; therefore the parallelogram AC has to CF the ratio which is compounded of ratios that are the ſame with the ratios of the ſides. And to this demonſtration agrees the enunciation which is at preſent in the text, viz. Equiangular parallelograms



rallelograms have to one another the ratio which is compounded of the ratios of the ſides: For the vulgar reading, "which "is compounded of their ſides," is abſurd. But, in this edition, we have kept the demonſtration which is in the Greek text, though not ſo ſhort as Candallas; becauſe the way of finding the ratio which is compounded of the ratios of the ſides, that is, of finding the ratio of the parallelograms, is ſhewn in that, but not in Candalla's demonſtration; whereby beginners may learn, in like caſes, how to find the ratio which is compounded of two or more given ratios.

From what has been ſaid, it may be obſerved, that in any magnitudes whatever of the ſame kind A, B, C, D, &c. the ratio compounded of the ratios of the firſt to the ſecond, of the ſecond to the third, and ſo on to the laſt, is only a name or expreſſion by which the ratio which the firſt A has to the laſt D is ſignified, and by which at the ſame time the ratios of all the magnitudes A to B, B to C, C to D from the firſt to the laſt, to one another, whether they be the ſame, or be not the ſame, are indicated; as in magnitudes which are continual proportionals A, B, C, D, &c. the duplicate ratio of the firſt to the ſecond is only a name, or expreſſion by which the ratio of the firſt A to the third C is ſignified, and by which, at the ſame time, is ſhown that there are two ratios of the magnitudes from the firſt to the laſt, viz. of the firſt A to the ſecond B, and of the ſecond B to the third or laſt C, which are the ſame with one another; and the triplicate ratio of the firſt to the ſecond is a name or expreſſion by which the ratio of the firſt A to the fourth D is ſignified, and by which, at the ſame time, is ſhown that there are three ratios of the magnitudes from the firſt to the laſt, viz. of the firſt A to the ſecond B, and of B to the third C, and of C to the fourth or laſt D, which are all the ſame with one another; and ſo in the caſe of any other multiplicate ratios. And that this is the right explication of the meaning of theſe ratios is plain from the definitions of duplicate and triplicate ratio in which Euclid makes uſe of the word λεγεται, is ſaid to be, or is called; which word, he, no doubt, made uſe of alſo in the definition of compound ratio, which Theon, or ſome other, has expunged from the elements; for the very ſame word is ſtill retained in the wrong definition of compound ratio, which is now the 5th of the 6th book: But in the citation of theſe definitions it is ſometimes retained, as in the demonſtration of prop. 19. book 6,

6. “ the firſt is ſaid to have, ἐχειν λεγεται, to the third the duplicate ratio,” &c. which is wrong tranſlated by Commandine and others, “ has” inſtead of “ is ſaid to have :” and ſometimes it is left out, as in the demonſtration of prop. 33. of the 11th book, in which we find “ the firſt has, εχει, to the third “ the triplicate ratio ;” but without doubt εχει, “ has,” in this place ſignifies the ſame as εχειν λεγεται, is ſaid to have : “ So “ likewiſe in prop. 23. B. 6. we find this citation, “ but the “ ratio of K to M is compounded, συγκειται, of the ratio of K to “ L, and the ratio of L to M,” which is a ſhorter way of expreſſing the ſame thing, which, according to the definition, ought to have been expreſſed by συγκεισθαι λεγεται, is ſaid to be compounded.

From theſe remarks, together with the propoſitions ſubjoined to the 5th book, all that is found concerning compound ratio, either in the ancient or modern geometers, may be underſtood and explained.

PROP. XXIV. B. VI.

It ſeems that ſome unſkilful editor has made up this demonſtration as we now have it, out of two others ; one of which may be made from the 2d prop. and the other from the 4th of this book : For after he has, from the 2d of this book, and compoſition and permutation, demonſtrated that the ſides about the angle common to the two parallelograms are proportionals, he might have immediately concluded that the ſides about the other equal angles were proportionals, viz. from prop. 34. B. 1. and prop. 7. book 5. This he does not, but proceeds to ſhow that the triangles and parallelograms are equiangular ; and in a tedious way, by help of prop. 4. of this book, and the 22d of book 5. deduces the ſame concluſion : From which it is plain that this ill compoſed demonſtration is not Euclid's : Theſe ſuperfluous things are now left out, and a more ſimple demonſtration is given from the 4th prop. of this book, the ſame which is in the tranſlation from the Arabic, by help of the 2d prop. and compoſition ; but in this the author neglects permutation, and does not ſhow the parallelograms to be equiangular, as is proper to do for the ſake of beginners.

PROP. XXV. B. VI.

It is very evident that the demonſtration which Euclid had given of this propoſition has been vitiated by ſome unſkilful hand : For, after this editor had demonſtrated that “ as the “ rectilineal figure ABC is to the rectilineal KGH, ſo is the “ parallelogram

Book VI. "parallelogram BE to the parallelogram EF;" nothing more ſhould have been added but this, "and the rectilineal figure "ABC is equal to the parallelogram BE; therefore the recti"lineal KGH is equal to the parallelogram EF," viz. from prop. 14. book 5. But betwixt theſe two ſentences he has inſerted this; "wherefore, by permutation, as the rectilineal fi"gure ABC to the parallelogram BE, ſo is the rectilineal KGH "to the parallelogram EF;" by which, it is plain, he thought it was not ſo evident to conclude that the ſecond of four proportionals is equal to the fourth from the equality of the firſt and third, which is a thing demonſtrated in the 14th prop. of B. 5. as to conclude that the third is equal to the fourth, from the equality of the firſt and ſecond, which is no where demonſtrated in the elements as we now have them: But though this propoſition, viz. the third of four proportionals is equal to the fourth, if the firſt be equal to the ſecond, had been given in the elements by Euclid, as very probably it was, yet he would not have made uſe of it in this place; becauſe, as was ſaid, the concluſion could have been immediately deduced without this ſuperfluous ſtep by permutation: This we have ſhown at the greater length, both becauſe it affords a certain proof of the vitiation of the text of Euclid; for the very ſame blunder is found twice in the Greek text of prop. 23. book 11. and twice in prop. 2. B. 12. and in the 5. 11. 12. and 18th of that book; in which places of book 12. except the laſt of them, it is rightly left out in the Oxford edition of Commandine's tranſlation; And alſo that geometers may beware of making uſe of permutation in the like caſes: for the moderns not unfrequently commit this miſtake, and among others Commandine himſelf in his commentary on prop. 5. book 3. p. 6. b. of Pappus Alexandrinus, and in other places: The vulgar notion of proportionals has, it ſeems, preoccupied many ſo much, that they do not ſufficiently underſtand the true nature of them.

Beſides, though the rectilineal figure ABC, to which another is to be made ſimilar, may be of any kind whatever; yet in the demonſtration the Greek text has "triangle" inſtead of "rec"tilineal figure," which error is corrected in the above-named Oxford edition.

PROP. XXVII. B. VI.

The ſecond caſe of this has 'αλλως, otherwiſe, prefixed to it, as if it was a different demonſtration, which probably has been done by ſome unſkilful librarian. Dr Gregory has rightly ly left it out: The ſcheme of this ſecond caſe ought to be marked with the ſame letters of the alphabet which are in the ſcheme of the firſt, as is now done.

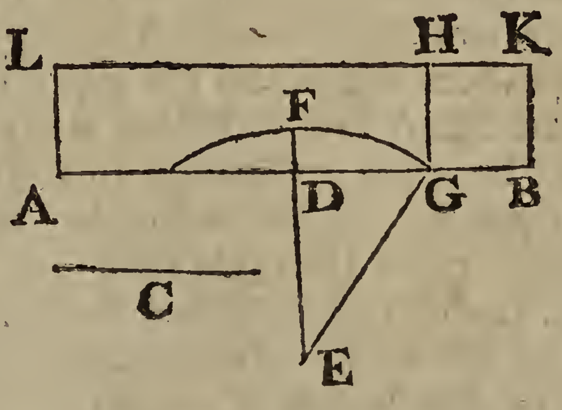
PROP. XXVIII. and XXIX. B. VI.

Theſe two problems, to the firſt of which the 27th prop. is neceſſary, are the moſt general and uſeful of all in the elements, and are moſt frequently made uſe of by the ancient geometers in the ſolution of other problems; and therefore are very ignorantly left out by Tacquet and Dechales in their editions of the elements, who pretend that they are ſcarce of any uſe: The caſes of theſe problems, wherein it is required to apply a rectangle which ſhall be equal to a given ſquare; to a given ſtraight line, either deficient or exceeding by a ſquare; as alſo to apply a rectangle which ſhall be equal to another given, to a given ſtraight line, deficient or exceeding by a ſquare; are very often made uſe of by geometers: And, on this account, it is thought proper, for the ſake of beginners, to give their conſtructions, as follows:

1. To apply a rectangle which ſhall be equal to a given ſquare, to a given ſtraight line, deficient by a ſquare: But the given ſquare muſt not be greater than that upon the half of the given line.

Let AB be the given ſtraight line, and let the ſquare upon the given ſtraight line C be that to which the rectangle to be applied muſt be equal, and this ſquare, by the determination, is not greater than that upon half of the ſtraight line AB.

Biſect AB in D, and if the ſquare upon AD be equal to the ſquare upon C, the thing required is done: But if it be not equal to it AD muſt be greater than C, according to the determination: Draw DE at right angles to AB, and make it equal to C; produce ED to F, ſo that EF be equal to AD or DB, and from the centre E, at the diſtance EF, deſcribe a circle meeting AB in G, and upon GB deſcribe the ſquare GBKH, and complete the rectangle AGHL; alſo join EG: And becauſe AB is biſected in D, the rectangle AG, GB together with the ſquare of DG is equal [a] to (the ſquare of DB, that is, of EF or EG, that is, to)



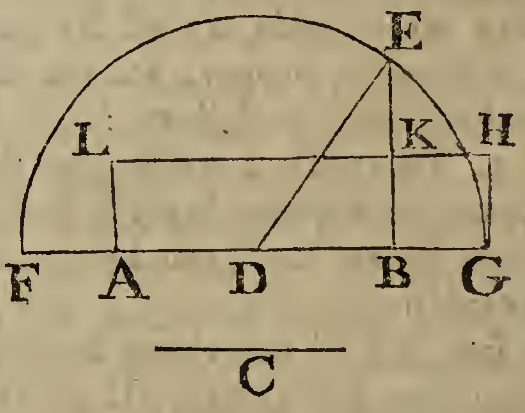
[a] 5. 2.

to) the ſquares of ED, DG: Take away the ſquare of DG from each of theſe equals; therefore the remaining rectangle AG, GB is equal to the ſquare of ED, that is, of C: But the rectangle AG, GB is the rectangle AH, becauſe GH is equal to GB: therefore the rectangle AH is equal to the given ſquare upon the ſtraight line C. Wherefore the rectangle AH equal to the given ſquare upon C, has been applied to the given ſtraight line AB, deficient by the ſquare GK. Which was to be done.

2. To apply a rectangle which ſhall be equal to a given ſquare, to a given ſtraight line, exceeding by a ſquare.

Let AB be the given ſtraight line, and let the ſquare upon the given ſtraight line C be that to which the rectangle to be applied muſt be equal.

Biſect AB in D, and draw BE at right angles to it, ſo that BE be equal to C; and having joined DE, from the centre D at the diſtance DE deſcribe a circle meeting AB produced in G; upon BG deſcribe the ſquare BGHK, and complete the rectangle AGHL. And becauſe AB is biſected in D, and produced to G, the rectangle AG, GB together with the ſquare of DB is equal[a] to (the ſquare of DG, or DE, that is, to) the ſquares of EB, BD. From each of theſe equals take the ſquare of DB; therefore the remaining rectangle AG, GB is equal to the ſquare of BE, that is, to the ſquare upon C. But the rectangle AG, GB is the rectangle AH, becauſe GH is equal to GB. Therefore the rectangle AH is equal to the ſquare upon C. Wherefore the rectangle AH, equal to the given ſquare upon C, has been applied to the given ſtraight line AB, exceeding by the ſquare GK. Which was to be done.



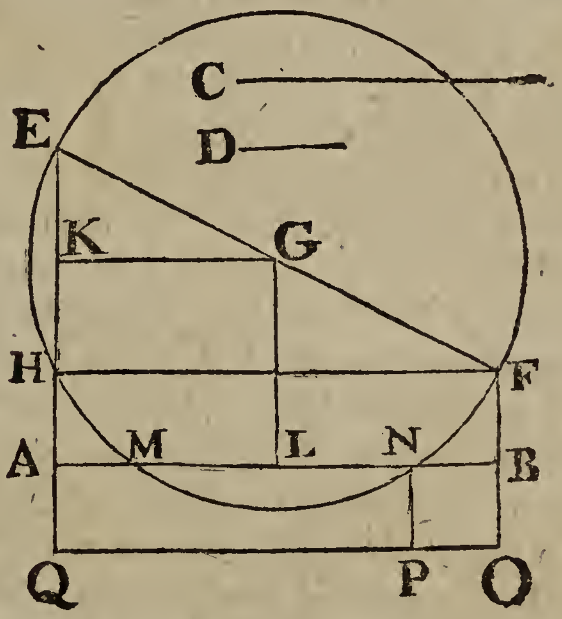
[a] 6. 2.

3. To apply a rectangle to a given ſtraight line which ſhall be equal to a given rectangle, and be deficient by a ſquare. But the given rectangle muſt not be greater than the ſquare upon the half of the given ſtraight line.

Let AB be the given ſtraight line, and let the given rectangle be that which is contained by the ſtraight lines C, D, which is not greater than the ſquare upon the half of AB; it is required to apply to AB a rectangle equal to the rectangle C, D, deficient by a ſquare.

Draw AE, BF at right angles to AB, upon the ſame ſide of it, and make AE equal to C, and BF to D: Join EF and biſect it in G; and from the centre G, at the diſtance GE, deſcribe a circle meeting AE again in H: join HF and draw GK parallel to it, and GL parallel to AE meeting AB in L.

Becauſe the angle EHF in a ſemicircle is equal to the right angle EAB, AB and HF are parallels, and AH and BF are parallels; wherefore AH is equal to BF, and the rectangle EA, AH equal to the rectangle EA, BF, that is to the rectangle C, D: And becauſe EG, GF are equal to one another, and AE, LG, BF parallels: therefore AL and LB are equal; alſo EK is equal to KH[a], and the rectangle C, D from the determination, is not greater than the ſquare of AL the half of AB; wherefore the rectangle EA, AH is not greater than the ſquare of AL, that is of KG: Add to each the ſquare of KE; therefore the ſquare[b] of AK is not greater than the ſquares of EK, KG, that is, than the ſquare of EG; and conſequently the ſtraight line AK or GL is not greater than GE. Now, if GE be equal to GL, the circle EHF touches AB in L, and therefore the ſquare of AL is[c] equal to the rectangle EA, AH, that is to the given rectangle C, D; and that which was required is done: But if EG, GL be unequal, EG muſt be the greater: and therefore the circle EHF cuts the ſtraight line AB: let it cut it in the points M, N, and upon NB deſcribe the ſquare NBOP, and complete the rectangle ANPQ: Becauſe LM is equal to[d] LN, and it has been proved that AL is equal to LB; therefore AM is equal to NB, and the rectangle AN, NB equal to the rectangle NA, AM, that is, to the rectangle[e] EA, AH or the rectangle C, D: But the rectangle AN, NB is the rectangle AP, becauſe PN is equal to NB: Therefore the rectangle AP is equal to the rectangle C, D; and the rectangle AP equal to the given rectangle C, D has been applied to the given ſtraight line AB, deficient by the ſquare BP. Which was to be done.



[a] 3. 3.

[b] 6. 2.

[c] 36. 3.

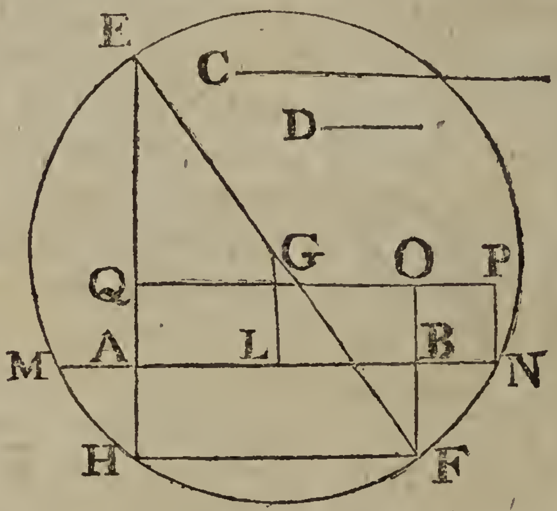
[d] 3. 3.

[e] Cor. 36. 3.

4. To apply a rectangle to a given ſtraight line that ſhall be equal to a given rectangle, exceeding by a ſquare.

Let AB be the given ſtraight line, and the rectangle C, D the given rectangle, it is required to apply a rectangle to AB equal to C, D, exceeding by a ſquare.

Draw AE, BF at right angles to AB, on the contrary ſides of it, and make AE equal to C, and BF equal to D: Join EF, and biſect it in G; and from the centre G, at the diſtance GE, deſcribe a circle meeting AE again in H; join HF, and draw GL parallel to AE; let the circle meet AB produced in M, N, and upon BN deſcribe the ſquare BNOP, and complete the rectangle ANPQ; becauſe the angle EHF in a ſemicircle is equal to the right angle EAB, AB and HF are parallels, and therefore AH and BF are equal, and the rectangle EA, AH equal to the rectangle EA, BF, that is, to the rectangle C, D: And becauſe ML is equal to LN, and AL to LB, therefore MA is equal to BN, and the rectangle AN, NB to MA, AN, that is, a 35. 3. [a] to the rectangle EA, AH, or the rectangle C, D: Therefore the rectangle AN, NB, that is, AP, is equal to the rectangle C, D; and to the given ſtraight line AB the rectangle AP has been applied equal to the given rectangle C, D, exceeding by the ſquare BP. Which was to be done.



Willebrordus Snellius was the firſt, as far as I know, who gave theſe conſtructions of the 3d and 4th problems in his Apollonius Batavus: And afterwards the learned Dr Halley gave them in the Scholium of the 18th prop. of the 8th book of Apollonius's conics reſtored by him.

The 3d problem is otherwiſe enunciated thus: To cut a given ſtraight line AB in the point N, ſo as to make the rectangle AN, NB equal to a given ſpace: Or, which is the ſame thing, having given AB the ſum of the ſides of a rectangle, and the magnitude of it being likewiſe given, to find its ſides.

And the 4th problem is the ſame with this, To find a point N in the given ſtraight line AB produced, ſo as to make the rectangle AN, NB equal to a given ſpace: Or, which is the ſame thing, having given AB the difference of the ſides of a rectangle, and the magnitude of it, to find the ſides.

PROP. XXXI. B. VI.

In the demonſtration of this, the inverſion of proportionals is twice neglected, and is now added, that the concluſion may be legitimately made by help of the 24th prop. of B. 5. as Clavius had done.

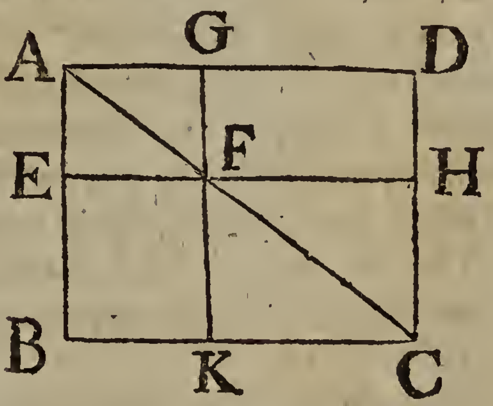
PROP. XXXII. B. VI.

The enunciation of the preceding 26th prop. is not general enough: becauſe not only two ſimilar parallelograms that have an angle common to both, are about the ſame diameter; but likewiſe two ſimilar parallelograms that have vertically oppoſite angles, have their diameters in the ſame ſtraight line: But there ſeems to have been another, and that a direct demonſtration of theſe caſes, to which this 32d propoſition was needful: And the 32d may be otherwiſe, and ſomething more briefly demonſtrated as follows.

PROP. XXXII. B. VI.

If two triangles which have two ſides of the one, &c.

Let GAF, HFC be two triangles which have two ſides AG, GF, proportional to the two ſides FH, HC, viz. AG to GF, as FH to HC; and let AG be parallel to FH, and GF to HC; AF and FC are in a ſtraight line.



Draw CK parallel [a] to FH, and let it meet GF produced in K: a 31. 1.
Becauſe AG, KC are each of them parallel to FH, they are parallel [b] b 30. 1.
to one another, and therefore the alternate angles AGF, FKC are equal: And AG is to GF, as (FH to HC, that is [c]) CK to c 34. 1.
KF; wherefore the triangles AGF, CKF are equiangular [d] d 6. 6.
and the angle AFG equal to the angle CFK: But GFK is a ſtraight line, therefore AF and FC are in a ſtraight line [e]. e 14. 1.

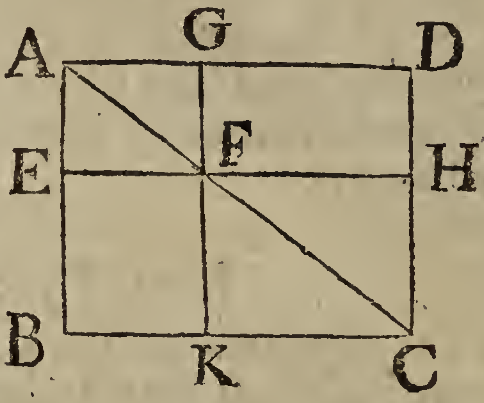
The 26th prop. is demonſtrated from the 32, as follows.

If two ſimilar and ſimilarly placed parallelograms have an angle common to both, or vertically oppoſite angles; their diameters are in the ſame ſtraight line.

Firſt,

Book VI. Firſt, Let the parallelograms ABCD, AEFG have the angle BAD common to both, and be ſimilar, and ſimilarly placed ABCD, AEFG are about the ſame diameter.

Produce EF, GF, to H, K, and join FA, FC; then becauſe the parallelograms ABCD, AEFG are ſimilar, DA is to AB, as GA to AE: wherefore the remainder DG is [a] to the remainder EB, as GA to AE: But DG is equal to FH, EB to HC, and AE to GF: Therefore as FH to HC, ſo is AG to GF; and FH, HC are parallel to AG, GF; and the triangles AGF, FHC are joined at one angle, in the point F; wherefore AF, FC are in the ſame ſtraight line [b].



a Cor. 19. 5.

b 32. 6.

Next, Let the parallelograms KFHC, GFEA, which are ſimilar and ſimilarly placed, have their angles KFH, GFE vertically oppoſite; their diameters AF, FC are in the ſame ſtraight line.

Becauſe AG, GF are parallel to FH, HC; and that AG is to GF, as FH to HC; therefore AF, FC are in the ſame ſtraight line [b].

PROP. XXXIII. B. VI.

The words, "becauſe they are at the centre," are left out as the addition of ſome unſkilful hand.

In the Greek, as alſo in the Latin tranſlation, the words α ετυχε, "any whatever," are left out in the demonſtration of both parts of the propoſition, and are now added as quite neceſſary; and, in the demonſtration of the ſecond part, where the triangle BGC is proved to be equal to CGK, the illative particle αρα in the Greek text ought to be omitted.

The ſecond part of the propoſition is an addition of Theon's, as he tells in his commentary on Ptolomy's Μεγαλη Συνταξις, p. 50.

PROP. B. C. D. B. VI.

Theſe three propoſitions are added, becauſe they are frequently made uſe of by geometers.

DEF.

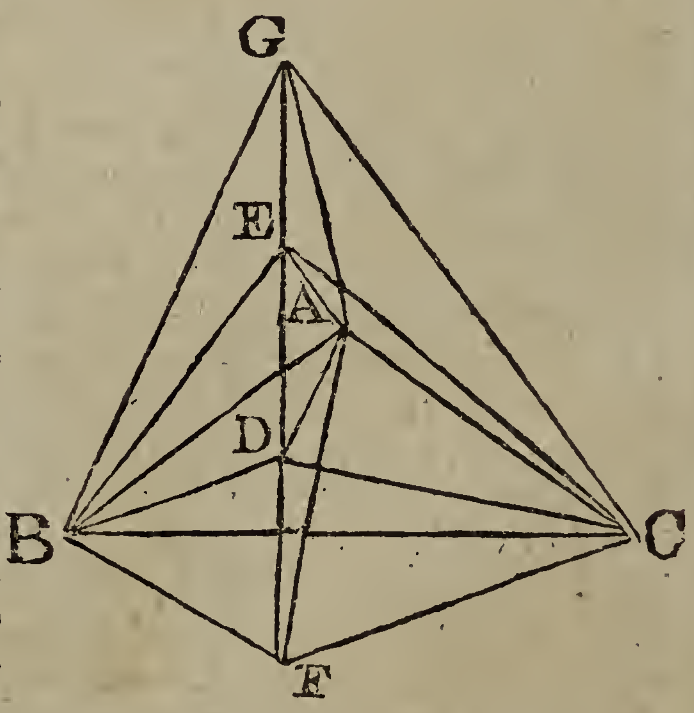
DEF. IX. and XI. B. XI.

THE ſimilitude of plane figures is defined from the equality of their angles, and the proportionality of the ſides about the equal angles; for from the proportionality of the ſides only, or only from the equality of the angles, the ſimilitude of the figures does not follow, except in the caſe when the figures are triangles: The ſimilar poſition of the ſides which contain the figures, to one another, depending partly upon each of theſe: And for the ſame reaſon, thoſe are ſimilar ſolid figures which have all their ſolid angles equal, each to each, and are contained by the ſame number of ſimilar plane figures: For there are ſome ſolid figures contained by ſimilar plane figures, of the ſame number, and even of the ſame magnitude, that are neither ſimilar nor equal, as ſhall be demonſtrated after the notes on the 10th definition: Upon this account it was neceſſary to amend the definition of ſimilar ſolid figures, and to place the definition of a ſolid angle before it: and from this and the 10th definition, it is ſufficiently plain how much the elements have been ſpoiled by unſkilful editors.

DEF. X. B. XI.

Since the meaning of the word "equal" is known and eſtabliſhed before it comes to be uſed in this definition: therefore the propoſition which is the 10th definition of this book, is a theorem, the truth or falſehood of which ought to be demonſtrated, not aſſumed; ſo that Theon, or ſome other Editor, has ignorantly turned a theorem which ought to be demonſtrated into this 10th definition: That figures are ſimilar, ought to be proved from the definition of ſimilar figures; that they are equal ought to be demonſtrated from the axiom, "Magnitudes that wholly coincide, are equal "to one another;" or from prop. A. of book 5. or the 9th prop. or the 14th of the ſame book, from one of which the equality of all kind of figures muſt ultimately be deduced. In the preceding books, Euclid has given no definition of equal figures, and it is certain he did not give this: For what is called the firſt def. of the third book, is really a theorem in which theſe circles are ſaid to be equal, that have the ſtraight lines from their centres to the circumferences equal, which is plain, from the definition of a circle; and therefore has by ſome editor been improperly placed among the definitions. The equality of figures ought not to be defined, but demonſtrated: Therefore, though it were true, that ſolid figures contained by the ſame number of ſimilar and equal plane figures are equal to one another, yet he would juſtly deſerve to be blamed who would make a definition of this propoſition which ought to be demonſtrated. But if this propoſition be not true, muſt it not be confeſſed, that geometers have, for theſe thirteen hundred years, been miſtaken in this elementary matter? And this ſhould teach us modeſty, and to acknowledge how little, through the weakneſs of our minds, we are able to prevent miſtakes even in the principles of ſciences which are juſtly reckoned amongſt the moſt certain; for that the propoſition is not univerſally true, can be ſhewn by many examples: The following is ſufficient.

Let there be any plane rectilineal figure, as the triangle ABC, and from a point D within it draw [a] the ſtraight line DE at right angles to the plane ABC; in DE take DE, DF equal to one another, upon the oppoſite ſides of the plane, and let G be any point in EF; join DA, DB, DC; EA EB, EC; FA, FB, FC; GA, GB, GC: Becauſe the ſtraight line EDF is at right angles to the plane ABC, it makes right angles with DA, DB, DC which it meets in that plane; and in the triangles EDB, FDB, ED and DB are equal to FD and DB, each to each, and they contain right angles; therefore the baſe EB is equal [b] to the baſe FB; in the ſame manner EA is equal to FA, and EC to FC: And in the triangles EBA, FBA, EB, BA are equal to FB, BA, and the baſe EA is equal to the baſe FA; wherefore the angle EBA is equal [c] to the angle FBA, and the triangle EBA equal [b] to the triangle FBA, and the other angles equal to the other angles; therefore theſe triangles are ſimilar [d]: In the ſame manner the triangle EBC is ſimilar to the



a 12. 11.
b 4. 1.
c 8. 1.
d 4. 6. 1. def.

the triangle FBC, and the triangle EAC to FAC; therefore there are two ſolid figures each of which is contained by ſix triangles, one of them by three triangles, the common vertex of which is the point G, and their baſes the ſtraight lines AB, BC, CA and by three other triangles the common vertex of which is the point E and their baſis the ſame lines AB, BC, CA: The other ſolid is contained by the ſame three triangles the common vertex of which is G, and their baſes AB, BC, CA; and by three other triangles of which the common vertex is the point F, and their baſes the ſame ſtraight lines AB, BC, CA: Now the three triangles GAB, GBC, GCA are common to both ſolids, and the three others EAB, EBC, ECA of the firſt ſolid have been ſhown equal and ſimilar to the three others FAB, FBC, FCA of the other ſolid, each to each; therefore theſe two ſolids are contained by the ſame number of equal and ſimilar planes: But that they are not equal is manifeſt, becauſe the firſt of them is contained in the other: Therefore it is not univerſally true that ſolids are equal which are contained by the ſame number of equal and ſimilar planes.

Cor. From this it appears that two unequal ſolid angles may be contained by the ſame number of equal plane angles.

For the ſolid angle at B, which is contained by the four plane angles EBA, EBC, GBA, GBC is not equal to the ſolid angle at the ſame point B which is contained by the four plane angles FBA, FBC, GBA, GBC; for this laſt contains the other: And each of them is contained by four plane angles, which are equal to one another, each to each, or are the ſelf ſame; as has been proved: And indeed there may be innumerable ſolid angles all unequal to one another, which are each of them contained by plane angles that are equal to one another, each to each: It is likewiſe manifeſt that the before-mentioned ſolids are not ſimilar, ſince their ſolid angles are not all equal.

And that there may be innumerable ſolid angles all unequal to one another, which are each of them contained by the ſame plane angles diſpoſed in the ſame order, will be plain from the three following propoſitions.

PROP. I. PROBLEM.

Three magnitudes, A, B, C being given, to find a fourth ſuch, that every three ſhall be greater than the remaining one.

Let D be the fourth: therefore D muſt be leſs than A, B, C together: Of the three A, B, C, let A be that which is not leſs than either of the two B and C: And firſt, let B and C together be not leſs than A; therefore B, C, D together are greater than A; and becauſe A is not leſs than B; A, C, D together are greater than B: In the like manner A, B, D together are greater than C: Wherefore in the caſe in which B and C together are not leſs than A, any magnitude D which is leſs than A, B, C together will anſwer the problem.

But if B and C together be leſs than A; then, becauſe it is required that B, C, D together be greater than A, from each of theſe taking away B, C, the remaining one D muſt be greater than the exceſs of A above B and C: Take therefore any magnitude D which is leſs than A, B, C together, but greater than the exceſs of A above B and C: Then B, C, D together are greater than A; and becauſe A is greater than either B or C, much more will A and D, together with either of the two B, C be greater than the other: And, by the conſtruction, A, B, C are together greater than D.

Cor. If beſides it be required, that A and B together ſhall not be leſs than C and D together; the exceſs of A and B together above C muſt not be leſs than D, that is, D muſt not be greater than that exceſs.

PROP. II. PROBLEM.

Four magnitudes A, B, C, D being given, of which A and B together are not leſs than C and D together, and ſuch that any three of them whatever are greater than the fourth; it is required to find a fifth magnitude E ſuch, that any two of the three A, B, E ſhall be greater than the third, and alſo that any two of the three C, D, E ſhall be greater than the third. Let A be not leſs than B: And C not leſs than D.

Firſt, Let the exceſs of C above D be not leſs than the exceſs of A above B: It is plain that a magnitude E can be taken which is leſs than the ſum of C and D, but greater than the exceſs of C above D; let it be taken; then E is greater likewiſe than the exceſs of A above B; wherefore E and B together are greater than A; and A is not leſs than B; therefore A and E together are greater than B; And, by the hypotheſis, A and B together are not leſs than C and D together, and C and D together are greater than E; therefore likewiſe A and B are greater than E.

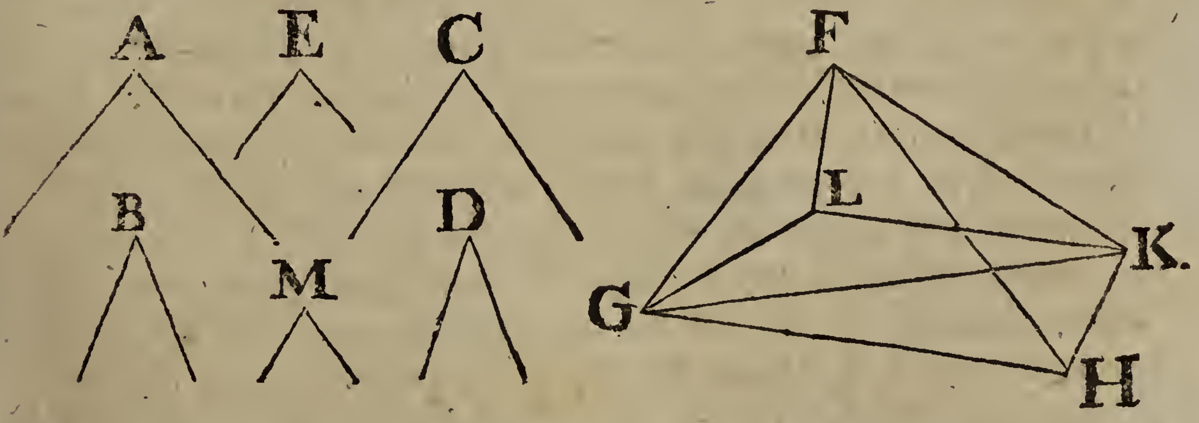
But let the exceſs of A above B be greater than the exceſs of C above D: And becauſe, by the hypotheſis, the three B, C, D are together greater than the fourth A; C and D together are greater than the exceſs of A above B: Therefore a magnitude may be taken which is leſs than C and D together, but greater than the exceſs of A above B. Let this magnitude be E; and becauſe E is greater than the exceſs of A above B, B together with E is greater than A: And, as in the preceding caſe, it may be ſhown that A together with E is greater than B, and that A together with B is greater than E: Therefore, in each of the caſes, it has been ſhown that any two of the three A, B, E are greater than the third.

And becauſe in each of the caſes E is greater than the exceſs of C above D, E together with D is greater than C; and by the hypotheſis, C is not leſs than D; therefore E together with C is greater than D; and, by the conſtruction, C and D together are greater than E: Therefore any two of the three, C, D, E are greater than the third.

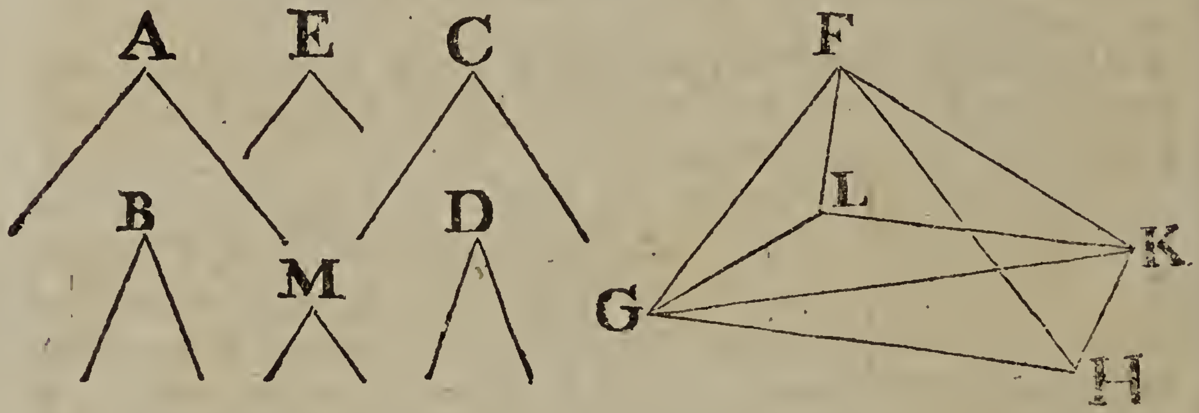
PROP. III. THEOREM.

There may be innumerable ſolid angles all unequal to one another, each of which is contained by the ſame four plane angles, placed in the ſame order.

Take three plane angles, A, B, C, of which A is not leſs than either of the other two, and ſuch, that A and B together are leſs than two right angles: and by problem 1. and its corollary, find a fourth angle D ſuch, that any three whatever of the angles A, B, C, D be greater than the remaining angle, and ſuch, that A and B together be not leſs than C and D together: And by problem 2. find a fifth angle E ſuch that any two of the angles A, B, E be greater than the third and alſo that any two of the angles C, D, E be greater than

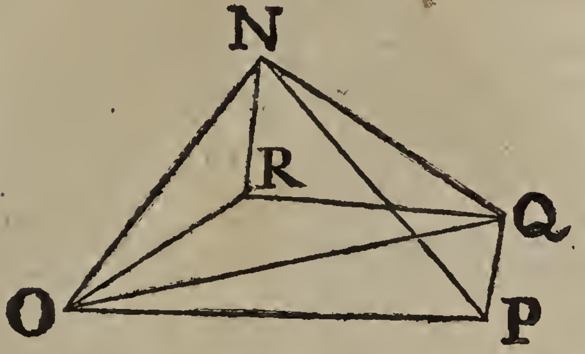


the third: And becauſe A and B together are leſs than two right angles, the double of A and B together is leſs than four right angles: But A and B together are greater than the angle E; wherefore the double of A, B together is greater than the three angles A, B, E together, which three are conſequently leſs than four right angles; and every two of the ſame angles A, B, E are greater than the third; therefore, by prop. 23. 11. a ſolid angle may be made contained by three plane angles equal to the angles A, B, E, each to each: Let this be the angle F contained by the three plane angles GFH, HFK, GFK which are equal to the angles A, B, E, each to each: And becauſe the angles C, D together are not greater than the angles A, B together, therefore the angles C, D, E are not greater than the angles A, B, E: But theſe laſt three are leſs than four right angles, as has been demonſtrated: wherefore alſo the angles C, D, E are together leſs than four right angles, and every two of them are greater than the third; therefore a ſolid angle may be made, which ſhall be contained by three plane angles equal to the angles C, D, E, each to each[a]: And by prop. 26. 11. at the point F in the ſtraight line FG a ſolid angle may be made equal to that which is contained by the three plane angles that are equal to the angles C, D, E: Let this be made, and let the angle GFK, which is equal to E, be one of the three; and let KFL, GFL be the other two which are equal to the angles, C, D, each to each. Thus there is a ſolid angle conſtituted at the point F contained by the four plane angles GFH, HFK, KFL, GFL which are equal to the angles A, B, C, D, each to each.



[a] 23. 11.

Again, Find another angle M ſuch, that every two of the three angles A, B, M be greater than the third, and alſo every two of the three C, D, M be greater than the third: And,



And, as in the preceding part, it may be demonſtrated that the three A, B, M are leſs than four right angles, as alſo that the three C, D, M are leſs than four right angles. Make therefore [a] a ſolid angle at N contained by the three plane angles ONP, PNQ, ONQ, which are equal to A, B, M, each to each: And by prop. 26. 11. make at the point N in the ſtraight line ON a ſolid angle contained by three plane angles of which one is the angle ONQ equal to M, and the other two are the angles QNR, ONR which are equal to the angles C, D, each to each. Thus, at the point N, there is a ſolid angle contained by the four plane angles ONP, PNQ, QNR, ONR which are equal to the angles A, B, C, D each to each. And that the two ſolid angles at the points F, N, each of which is contained by the above named four plane angles, are not equal to one another, or that they cannot coincide, will be plain by conſidering that the angles GFK, ONQ; that is, the angles E, M, are unequal by the conſtruction; and therefore the ſtraight lines GF, FK cannot coincide with ON, NQ, nor conſequently can the ſolid angles, which therefore are unequal.

[a] 23. 11.

And becauſe from the four plane angles A, B, C, D, there can be found innumerable other angles that will ſerve the ſame purpoſe with the angles E and M; it is plain that innumerable other ſolid angles may be conſtituted which are each contained by the ſame four plane angles, and all of them unequal to one another, Q. E. D.

And from this it appears that Clavius and other authors are miſtaken, who aſſert that thoſe ſolid angles are equal which are contained by the ſame number of plane angles that are equal to one another, each to each. Alſo it is plain that the 26th prop. of book 11. is by no means ſufficiently demonſtrated, becauſe the equality of two ſolid angles, whereof each is contained by three plane angles which are equal to one another, each to each, is only aſſumed, and not demonſtrated.

PROP.

PROP. I. B. XI.

The words at the end of this, " for a ſtraight line cannot " meet a ſtraight line in more than one point," are left out, as an addition by ſome unſkilful hand; for this is to be demonſtrated, not aſſumed.

Mr Thomas Simpſon, in his notes at the end of the 2d edition of his elements of geometry, p. 262. after repeating the words of this note, adds, " Now, can it poſſibly ſhow any want " of ſkill in an editor (he means Euclid or Theon) to refer to " an axiom which Euclid himſelf hath laid down, Book 1. ' No. 14." he means Barrow's Euclid, for it is the 10th in the Greek, " and not to have demonſtrated, what no man can de- " monſtrate?" But all that in this caſe can follow from that axiom is, that, if two ſtraight lines could meet each other in two points, the parts of them betwixt theſe points muſt coincide, and ſo they would have a ſegment betwixt theſe points common to both. Now, as it has not been ſhown in Euclid, that they cannot have a common ſegment, this does not prove that they cannot meet in two points from which their not having a common ſegment, is deduced in the Greek edition: But, on the contrary, becauſe they cannot have a common ſegment, as is ſhown in Cor. of 11th prop. Book 1. of 4to edition, it follows plainly that they cannot meet in two points, which the remarker ſays no man can demonſtrate.

Mr Simpſon, in the ſame notes, p. 265. juſtly obſerves, that in the corollary of prop. 11. Book 1. 4to edit. the ſtraight lines AB, BD, BC, are ſuppoſed to be all in the ſame plane, which cannot be aſſumed in 1ſt prop. book 11. This, ſoon after the 4to edition was publiſhed, I obſerved and corrected as it is now in this edition: He is miſtaken in thinking the 10th axiom he mentions here to be Euclid's; it is none of Euclid's, but is the 10th in Dr Barrow's edition, who had it from Herigon's Curſus, vol. 1. and in place of it the corollary of 10th prop. Book 1. was added.

PROP. II. B. XI.

This propoſition ſeems to have been changed and vitiated by ſome editor; for all the figures defined in the 1ſt book of the elements, and among them triangles, are by the hypotheſis, plane figures; that is, ſuch as are deſcribed in a plane; wherefore the ſecond part of the enunciation needs no demonſtration. Beſides, a convex ſuperficies may be terminated by

three

three ſtraight lines meeting one another: The thing that ſhould have been demonſtrated is, that two, or three ſtraight lines, that meet one another, are in one plane. And as this is not ſufficiently done, the enunciation and demonſtration are changed into thoſe now put into the text.

PROP. III. B. XI.

In this propoſition the following words near to the end of it are left out, viz. "therefore DEB, DFB are not ſtraight lines; "in the like manner it may be demonſtrated that there can be "no other ſtraight line between the points D, B:" Becauſe from this, that two lines include a ſpace, it only follows that one of them is not a ſtraight line: And the force of the argument lies in this, viz. if the common ſection of the planes be not a ſtraight line, then two ſtraight lines could include a ſpace, which is abſurd; therefore the common ſection is a ſtraight line.

PROP. IV. B. XI.

The words "and the triangle AED to the triangle BEC" are omitted, becauſe the whole concluſion of the 4th prop. B. 1. has been ſo often repeated in the preceding books, it was needleſs to repeat it here.

PROP. V. B. XI.

In this, near to the end, ’επιπεδ̀ω, ought to be left out in the Greek text: And the word "plane" is rightly left out in the Oxford edition of Commandine's tranſlation.

PROP. VII. B. XI.

This propoſition has been put into this book by ſome unſkilful editor, as is evident from this, that ſtraight lines which are drawn from one point to another in a plane, are, in the preceding books, ſuppoſed to be in that plane: And if they were not, ſome demonſtrations in which one ſtraight line is ſuppoſed to meet another would not be concluſive, becauſe theſe lines would not meet one another: For inſtance, in prop. 30. B. 1. the ſtraight line GK would not meet EF, if GK were not in the plane in which are the parallels AB, CD, and in which, by hypotheſis, the ſtraight line EF is: Beſides, this 7th propoſition is demonſtrated by the preceding 3. in which the very thing which is propoſed to be demonſtrated in the 7th, is twice aſſumed, viz. that the ſtraight line drawn from one point to another in a plane, is in that plane; and the ſame thing is aſſumed in the preceding 6th prop. in which the ſtraight line which joins the points B, D that are in the plane to which AB and CD are at right angles, is ſuppoſed to be in that plane: And the 7th, of which another demonſtration is given, is kept in the book merely to preſerve the number of the propoſitions; for it is evident from the 7th and 35th definitions of the 1ſt book, though it had not been in the elements.

PROP. VIII. B. XI.

In the Greek, and in Commandine's and Dr Gregory's tranſlations, near to the end of this propoſition, are the following words: "But DC is in the plane through BA, AD," inſtead of which, in the Oxford edition of Commandine's tranſlation is rightly put "but DC is in the plane through BD, "DA:" But all the editions have the following words, viz. "becauſe AB, BD are in the plane thro' BD, DA, and DC "is in the plane in which are AB, BD," which are manifeſtly corrupted, or have been added to the text; for there was not the leaſt neceſſity to go ſo far about to ſhow that DC is in the ſame plane in which are BD, DA becauſe it immediately follows from prop. 7. preceding, that BD, DA, are in the plane in which are the parallels AB, CD: Therefore, inſtead of theſe words, there ought only to be "becauſe all three are in the plane in which "are the parallels AB, CD."

PROP. XV. B. XI.

After the words, "and becauſe BA is parallel to GH," the following are added, "for each of them is parallel to DE, and "are not both in the ſame plane with it," as being manifeſtly forgotten to be put into the text.

PROP. XVI. B. XI.

In this, near to the end, inſtead of the words, "but ſtraight "lines which meet neither way" ought to be read, "but "ſtraight lines in the ſame plane which produced meet neither "way:" Becauſe, though in citing this definition in prop. 27. book 1. it was not neceſſary to mention the words, "in the "ſame plane," all the ſtraight lines in the books preceding this being in the ſame plane; yet here it was quite neceſſary.

PROP. XX. B. XI.

In this near the beginning, are the words, "But if not, let "BAC be the greater:" But the angle BAC may happen to be equal to one of the other two: Wherefore this place ſhould be read thus, " But if not, let the angle BAC be not leſs than " either of the other two, but greater than DAB."

At the end of this propoſition it is ſaid, " in the ſame manner it may be demonſtrated," though there is no need of any demonſtration; becauſe the angle BAC being not leſs than either of the other two, it is evident that BAC together with one of them is greater than the other.

PROP. XXII. B. XI.

And likewiſe in this, near the beginning, it is ſaid, " But " if not, let the angles at B, E, H be unequal, and let the angle at B be greater than either of thoſe at E, H:" Which words manifeſtly ſhow this place to be vitiated, becauſe the angle at B may be equal to one of the other two. They ought therefore to be read thus, " But if not, let the angles at B, E " H be unequal, and let the angle at B be not leſs than either " of the other two at E, H: Therefore the ſtraight line AC " is not leſs than either of the two DF, GK."

PROP. XXIII. B. XI.

The demonſtration of this is made ſomething ſhorter, by not repeating in the third caſe the things which were demonſtrated in the firſt; and by making uſe of the conſtruction which Campanus has given; but he does not demonſtrate the ſecond and third caſes: The conſtruction and demonſtration of the third caſe are made a little more ſimple than in the Greek text.

PROP. XXIV. B. XI.

The word " ſimilar" is added to the enunciation of this propoſition, becauſe the planes containing the ſolids which are to be demonſtrated to be equal to one another, in the 25th propoſition, ought to be ſimilar and equal, that the equality of the ſolids may be inferred from prop. C. of this book: And, in the Oxford edition of Commandine's tranſlation, a corrollary is added to prop. 24. to ſhow that the parallelograms mentioned in this propoſition are ſimilar, that the equality of the ſolids in prop. 25. may be deduced from the 10th def. of book 11.

PROP. XXV. and XXVI. B. XI.

In the 25th prop. ſolid figures which are contained by the ſame number of ſimilar and equal plane figures, are ſuppoſed to be equal to one another. And it ſeems that Theon, or ſome other

Book XI. other editor, that he might ſave himſelf the trouble of demonſtrating the ſolid figures mentioned in this propoſition to be equal to one another, has inſerted the 10th def. of this book, to ſerve inſtead of a demonſtration; which was very ignorantly done.

Likewiſe in the 26th prop. two ſolid angles are ſuppoſed to be equal: If each of them be contained by three plane angles which are equal to one another, each to each. And it is ſtrange enough that none of the commentators on Euclid have, as far as I know, perceived that ſomething is wanting in the demonſtrations of theſe two propoſitions. Clavius, indeed, in a note upon the 11th def. of this book, affirms, that it is evident that thoſe ſolid angles are equal which are contained by the ſame number of plane angles, equal to one another, each to each, becauſe they will coincide, if they be conceived to be placed within one another; but this is ſaid without any proof, nor is it always true, except when the ſolid angles are contained by three plane angles only, which are equal to one another, each to each: And in this caſe the propoſition is the ſame with this, that two ſpherical triangles that are equilateral to one another, are alſo equiangular to one another, and can coincide: which ought not to be granted without a demonſtration. Euclid does not aſſume this in the caſe of rectilineal triangles, but demonſtrates in prop. 8. book 1. that triangles which are equilateral to one another are alſo equiangular to one another; and from this their total equality appears by prop. 4. book 1. And Menelaus, in the 4th prop. of his firſt book of ſpherics, explicitly demonſtrates that ſpherical triangles which are mutually equilateral, are alſo equiangular to one another; from which it is eaſy to ſhow that they muſt coincide, providing they have their ſides diſpoſed in the ſame order and ſituation.

To ſupply theſe defects, it was neceſſary to add the three propoſitions marked A, B, C to this book. For the 25th, 26th and 28th propoſitions of it, and conſequently eight others, viz. the 27th, 31ſt, 32d, 33d, 34th, 36th, 37th, and 40th of the ſame, which depend upon them, have hitherto ſtood upon an infirm foundation; as alſo, the 8th, 12th, Cor. of 17th and 18th of 12th book, which depend upon the 9th definition. For it has been ſhown in the notes on def. 10. of this book, that ſolid figures which are contained by the ſame number of ſimilar and equal plane figures, as alſo ſolid angles that are contained by the ſame number of equal plane angles, are not always equal to one another.

It is to be obſerved that Tacquet, in his Euclid, defines equal ſolid angles to be ſuch, "as being put within one another do "coincide:" But this is an axiom, not a definition; for it is true of all magnitudes whatever. He made this uſeleſs definition, that by it he might demonſtrate the 36th prop. of this book, without the help of the 35th of the ſame: Concerning which demonſtration, ſee the note upon prop. 36.

PROP. XXVIII. B. XI.

In this it ought to have been demonſtrated, not aſſumed, that the diagonals are in one plane. Clavius has ſupplied this defect.

PROP. XXIX. B. XI.

There are three caſes of this propoſition; the firſt is, when the two parallelograms oppoſite to the baſe AB have a ſide common to both; the ſecond is, when theſe parallelograms are ſeparated from one another; and the third, when there is a part of them common to both; and to this laſt only, the demonſtration that has hitherto been in the elements does agree. The firſt caſe is immediately deduced from the preceding 28th prop. which ſeems for this purpoſe to have been premiſed to this 29th, for it is neceſſary to none but to it, and to the 40th of this book, as we now have it, to which laſt, it would, without doubt, have been premiſed, if Euclid had not made uſe of it in the 29th; but ſome unſkilful editor has taken it away from the elements, and has mutilated Euclid's demonſtration of the other two caſes, which is now reſtored, and ſerves for both at once.

PROP. XXX. B. XI.

In the demonſtration of this, the oppoſite planes of the ſolid CP, in the figure in this edition, that is, of the ſolid CO in Commandine's figure, are not proved to be parallel; which it is proper to do for the ſake of learners.

PROP. XXXI. B. XI.

There are two caſes of this propoſition; the firſt is, when the inſiſting ſtraight lines are at right angles to the baſes; the other, when they are not: The firſt caſe is divided again into two others, one of which is, when the baſes are equiangular parallelograms; the other, when they are not equiangular.

The Greek editor makes no mention of the firſt of theſe two laſt caſes, but has inſerted the demonſtration of it as a part of that of the other: And therefore ſhould have taken notice of it in a corollary; but we thought it better to give theſe two caſes ſeparately: The demonſtration alſo is made ſomething ſhorter by following the way Euclid has made uſe of in prop. 14. book 6. Beſides, in the demonſtration of the caſe in which the inſiſting ſtraight lines are not at right angles to the baſes, the editor does not prove that the ſolids deſcribed in the conſtruction, are parallelepipeds, which it is not to be thought that Euclid neglected: Alſo the words, "of which the inſiſting ſtraight "lines are not in the ſame ſtraight lines," have been added by ſome unſkilful hand; for they may be in the ſame ſtraight lines.

PROP. XXXII. B. XI.

The editor has forgot to order the parallelogram FH to be applied in the angle FGH equal to the angle LCG, which is neceſſary. Clavius has ſupplied this.

Alſo, in the conſtruction, it is required to complete the ſolid of which the baſe is FH, and altitude the ſame with that of the ſolid CD: But this does not determine the ſolid to be completed, ſince there may be innumerable ſolids upon the ſame baſe, and of the ſame altitude: It ought therefore to be ſaid, "complete the ſolid of which the baſe is FH, and one of "its inſiſting ſtraight lines is FD:" The ſame correction muſt be made in the following propoſition 33.

PROP. D. B. XI.

It is very probable that Euclid gave this propoſition a place in the elements, ſince he gave the like propoſition concerning equiangular parallelograms in the 23d. B. 6.

PROP. XXXIV. B. XI.

In this the words, ὧν αἱ ἐφεστῶσαι οὐκ εισιν ἐπι τῶν αυτῶν ευθειῶν, "of which the inſiſting ſtraight lines are not in the ſame "ſtraight lines," are thrice repeated; but theſe words ought either to be left out, as they are by Clavius, or, in place of them, ought to be put, "whether the inſiſting ſtraight lines be, "or be not in the ſame ſtraight lines:" For the other caſe is without any reaſon excluded; alſo the words, ὧν τα ὕψη, of

“ which the altitudes,” are twice put for ὧν ἁι ἐφεστωσαι, “ of “ which the inſiſting ſtraight lines ;” which is a plain miſtake : For the altitude is always at right angles to the baſe.

PROP. XXXV. B. XI.

The angles ABH, DEM are demonſtrated to be right angles in a ſhorter way than in the Greek ; and in the ſame way ACH, DFM may be demonſtrated to be right angles : Alſo the repetition of the ſame demonſtration, which begins with “ in the ſame manner,” is left out, as it was probably added to the text by ſome editor : for the words “ in like manner “ we may demonſtrate,” are not inſerted except when the demonſtration is not given, or when it is ſomething different from the other if it be given, as in prop. 26. of this book. Companus has not this repetition.

We have given another demonſtration of the corollary, beſides the one in the original, by help of which the following 36th prop. may be demonſtrated without the 35th.

PROP. XXXVI. B. XI.

Tacquet in his Euclid demonſtrates this propoſition without the help of the 35th ; but it is plain, that the ſolids mentioned in the Greek text in the enunciation of the propoſition as equiangular, are ſuch that their ſolid angles are contained by three plane angles equal to one another, each to each ; as is evident from the conſtruction. Now Tacquet does not demonſtrate, but aſſumes theſe ſolid angles to be equal to one another ; for he ſuppoſes the ſolids to be already made, and does not give the conſtruction by which they are made : But, by the ſecond demonſtration of the preceding corollary, his demonſtration is rendered legitimate likewiſe in the caſe where the ſolids are conſtructed as in the text.

PROP. XXXVII. B. XI.

In this it is aſſumed that the ratios which are triplicate of thoſe ratios which are the ſame with one another, are likewiſe the ſame with one another ; and that thoſe ratios are the ſame with one another, of which the triplicate ratios are the ſame with one another ; but this ought not to be granted without a demonſtration ; nor did Euclid aſſume the firſt and eaſieſt of theſe two propoſitions, but demonſtrated it in the caſe of duplicate ratios, in the 22d prop. book 6. On this account, another demonſtration is given of this propoſition like to that which Euclid gives in prop. 22. book 6. as Clavius has done.

PROP. XXXVIII. B. XI.

When it is required to draw a perpendicular from a point in one plane which is at right angles to another plane, unto this laſt plane, it is done by drawing a perpendicular from the point to the common ſection of the planes; for this perpendicular will be perpendicular to the plane, by Def. 4. of this book: And it would be fooliſh in this caſe to do it by the 11th prop. of the ſame: But Euclid[a], Apollonius, and other geometers, when they have occaſion for this problem, direct a perpendicular to be drawn from the point to the plane, and conclude that it will fall upon the common ſection of the planes, becauſe this is the very ſame thing as if they had made uſe of the conſtruction above mentioned, and then concluded that the ſtraight line muſt be perpendicular to the plane; but is expreſſed in fewer words: Some editor, not perceiving this, thought it was neceſſary to add this propoſition, which can never be of any uſe to the 11th book, and its being near to the end among propoſitions with which it has no connection, is a mark of its having been added to the text.

[a] 17. 12. in other editions.

PROP. XXXIX. B. XI.

In this it is ſuppoſed, that the ſtraight lines which biſect the ſides of the oppoſite planes, are in one plane, which ought to have been demonſtrated; as is now done.

B. XII.

Book XII.

THE learned Mr Moor, profeſſor of Greek in the Univerſity of Glaſgow, obſerved to me, that it plainly appears from Archimedes's epiſtle to Doſitheus, prefixed to his books of the Sphere and Cylinder, which epiſtle he has reſtored from ancient manuſcripts, that Eudoxus was the author of the chief propoſitions in this 12th book.

PROP. II. B. XII.

At the beginning of this it is ſaid, "if it be not ſo, the ſquare "of BD ſhall be to the ſquare of FH, as the circle ABCD is "to ſome ſpace either leſs than the circle EFGH, or greater "than it:" And the like is to be found near to the end of this propoſition, as alſo in prop. 5. 11. 12. 18. of this book: Concerning which, it is to be obſerved, that in the demonſtration of theorems, it is ſufficient, in this and the like caſes, that a thing made uſe of in the reaſoning can poſſibly exiſt, providing this be evident, though it cannot be exhibited or found by a geometrical conſtruction: So, in this place, it is aſſumed, that there may be a fourth proportional to theſe three magnitudes, viz. the ſquares of BD, FH, and the circle ABCD; becauſe it is evident that there is ſome ſquare equal to the circle ABCD though it cannot be found geometrically; and to the three rectilineal figures, viz. the ſquares of BD, FH, and the ſquare which is equal to the circle ABCD, there is a fourth ſquare proportional; becauſe to the three ſtraight lines which are their ſides, there is a fourth ſtraight line proportional [a], and this fourth ſquare, or a ſpace equal to it, is the ſpace which in this propoſition is denoted by the letter S: And the like is to be underſtood in the other places above cited: And it is probable that this has been ſhewn by Euclid, but left out by ſome editor; for the lemma which ſome unſkilful hand has added to this propoſition explains nothing of it.

a 12. 6.

PROP. III. B. XII.

In the Greek text and the tranſlations, it is ſaid, "and "becauſe the two ſtraight lines BA, AC which meet one an"other," &c. here the angles BAC, KHL are demonſtrated to be equal to one another by 10th prop. B. 11. which had been done before: Becauſe the triangle EAG was proved to be ſimilar to the triangle KHL: This repetition is left out, and the triangles BAC, KHL are proved to be ſimilar in a ſhorter way by prop. 21. B. 6.

PROP. IV. B. XII.

A few things in this are more fully explained than in the Greek text.

PROP. V. B. XII.

In this, near to the end, are the words, ὡς ἔμπροσθεν ’εδείχθη, "as was before ſhown," and the ſame are found again in the end of prop. 18. of this book; but the demonſtration referred to, except it be the uſeleſs lemma annexed to the 2d prop. is no where in theſe elements, and has been perhaps left out by ſome editor who has forgot to cancel thoſe words alſo.

PROP. VI. B. XII.

A ſhorter demonſtration is given of this; and that which is in the Greek text may be made ſhorter by a ſtep than it is: For the author of it makes uſe of the 22d prop. of B. 5. twice: Whereas once would have ſerved his purpoſe; becauſe that propoſition extends to any number of magnitudes which are proportionals taken two and two, as well as to three which are proportional to other three.

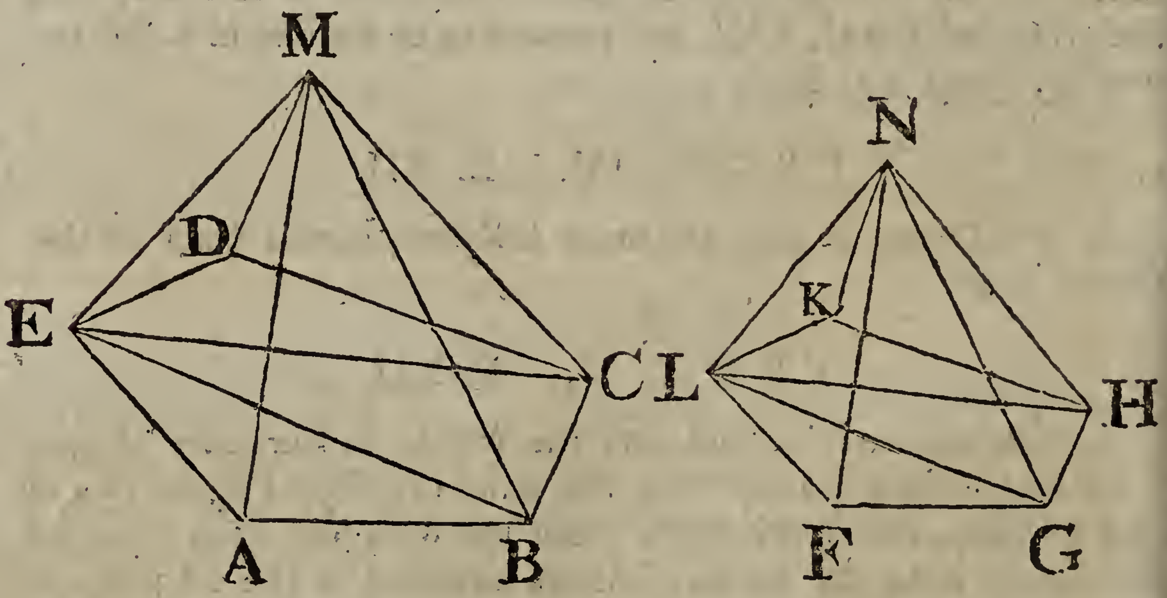
COR. PROP. VIII. B. XII.

The demonſtration of this is imperfect, becauſe it is not ſhown, that the triangular pyramids into which thoſe upon multangular baſes are divided, are ſimilar to one another, as ought neceſſarily to have been done, and is done in the like caſe in prop. 12. of this book: The full demonſtration of the corollary is as follows:

Upon the polygonal baſes ABCDE, FGHKL, let there be ſimilar and ſimilarly ſituated pyramids which have the points M, N for their vertices: The pyramid ABCDEM has to the pyramid FGHKLN the triplicate ratio of that which the ſide AB has to the homologous ſide FG.

Let the polygons be divided into the triangles ABE, EBC, ECD; FGL, LGH, LHK, which are ſimilar [a] each to each; And becauſe the pyramids are ſimilar, therefore [b] the triangle EAM is ſimilar to the triangle LFN, and the triangle ABM to FGN: Wherefore [c] ME is to EA, as NL to LF; and as

a 20. 6.
b 11. def. 11.
c 4. 6.



AE to EB, ſo is FL to LG, becauſe the triangles EAB, LFG are ſimilar; therefore, ex æquali, as ME to EB, ſo is NL to

LG; In like manner it may be ſhewn that EB is to BM, as LG to GN; therefore, again, ex æquali, as EM to MB, ſo is LN to NG: Wherefore the triangles EMB, LNG having their ſides proportionals are [d] equiangular, and ſimilar to one another: Therefore the pyramids which have the triangles EAB, LFG for their baſes, and the points M, N for their vertices, are ſimilar [b] to one another, for their ſolid angles are [e] equal, and the ſolids themſelves are contained by the ſame number of ſimilar planes: In the ſame manner the pyramid EBCM may be ſhewn to be ſimilar to the pyramid LGHN, and the pyramid ECDM to LHKN: And becauſe the pyramids EABM, LFGN are ſimilar, and have triangular baſes, the pyramid EABM has [f] to LFGN the triplicate ratio of that which EB has to the homologous ſide LG. And, in the ſame manner, the pyramid EBCM has to the pyramid LGHN the triplicate ratio of that which EB has to LG: Therefore as the pyramid EABM is to the pyramid LFGN, ſo is the pyramid EBCM to the pyramid LGHN: In like manner, as the pyramid EBCM is to LGHN, ſo is the pyramid ECDM to the pyramid LHKN: And as one of the antecedents is to one of the conſequents, ſo are all the antecedents to all the conſequents: Therefore as the pyramid EABM to the pyramid LFGN, ſo is the whole pyramid ABCDEM to the whole pyramid FGHKLN: And the pyramid EABM has to the pyramid LFGN the triplicate ratio of that which AB has to FG; therefore the whole pyramid has to the whole pyramid the triplicate ratio of that which AB has to the homologous ſide FG. Q. E. D.

[d] 5. 6.

[b] 11. def. 11.

[e] B. 11.

[f] 8. 12.

PROP. XI. and XII. B. XII.

The order of the letters of the alphabet is not obſerved in theſe two propoſitions, according to Euclid's manner, and is now reſtored: By which means, the firſt part of prop. 12. may be demonſtrated in the ſame words with the firſt part of prop. 11.; on this account the demonſtration of that firſt part is left out, and aſſumed from prop. 11.

PROP. XII. B. XII.

In this propoſition the common ſection of a plane parallel to the baſes of a cylinder, with the cylinder itſelf, is ſuppoſed to be a circle, and it was thought proper briefly to demonſtrate it; from whence it is ſufficiently manifeſt, that this plane divides the cylinder into two others: And the ſame thing is underſtood to be ſupplied in prop. 14.

PROP. XV. B. XII.

"And complete the cylinders AX, EO," both the enunciation and expoſition of the propoſition repreſent the cylinders as well as the cones, as already deſcribed: Wherefore the reading ought rather to be, "and let the cones be ALC, ENG; "and the cylinders AX, EO."

The firſt caſe in the ſecond part of the demonſtration is wanting; and ſomething alſo in the ſecond caſe of that part, before the repetition of the conſtruction is mentioned; which are now added.

PROP. XVII. B. XII.

In the enunciation of this propoſition, the Greek words *εἰς την μειζονα σφαιραν στερεον πολυεδρον εγγραψαι μη ψαῦον τῆς ελασσονος σφαιρας κατα την επιφανειαν*, are thus tranſlated by Commandine and others, "in majori ſolidum polyhedrum deſcribere quod "minoris ſphæræ ſuperficiem non tangat;" that is, "to de-"ſcribe in the greater ſphere a ſolid polyhedron which ſhall "not meet the ſuperficies of the leſſer ſphere:" Whereby they refer the words *κατα την ἐπιφανειαν* to theſe next to them *της ελασσονος σφαιρας*: But they ought by no means to be thus tranſlated, for the ſolid polyhedron doth not only meet the ſuperficies of the leſſer ſphere, but pervades the whole of that ſphere: Therefore the foreſaid words are to be referred to *το στερεον πολυεδρον*, and ought thus to be tranſlated, viz. to deſcribe in the greater ſphere a ſolid polyhedron whoſe ſuperficies ſhall not meet the leſſer ſphere: as the meaning of the propoſition neceſſarily requires.

The demonſtration of the propoſition is ſpoiled and mutilated: For ſome eaſy things are very explicitly demonſtrated, while others not ſo obvious are not ſufficiently explained; for example, when it is affirmed, that the ſquare of KB is greater than the double of the ſquare of BZ, in the firſt demonſtration; and that the angle BZK is obtuſe, in the ſecond: Both which ought to have been demonſtrated: Beſides, in the firſt demonſtration, it is ſaid, "draw KΩ from the point K perpen-"dicular to BD; whereas it ought to have been ſaid, "join "KV," and it ſhould have been demonſtrated that KV is perpendicular to BD: For it is evident from the figure in Hervagius's and Gregory's editions, and from the words of the demonſtration, that the Greek Editor did not perceive that the perpendicular drawn from the point K to the ſtraight line BD muſt neceſſarily fall upon the point V, for in the figure it is made to fall upon the point Ω a different point from V, which is likewiſe ſuppoſed in the demonſtration. Commandine ſeems to have been aware of this; for in this figure he marks one and the ſame point with the two letters V, Ω; and before Commandine, the learned John Dee, in the commentary he annexes to this propoſition in Henry Billinſley's tranſlation of the Elements printed at London, ann. 1570, expreſsly takes notice of this error, and gives a demonſtration ſuited to the conſtruction in the Greek text, by which he ſhews that the perpendicular drawn from the point K to BD, muſt neceſſarily fall upon the point V.

Likewiſe it is not demonſtrated that the quadrilateral figures SOPT, TPRY, and the triangle YRX do not meet the leſſer ſphere. as was neceſſary to have been done: Only Clavius, as far as I know, has obſerved this, and demonſtrated it by a lemma, which is now premiſed to this propoſition, ſomething altered and more briefly demonſtrated.

In the corollary of this propoſition, it is ſuppoſed that a ſolid polyhedron is deſcribed in the other ſphere ſimilar to that which is deſcribed in the ſphere BCDE; but, as the conſtruction by which this may be done is not given, it was thought proper to give it, and to demonſtrate, that the pyramids in it are ſimilar to thoſe of the ſame order in the ſolid polyhedron deſcribed in the ſphere BCDE.

From the preceding notes, it is ſufficiently evident how much the elements of Euclid, who was a moſt accurate geometer, have been vitiated and mutilated by ignorant editors. The opinion which the greateſt part of learned men have entertained concerning the preſent Greek edition, viz. that it is very little or nothing different from the genuine work of Euclid, has without doubt deceived them, and made them leſs attentive and accurate in examining that edition; whereby ſeveral errors, ſome of them groſs enough, have eſcaped their notice from the age in which Theon lived to this time. Upon which account there is ſome ground to hope that the pains we have taken in correcting thoſe errors, and freeing the Elements as far as we could from blemiſhes, will not be unacceptable to good judges who can diſcern when demonſtrations are legitimate, and when they are not.

The objections which, ſince the firſt edition, have been made againſt ſome things in the notes, eſpecially againſt the doctrine of proportionals, have either been fully anſwered in Dr Barrow's Lect. Mathemat. and in theſe notes; or are ſuch, except one which has been taken notice of in the note on prop. 1. book 11. as ſhew that the perſon who made them has not ſufficiently conſidered the things againſt which they are brought; ſo that it is not neceſſary to make any further anſwer to theſe objections and others like them againſt Euclid's definition of proportionals; of which definition Dr Barrow juſtly ſays in page 297. of the above named book, that "Niſi machinis impulſa validioribus æternum perſiſtet inconcuſſa."

FINIS.

EUCLID's

DATA.

IN THIS EDITION
SEVERAL ERRORS ARE CORRECTED,
AND
SOME PROPOSITIONS ADDED.

BY ROBERT SIMSON, M. D.

Emeritus Profeſſor of Mathematics in the Univerſity of Glaſgow.

EDINBURGH:
Printed for F. Wingrave, London; and E. Balfour, Edinburgh.
M.DCC.XCIV.

PREFACE.

EUCLID's DATA is the firſt in order of the books written by the ancient geometers to facilitate and promote the method of reſolution or analyſis. In the general, a thing is ſaid to be given which is either actually exhibited, or can be found out, that is, which is either known by hypotheſis, or that can be demonſtrated to be known; and the propoſitions in the book of Euclid's Data ſhew what things can be found out or known from thoſe that by hypotheſis are already known; ſo that in the analyſis or inveſtigation of a problem, from the things that are laid down to be known or given, by the help of theſe propoſitions other things are demonſtrated to be given, and from theſe, other things are again ſhewn to be given, and ſo on, until that which was propoſed to be found out in the problem is demonſtrated to be given, and when this is done, the problem is ſolved, and its compoſition is made and derived from the compoſitions of the Data which were made uſe of in the analyſis. And thus the Data of Euclid are of the moſt general and neceſſary uſe in the ſolution of problems of every kind.

Euclid is reckoned to be the author of the Book of the Data, both by the ancient and modern geometers; and there ſeems to be no doubt of his having written a book on this ſubject, but which, in the courſe of ſo many ages, has been much vitiated by unſkilful editors in ſeveral places, both in the order of the propoſitions, and in the definitions and demonſtrations themſelves.

ſelves. To correct the errors which are now found in it, and bring it nearer to the accuracy with which it was, no doubt, at firſt written by Euclid, is the deſign of this edition, that ſo it may be rendered more uſeful to geometers, at leaſt to beginners who deſire to learn the inveſtigatory method of the ancients. And for their ſakes, the compoſitions of moſt of the Data are ſubjoined to their demonſtrations, that the compoſitions of problems ſolved by help of the Data may be the more eaſily made.

Marinus the philoſopher's preface, which, in the Greek edition, is prefixed to the Data, is here left out, as being of no uſe to underſtand them. At the end of it, he ſays, that Euclid has not uſed the ſynthetical, but the analytical method in delivering them; in which he is quite miſtaken; for, in the analyſis of a theorem, the thing to be demonſtrated is aſſumed in the analyſis; but in the demonſtrations of the Data, the thing to be demonſtrated, which is, that ſomething or other is given, is never once aſſumed in the demonſtration, from which it is manifeſt, that every one of them is demonſtrated ſynthetically; though indeed, if a propoſition of the Data be turned into a problem, for example the 84th or 85th in the former editions, which here are the 85th and 86th, the demonſtration of the propoſition becomes the analyſis of the problem.

Wherein this edition differs from the Greek, and the reaſons of the alterations from it, will be ſhewn in the notes at the end of the Data.

EUCLID's DATA.

DEFINITIONS.

I.

SPACES, lines, and angles, are ſaid to be given in magnitude, when equals to them can be found.

II.

A ratio is ſaid to be given, when a ratio of a given magnitude to a given magnitude which is the ſame ratio with it can be found.

III.

Rectilineal figures are ſaid to be given in ſpecies, which have each of their angles given, and the ratios of their ſides given.

IV.

Points, lines, and ſpaces are ſaid to be given in poſition, which have always the ſame ſituation, and which are either actually exhibited, or can be found.

A.

An angle is ſaid to be given in poſition, which is contained by ſtraight lines given in poſition.

V.

A circle is ſaid to be given in magnitude, when a ſtraight line from its centre to the circumference is given in magnitude.

VI.

A circle is ſaid to be given in poſition and magnitude, the centre of which is given in poſition, and a ſtraight line from it to the circumference is given in magnitude.

VII.

Segments of circles are ſaid to be given in magnitude, when the angles in them, and their baſes, are given in magnitude.

VIII.

Segments of circles are ſaid to be given in poſition and magnitude, when the angles in them are given in magnitude, and their baſes are given both in poſition and magnitude.

IX.

A magnitude is ſaid to be greater than another by a given magnitude, when this given magnitude being taken from it, the remainder is equal to the other magnitude.

Book XII.

X.

A magnitude is ſaid to be leſs than another by a given magnitude, when this given magnitude being added to it, the whole is equal to the other magnitude.

* 1.

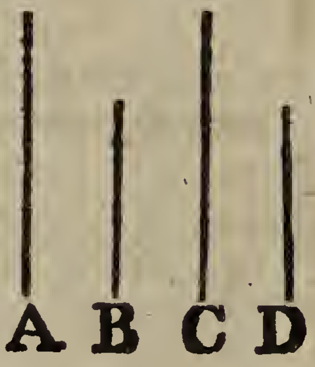
PROPOSITION I.

See N.

THE ratios of given magnitudes to one another is given.

Let A, B be two given magnitudes, the ratio of A to B is given.

Becauſe A is a given magnitude, there may
a 1. def. dat. [a] be found one equal to it; let this be C:
And becauſe B is given, one equal to it may
be found; let it be D; and ſince A is equal
b 7. 5. to C, and B to D; therefore [b] A is to B, as
C to D; and conſequently the ratio of A to
B is given, becauſe the ratio of the given
magnitudes C, D which is the ſame with it,
has been found.



2.

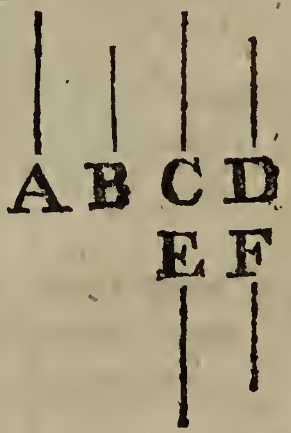
PROP. II.

See N.

IF a given magnitude has a given ratio to another magnitude, "and if unto the two magnitudes by "which the given ratio is exhibited, and the given "magnitude, a fourth proportional can be found;" the other magnitude is given.

Let the given magnitude A have a given ratio to the magnitude B; if a fourth proportional can be found to the three magnitudes above named, B is given in magnitude.

Becauſe A is given, a magnitude may be
a 1. def. found equal to it [a]; let this be C: And be-
cauſe the ratio of A to B is given, a ratio
which is the ſame with it may be found, let
this be the ratio of the given magnitude E
to the given magnitude F: Unto the magni-
tudes E, F, C find a fourth proportional
D, which, by the hypotheſis, can be done.
Wherefore, becauſe A is to B, as E to F; and
b 11. 5. as E to F, ſo is C to D; A is [b] to B, as C to



D.

---

* The figures in the margin ſhow the number of the propoſitions in the other editions.

D. But A is equal to C; therefore [c] B is equal to D. The magnitude B is therefore given [a], becauſe a magnitude D equal to it has been found. c 14. 5. a 1 def.

The limitation within the inverted commas is not in the Greek text, but is now neceſſarily added; and the ſame muſt be underſtood in all the propoſitions of the book which depend upon this ſecond propoſition, where it is not expreſsly mentioned. See the note upon it.

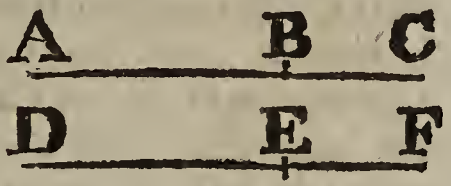
PROP. III.

3.

IF any given magnitudes be added together, their ſum ſhall be given.

Let any given magnitudes AB, BC be added together, their ſum AC is given.

Becauſe AB is given, a magnitude equal to it may [a] be found; let this be DE: And becauſe BC is given, one equal to it may be found; let this be EF: Wherefore, becauſe AB is equal to DE, and BC equal to EF; the whole AC is equal to the whole DF: AC is therefore given, becauſe DF has been found which is equal to it. a 1 def.



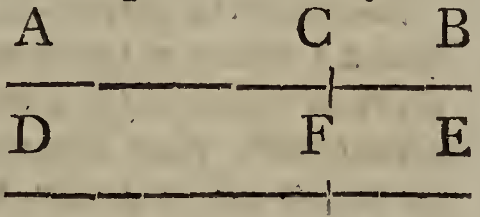
PROP. IV.

4.

IF a given magnitude be taken from a given magnitude; the remaining magnitude ſhall be given.

From the given magnitude AB, let the given magnitude AC be taken; the remaining magnitude CB is given.

Becauſe AB is given, a magnitude equal to it may [a] be found; let this be DE: And becauſe AC is given, one equal to it may be found; let this be DF: Wherefore becauſe AB is equal to DE, and AC to DF; the remainder CB is equal to the remainder FE. CB is therefore given [a], becauſe FE which is equal to it has been found. a 1 def.



12.

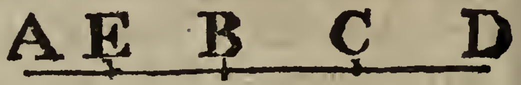
PROP. V.

See N. IF of three magnitudes, the firſt together with the ſecond be given, and alſo the ſecond together with the third; either the firſt is equal to the third, or one of them is greater than the other by a given magnitude.

Let AB, BC, CD be three magnitudes, of which AB together with BC, that is AC, is given; and alſo BC together with CD, that is BD, is given. Either AB is equal to CD, or one of them is greater than the other by a given magnitude.

Becauſe AC, BD are each of them given, they are either equal to one another, or not equal. Firſt, let them be equal, and becauſe AC is equal to BD, take away the common part BC; therefore the remainder AB is equal to the remainder CD.

But if they be unequal, let AC be greater than BD, and
make CE equal to BD. Therefore CE is given, becauſe BD
is given. And the whole AC is
a 4 dat. given; therefore [a] AE the remainder is given. And becauſe EC is
equal to BD, by taking BC from
both, the remainder EB is equal to the remainder CD. And
AE is given; wherefore AB exceeds EB, that is CD by the
given magnitude AE.



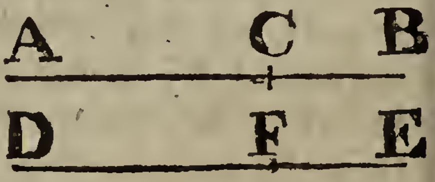
5.

PROP. VI.

See N. IF a magnitude has a given ratio to a part of it, it ſhall alſo have a given ratio to the remaining part of it.

Let the magnitude AB have a given ratio to AC a part of it; it has alſo a given ratio to the remainder BC.

Becauſe the ratio of AB to AC is given, a ratio may be
a 2. def. found [a] which is the ſame to it: Let this be the ratio of DE
a given magnitude to the given magnitude DF. And becauſe DE, DF are
b 4. dat. given, the remainder FE is [b] given:
And becauſe AB is to AC, as DE to
c E. 5. DF, by converſion [c] AB is to BC, as
DE to EF. Therefore the ratio of AB to BC is given, becauſe the ratio of the given magnitudes DE, EF, which is the ſame with it, has been found.



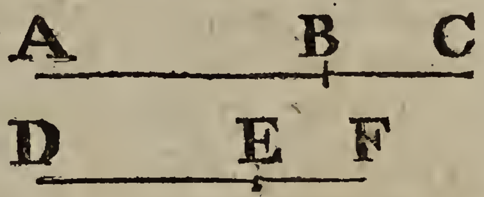
Cor. From this it follows, that the parts AC, CB have a
given ratio to one another: Becauſe as AB to BC, ſo is DE to
EF; by diviſion [d], AC is to CB, as DF to FE; and DF, d 17. 5.
FE are given; therefore [a] the ratio of AC to CB is given. a 2. def.

PROP. VII. 6.

IF two magnitudes which have a given ratio to one another, be added together; the whole magnitude ſhall have to each of them a given ratio. See N.

Let the magnitudes AB, BC which have a given ratio to one another, be added together; the whole AC has to each of the magnitudes AB, BC a given ratio.

Becauſe the ratio of AB to BC is given, a ratio may be
found [a] which is the ſame with it; let this be the ratio of the a 2. def.
given magnitudes DE, EF: And
becauſe DE, EF are given, the
whole DF is given [b]: And becauſe b 3. dat.
as AB to BC, ſo is DE to EF; by
compoſition [c] AC is to CB as DF to c 18. 5.
FE; and by converſion [d], AC is to AB, as DF to DE: d E. 5.
Wherefore becauſe AC is to each of the magnitudes AB, BC,
as DF to each of the others DE, EF; the ratio of AC to
each of the magnitudes AB, BC is given [a].

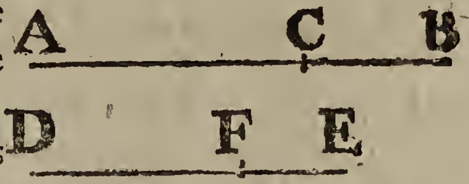


PROP. VIII. 7.

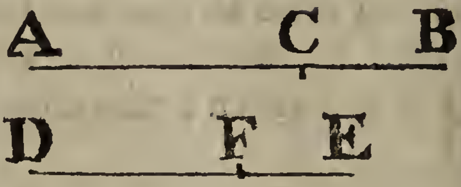
IF a given magnitude be divided into two parts which have a given ratio to one another, and if a fourth proportional can be found to the ſum of the two magnitudes by which the given ratio is exhibited, one of them, and the given magnitude; each of the parts is given. See N.

Let the given magnitude AB be divided into the parts AC, CB which have a given ratio to one another; if a fourth proportional can be found to the above named magnitudes; AC and CB are each of them given.

Becauſe the ratio of AC to CB is
given, the ratio of AB to BC is given [a]; therefore a ratio a 7. dat.



b 2. def. which is the ſame with it can be found [b], let this be the ratio
of the given magnitudes DE, EF:
And becauſe the given magnitude AB
has to BC the given ratio of DE to
EF, if unto DE, EF, AB a fourth
proportional can be found, this which
c 2. dat. is BC is given [c]; and becauſe AB is given, the other part AC
d 4. dat. is given [d].



In the ſame manner, and with the like limitation, if the difference AC of two magnitudes AB, BC which have a given ratio be given; each of the magnitudes AB, BC is given.

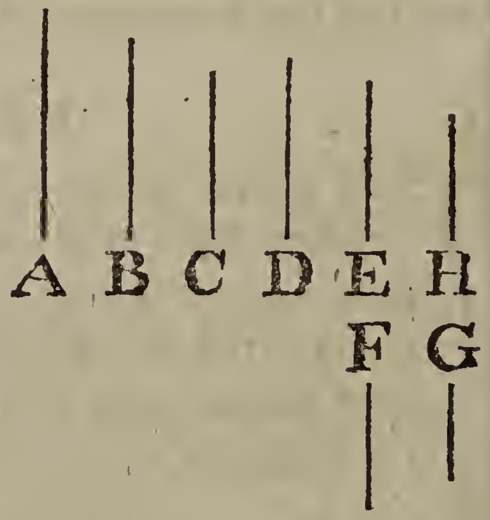
8.

PROP. IX.

MAGNITUDES which have given ratios to the ſame magnitude, have alſo a given ratio to one another.

Let A, C have each of them a given ratio to B; A has a given ratio to C.

Becauſe the ratio of A to B is given, a ratio which is the
a 2 def. ſame to it may be found [a]; let this be the ratio of the given
magnitudes D, E: And becauſe the ratio of B to C is given,
a ratio which is the ſame with it may be found [a]; let this be the ratio of the given magnitudes F, G: To F, G, E find a fourth proportional H, if it can be done; and becauſe as A is to B, ſo is D to E; and as B to C, ſo is (F to G, and ſo is) E to H; ex æquali, as A to C, ſo is D to H: Therefore the ratio of A to C is given [a], becauſe the ratio of the given magnitudes D and H, which is the ſame with it has been found: But if a fourth proportional to F, G, E cannot be found, then it can only be ſaid that the ratio of A to C is compounded of the ratios of A to B, and B to C, that is, of the given ratios of D to E, and F to G.



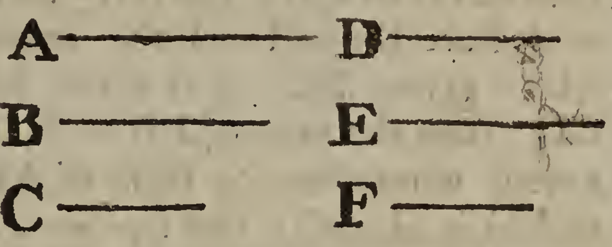
PROP.

PROP. X.

9.

IF two or more magnitudes have given ratios to one another, and if they have given ratios, though they be not the ſame, to ſome other magnitudes; theſe other magnitudes ſhall alſo have given ratios to one another.

Let two or more magnitudes A, B, C have given ratios to one another; and let them have given ratios, though they be not the ſame, to ſome other magnitudes D, E, F: The magnitudes D, E, F have given ratios to one another.



Becauſe the ratio of A to B is given, and likewiſe the ratio of A to D; therefore the ratio of D to B is given [a]; but the ratio of B to E is given; therefore [a] the ratio of D to E is given: And becauſe the ratio of B to C is given, and alſo the ratio of B to E; the ratio of E to C is given [a]: And the ratio of C to F is given; wherefore the ratio of E to F is given: D, E, F have therefore given ratios to one another.

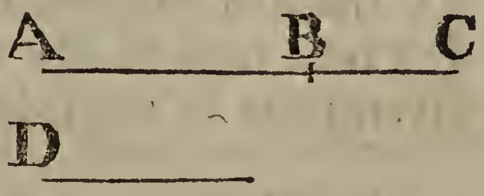
a 9. dat.

PROP. XI.

22.

IF two magnitudes have each of them a given ratio to another magnitude, both of them together ſhall have a given ratio to that other.

Let the magnitudes AB, BC have a given ratio to the magnitude D; AC has a given ratio to the ſame D.



Becauſe AB, BC have each of them a given ratio to D, the ratio of AB to BC is given [a]: And by compoſition, the ratio of AC to CB is given [b]: But the ratio of BC to D is given; therefore [a] the ratio of AC to D is given.

a 9. dat.

b 7. dat.

23.

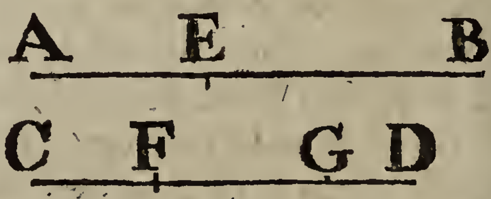
PROP. XII.

See N.

IF the whole have to the whole a given ratio, and the parts have to the parts given, but not the ſame, ratios: Every one of them, whole or part, ſhall have to every one a given ratio,

Let the whole AB have a given ratio to the whole CD, and the parts AE, EB have given, but not the ſame, ratios to the parts CF, FD: Every one ſhall have to every one, whole or part, a given ratio.

Becauſe the ratio of AE to CF is given, as AE to CF, ſo
make AB to CG; the ratio therefore of AB to CG is given;
wherefore the ratio of the remainder EB to the remainder
a 19. 5. FG is given, becauſe it is the ſame [a] with the ratio of AB to
CG: And the ratio of EB to FD is
given, wherefore the ratio of FD
b 9. dat. to FG is given [b]; and by conver-
ſion, the ratio of FD to DG is
c 6. dat. given [c]: And becauſe AB has to



each of the magnitudes CD, CG a given ratio, the ratio of
CD to CG is given [b]; and therefore [c] the ratio of CD to DG
is given: But the ratio of GD to DF is given, wherefore [b] the
d cor. 6. dat. ratio of CD to DF is given, and conſequently [d] the ratio of
CF to FD is given; but the ratio of CF to AE is given, as
e 10 dat. alſo the ratio of FD to EB; wherefore [e] the ratio of AE to
f 7. dat. EB is given; as alſo the ratio of AB to each of them [f]: The
ratio therefore of every one to every one is given.

24.

PROP. XIII.

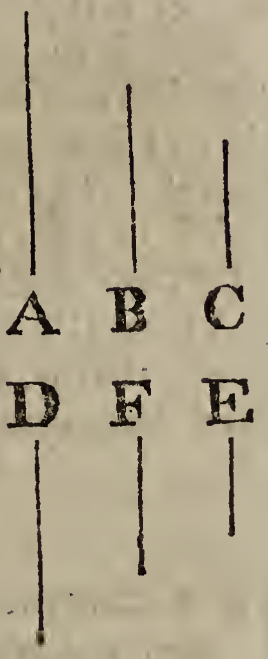
See N.

IF the firſt of three proportional ſtraight lines has a given ratio to the third, the firſt ſhall alſo have a given ratio to the ſecond.

Let A, B, C be three proportional ſtraight lines, that is, as A to B, ſo is B to C; if A has to C a given ratio, A ſhall alſo have to B a given ratio.

Becauſe the ratio of A to C is given, a ratio which is the
a 2 def. ſame with it may be found [a]; let this be the ratio of the gi-
b 13. 6. ven ſtraight lines D, E; and between D and E find a [b] mean
proportional

proportional F; therefore the rectangle contained by D and E is equal to the ſquare of F, and the rectangle D, E is given, becauſe its ſides D, E are given; wherefore the ſquare of F, and the ſtraight line F is given: And becauſe as A is to C, ſo is D to E; but as A to C, ſo is [c] the ſquare of A to the ſquare of B; and as D to E, ſo is [c] the ſquare of D to the ſquare of F: Therefore the ſquare [d] of A is to the ſquare of B, as the ſquare of D to the ſquare of F: As therefore [e] the ſtraight line A to the ſtraight line B, ſo is the ſtraight line D to the ſtraight line F: Therefore the ratio of A to B is given [a], becauſe the ratio of the given ſtraight lines D, F which is the ſame with it has been found.



c 2. cor. 20. 6.
d 11. 5.
e 22. 6.
a 2. def.

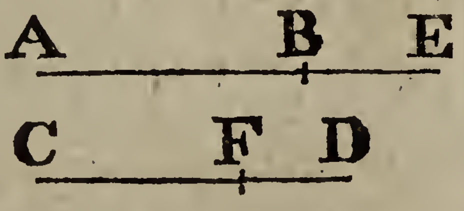
PROP. XIV.

A.
See N.

IF a magnitude together with a given magnitude has a given ratio to another magnitude; the exceſs of this other magnitude above a given magnitude has a given ratio to the firſt magnitude: And if the exceſs of a magnitude above a given magnitude has a given ratio to another magnitude; this other magnitude together with a given magnitude has a given ratio to the firſt magnitude.

Let the magnitude AB together with the given magnitude BE, that is AE, have a given ratio to the magnitude CD; the exceſs of CD above a given magnitude has a given ratio to AB.

Becauſe the ratio of AE to CD is given, as AE to CD, ſo make BE to FD; therefore the ratio of BE to FD is given, and BE is given; wherefore FD is given [a]: And becauſe as AE to CD, ſo is BE to FD, the remainder AB is [b] to the remainder CF, as AE to CD: But the ratio of AE to CD is given, therefore the ratio of AB to CF is given; that is, CF the exceſs of CD above the given magnitude FD has a given ratio to AB.



a 2. dat.
b 19. 5.

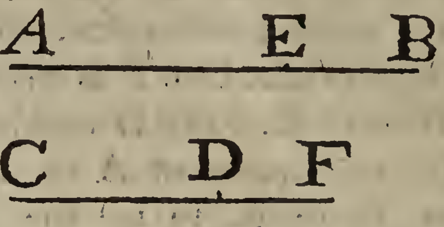
Next, let the exceſs of the magnitude AB above the given magnitue BE, that is, let AE have a given ratio to the magnitude

nitude CD : CD together with a given magnitude has a given ratio to AB.

Becauſe the ratio of AE to CD is given, as AE to CD, ſo make BE to FD; therefore the ratio of BE to FD is given and BE is given, wherefore FD is given [a]: And becauſe as AE to CD, ſo is BE to FD, AB is to CF, as [c] AE to CD: But the ratio of AE to CD is given, therefore the ratio of AB to CF is given: that is, CF which is equal to CD together with the given magnitude DF has a given ratio to AB.

a 2. dat.

c 12. 5.



PROP. XV.

B.

See N.

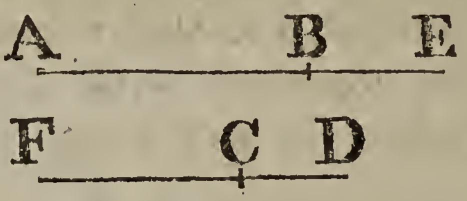
IF a magnitude together with that to which another magnitude has a given ratio, be given; the ſum of this other, and that to which the firſt magnitude has a given ratio is given.

Let AB, CD be two magnitudes of which AB together with BE to which CD has a given ratio, is given; CD is given together with that magnitude to which AB has a given ratio.

Becauſe the ratio of CD to BE is given, as BE to CD, ſo make AE to FD; therefore the ratio of AE to FD is given, and AE is given, wherefore [a] FD is given: And becauſe as BE to CD, ſo is AE to FD: AB is [b] to FC, as BE to CD: And the ratio of BE to CD is given, wherefore the ratio of AB to FC is given: And FD is given, that is CD together with FC to which AB has a given ratio is given.

a 2. dat.

b Cor. 19. 5.



PROP. XVI.

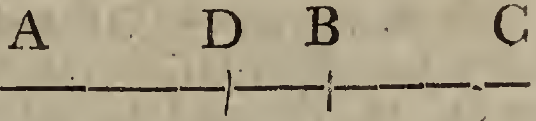
10.

See N.

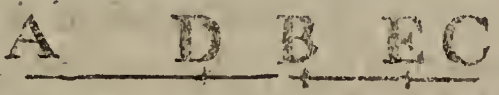
IF the exceſs of a magnitude above a given magnitude, has a given ratio to another magnitude; the exceſs of both together above a given magnitude ſhall have to that other a given ratio: And if the exceſs of two magnitudes together above a given magnitude, has to one of them a given ratio; either the exceſs of the other above a given magnitude has to that one a given ratio; or the other is given together with the magnitude to which that one has a given ratio.

Let the exceſs of the magnitude AB above a given magnitude, have a given ratio to the magnitude BC; the exceſs of AC, both of them together, above the given magnitude, has a given ratio to BC.

Let AD be the given magnitude, the exceſs of AB above which, viz. DB has a given ratio to BC: And becauſe DB has a given ratio to BC, the ratio of DC to CB is given[a], and AD is given; therefore DC, the exceſs of AC above the given magnitude AD, has a given ratio to BC.



Next, let the exceſs of two magnitudes AB, BC together, above a given magnitude, have to one of them BC a given ratio; either the exceſs of the other of them AB above the given magnitude ſhall have to BC a given ratio; or AB is given, together with the magnitude to which BC has a given ratio.



Let AD be the given magnitude, and firſt let it be leſs than AB; and becauſe DC the exceſs of AC above AD has a given ratio to BC, DB has[b] a given ratio to BC; that is, DB the exceſs of AB above the given magnitude AD, has a given ratio to BC.

But let the given magnitude be greater than AB, and make AE equal to it; and becauſe EC, the exceſs of AC above AE has to BC a given ratio, BC has[c] a given ratio to BE; and becauſe AE is given, AB together with BE to which BC has a given ratio is given.

PROP. XVII.

11.

See N.

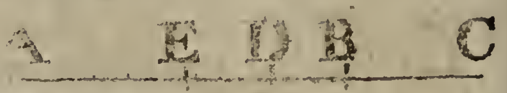
IF the exceſs of a magnitude above a given magnitude has a given ratio to another magnitude; the exceſs of the ſame firſt magnitude above a given magnitude, ſhall have a given ratio to both the magnitudes together. And if the exceſs of either of two magnitudes above a given magnitude has a given ratio to both magnitudes together; the exceſs of the ſame above a given magnitude ſhall have a given ratio to the other.

Let the exceſs of the magnitude AB above a given magnitude have a given ratio to the magnitude BC; the exceſs of AB above a given magnitude has a given ratio to AC.

[a] 7. dat.

[b] Cor. 6. dat.

[c] 6. dat.



Let AD be the given magnitude; and becauſe DB, the exceſs of AB above AD, has a given ratio to BC; the ratio of DC to DB is given[a]: Make the ratio of AD to DE the ſame with this ratio; therefore the ratio of AD to DE is given; and AD is given, wherefore[b] DE and the remainder AE are given: And becauſe as DC to DB, ſo is AD to DE, AC is[c] to EB, as DC to DB; and the ratio of DC to DB is given; wherefore the ratio of AC to EB is given: And becauſe the ratio of EB to AC is given, and that AE is given, therefore EB the exceſs of AB above the given magnitude AE, has a given ratio to AC.

[a] 7. dat.
[b] 2. dat.
[c] 12. 5.

Next, let the exceſs of AB above a given magnitude have a given ratio to AB and BC together, that is to AC; the exceſs of AB above a given magnitude has a given ratio to BC.

Let AE be the given magnitude; and becauſe EB the exceſs of AB above AE has to AC a given ratio, as AC to EB ſo make AD to DE; therefore the ratio of AD to DE is given, as alſo[d] the ratio of AD to AE: And AE is given, wherefore[b] AD is given: And becauſe, as the whole AC, to the whole EB, ſo is AD to DE, the remainder DC is[e] to the remainder DB, as AC to EB; and the ratio of AC to EB is given; wherefore the ratio of DC to DB is given, as alſo[f] the ratio of DB to BC: And AD is given; therefore DB, the exceſs of AB above a given maguitude AD, has a given ratio to BC.

[d] 6 dat.
[e] 19. 5.
[f] Cor. 6. dat.

14.

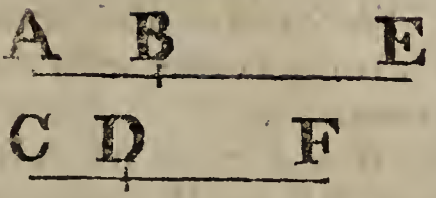
PROP. XVIII.

If to each of two magnitudes, which have a given ratio to one another, a given magnitude be added; the whole ſhall either have a given ratio to one another, or the exceſs of one of them above a given magnitude ſhall have a given ratio to the other.

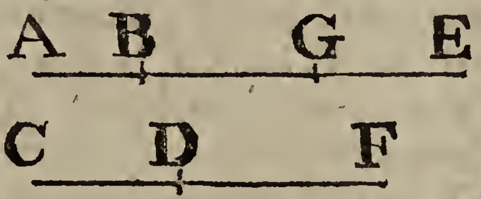
Let the two magnitudes AB, CD have a given ratio to one another, and to AB let the given magnitude BE be added, and the given magnitude DF to CD: The wholes AE, CF either have a given ratio to one another, or the exceſs of one of them above a given magnitude has a given ratio to the other[a].

[a] 1. dat.

Becauſe BE, DF are each of them given, their ratio is given and if this ratio be the ſame with the ratio of AB to CD, the ratio of AE to CF, which is the ſame [b] b 12. 5. with the given ratio of AB to CD, ſhall be given.



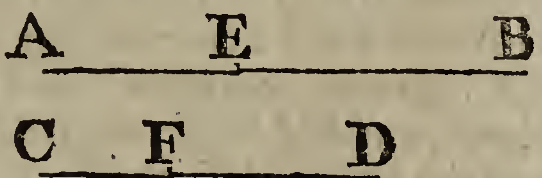
But if the ratio of BE to DF be not the ſame with the ratio of AB to CD, either it is greater than the ratio of AB to CD, or, by inverſion, the ratio of DF to BE is greater than the ratio of CD to AB: Firſt, let the ratio of BE to DF be greater than the ratio of AB to CD; and as AB to CD, ſo make BG to DF; therefore the ratio of BG to DF is given; and DF is given, therefore [c] BG is given: c 2 dat. And becauſe BE has a greater ratio to DF than (AB to CD, that is, than) BG to DF, BE is greater [d] than BG: And be- d 10. 5. cauſe as AB to CD, ſo is BG to DF; therefore AG is [b] to CF, as AB to CD: But the ratio of AB to CD is given, wherefore the ratio of AG to CF is given; and becauſe BE, BG are each of them given, GE is given: Therefore AG, the exceſs of AE above a given magnitude GE, has a given ratio to CF. The other caſe is demonſtrated in the ſame manner.



PROP. XIX. 15.

IF from each of two magnitudes, which have a given ratio to one another, a given magnitude be taken, the remainders ſhall either have a given ratio to one another, or the exceſs of one of them above a given magnitude, ſhall have a given ratio to the other.

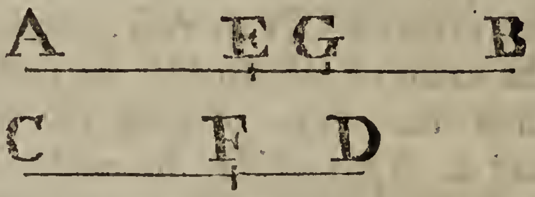
Let the magnitudes AB, CD have a given ratio to one another, and from AB let the given magnitude AE be taken, and from CD, the given magnitudes CF: The remainder EB, FD ſhall either have a given ratio to one another, or the exceſs of one of them above a given magnitude ſhall have a given ratio to the other.



Becauſe AE, CF are each of them given, their ratio is given [a]; and if this ratio be the ſame with the ratio of AB to a 1 dat.

CD, the ratio of the remainder EB to the remainder FD,
b 19. 5. which is the ſame [b] with the given ratio of AB to CD, ſhall be given.

But if the ratio of AB to CD be not the ſame with the ratio of AE to CF, either it is greater than the ratio of AE to CF, or, by inverſion, the ratio CD to AB is greater than the ratio of CF to AE: Firſt, let the ratio of AB to CD be greater than the ratio of AE to CF, and as AB to CD, ſo make AG to CF ; therefore the ratio of AG to CF is given, and
c 2 dat. CF is given, wherefore [c] AG is given: And becauſe the ratio of AB to CD, that is, the ratio of AG to CF, is greater than the ratio of AE to CF; AG is
d 10. 5. greater [d] than AE: And AG, AE are given, therefore the remainder EG is given; and as AB to CD, ſo is AG to CF, and ſo is [b] the remainder GB to the remainder FD; and the ratio of AB to CD is given: Wherefore the ratio of GB to FD is given; therefore GB, the exceſs of EB above a given magnitude EG, has a given ratio to FD. In the ſame manner the other caſe is demonſtrated.



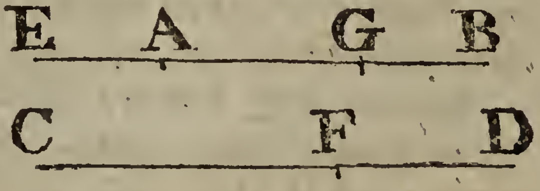
16.

PROP. XX.

IF to one of two magnitudes which have a given ratio to one another, a given magnitude be added, and from the other a given magnitude be taken; the exceſs of the ſum above a given magnitude ſhall have a given ratio to the remainder.

Let the two magnitudes AB, CD have a given ratio to one another, and to AB let the given magnitude EA be added, and from CD let the given magnitude CF be taken; the exceſs of the ſum EB above a given magnitude has a given ratio to the remainder FD.

Becauſe the ratio of AB to CD is given, make as AB to CD, ſo AG to CF: Therefore the ratio of AG to CF is given, and
a 2 dat. CF is given, wherefore [a] AG is given; and EA is given, therefore the whole EG is given: And becauſe as AB to CD, ſo is AG
b 19. 5. to CF, and ſo is [b] the remainder GB to the remainder FD; the ratio of GB to FD is given. And EG is given, therefore GB, the exceſs of the ſum EB above



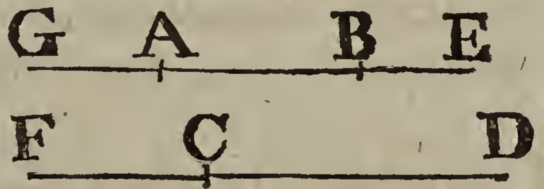
bove the given magnitude EG, has a given ratio to the remainder FD.

PROP. XXI. C.

IF two magnitudes have a given ratio to one another, See N. if a given magnitude be added to one of them, and the other be taken from a given magnitude; the ſum, together with the magnitude to which the remainder has a given ratio, is given: And the remainder is given together with the magnitude to which the ſum has a given ratio.

Let the two magnitudes AB, CD have a given ratio to one another; and to AB let the given magnitude BE be added, and let CD be taken from the given magnitude FD: The ſum AE is given together with the magnitude to which the remainder FC has a given ratio.

Becauſe the ratio of AB to CD is given, make as AB to CD, ſo GB to FD: Therefore the ratio of GB to FD is given, and FD is given, wherefore GB is given[a]; and BE is given, the whole GE is therefore given: and becauſe as AB to CD, ſo is GB to FD, and ſo is[b] GA to FC; the ratio of GA to FC is given: And AE together with GA is given, becauſe GE is given; therefore the ſum AE, together with GA, to which the remainder FC has a given ratio, is given. The ſecond part is manifeſt from prop. 15.



a 2 dat.

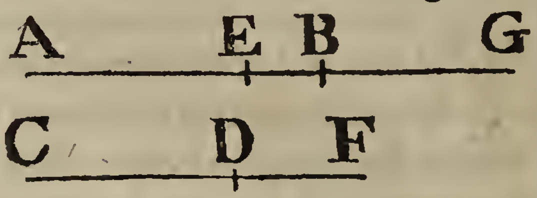
b. 19. 5.

PROB. XXII. D.

IF two magnitudes have a given ratio to one another, See N. if from one of them a given magnitude be taken, and the other be taken from a given magnitude; each of the remainders is given together with the magnitude to which the other remainder has a given ratio.

Let the two magnitudes AB, CD have a given ratio to one another, and from AB let the given magnitude AE be taken, and let CD be taken from the given magnitude CF: The remainder EB is given together with the magnitude to which the other remainder DF has a given ratio.

Becauſe the ratio of AB to CD is given, make as AB to CD, ſo AG to CF: The ratio of AG to CF is therefore given, and CF is given, wherefore[a] AG is given; and AE is given, and therefore the remainder EG is given: And becauſe as AB to CD, ſo is AG to CF: And ſo is[b] the remainder BG to the remainder DF; the ratio of BG to DF is given: And EB together with BG is given, becauſe EG is given: Therefore the remainder EB together with BG, to which DF the other remainder has a given ratio, is given. The ſecond part is plain from this and prop. 15.



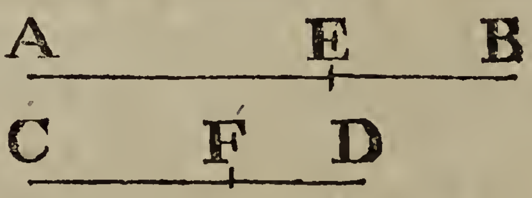
a 2. dat.
b. 19. 5.

20. PROP. XXIII.

See N. IF, from two given magnitudes there be taken magnitudes which have a given ratio to one another, the remainders ſhall either have a given ratio to one another, or the exceſs of one of them above a given magnitude ſhall have a given ratio to the other.

Let AB, CD be two given magnitudes, and from them let the magnitudes AE, CF, which have a given ratio to one another, be taken; the remainders EB, FD either have a given ratio to one another, or the exceſs of one of them above a given magnitude has a given ratio to the other.

Becauſe AB, CD are each of them given, the ratio of AB to CD is given: And if this ratio be the ſame with the ratio of AE to CF, then the remainder EB has[a] the ſame given ratio to the remainder FD.



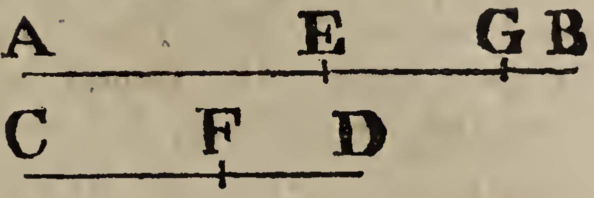
a 19. 5.

But if the ratio of AB to CD be not the ſame with the ratio of AE to CF, it is either greater than it, or, by inverſion, the ratio of CD to AB is greater than the ratio of CF to AE: Firſt, let the ratio of AB to CD be greater than the ratio of AE to CF; and as AE to CF, ſo make AG to CD; therefore the ratio of AG to CD is given, becauſe the ratio of AE to CF is given; and CD is given, wherefore[b] AG is given;

b. 2. dat.

given; and becauſe the ratio of AB to CD is greater than the ratio of (AE to CF, that is, than the ratio of) AG to CD; AB is greater [c] than AG: And AB, AG are given; therefore the remainder BG is given: And becauſe as AE to CF, ſo is AG to CD, and ſo is [a] EG to FD; the ratio of EG to FD is given: And GB is given; therefore EG, the exceſs of EB above a given magnitude GB, has a given ratio to FD. The other caſe is ſhown in the ſame way.

c 10. 5.
a 19. 5.



PROP. XXIV.

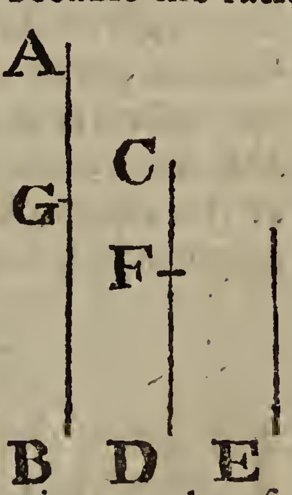
13.
See N.

IF there be three magnitudes, the firſt of which has a given ratio to the ſecond, and the exceſs of the ſecond above a given magnitude has a given ratio to the third; the exceſs of the firſt above a given magnitude ſhall alſo have a given ratio to the third.

Let AB, CD, E, be the three magnitudes of which AB has a given ratio to CD; and the exceſs of CD above a given magnitude has a given ratio to E: The exceſs of AB above a given magnitude has a given ratio to E.

Let CF be the given magnitude, the exceſs of CD above which, *viz.* FD has a given ratio to E: And becauſe the ratio of AB to CD is given, as AB to CD, ſo make AG to CF; therefore the ratio of AG to CF is given; and CF is given, wherefore [a] AG is given: And becauſe as AB to CD, ſo is AG to CF, and ſo is [b] GB to FD; the ratio of GB to FD is given. And the ratio of FD to E is given, wherefore [c] the ratio of GB to E is given, and AG is given; therefore GB the exceſs of AB above a given magnitude AG has a given ratio to E.

a 2. dat.
b 19. 5.
c 9. dat.



Cor. 1. And if the firſt has a given ratio to the ſecond, and the exceſs of the firſt above a given magnitude has a given ratio to the third; the exceſs of the ſecond above a given magnitude ſhall have a given ratio to the third. For, if the ſecond be called the firſt, and the firſt the ſecond, this corollary will be the ſame with the propoſition.

Cor. 2. Alſo, if the firſt has a given ratio to the ſecond, and the exceſs of the third above a given magnitude has alſo a given ratio to the ſecond, the ſame exceſs ſhall have a given ratio to the firſt; as is evident from the 9th dat.

17. PROP. XXV.

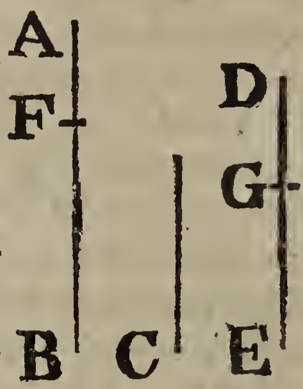
IF there be three magnitudes, the exceſs of the firſt whereof above a given magnitude has a given ratio to the ſecond; and the exceſs of the third above a given magnitude has a given ratio to the ſame ſecond: The firſt ſhall either have a given ratio to the third, or the exceſs of one of them above a given magnitude ſhall have a given ratio to the other.

Let AB, C, DE be three magnitudes, and let the exceſſes of each of the two AB, DE above given magnitudes have given ratios to C; AB, DE either have a given ratio to one another, or the exceſs of one of them above a given magnitude has a given ratio to the other.

Let FB the exceſs of AB above the given magnitude AF have a given ratio to C; and let GE the exceſs of DE above the given magnitude DG have a given ratio to C; and becauſe FB, GE have each of them a given ratio to C, they have a given ratio [a] to one another. But to FB, GE the given magnitudes AF, DG are added; therefore [b] the whole magnitudes AB, DE have either a given ratio to one another, or the exceſs of one of them above a given magnitude has a given ratio to the other.

[a] 9. dat.

[b] 18. dat.

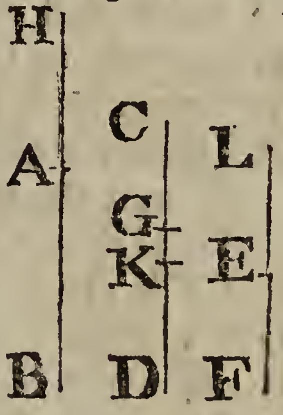


18. PROP. XXVI.

IF there be three magnitudes, the exceſſes of one of which above given magnitudes have given ratios to the other two magnitudes; theſe two ſhall either have a given ratio to one another, or the exceſs of one of them above a given magnitude ſhall have a given ratio to the other.

Let AB, CD, EF be three magnitudes, and let GD the exceſs of one of them CD above the given magnitude CG have a given ratio to AB; and alſo let KD the exceſs of the ſame CD above the given magnitude CK have a given ratio to EF, either AB has a given ratio to EF, or the exceſs of one of them above a given magnitude has a given ratio to the other.

Becauſe GD has a given ratio to AB, as GD to AB, ſo make CG to HA; therefore the ratio of CG to HA is given; and CG is given, wherefore [a] HA is given: And becauſe as GD to AB, ſo is CG to HA, and ſo is [b] CD to HB; the ratio of CD to HB is given: Alſo becauſe KD has a given ratio to EF, as KD to EF, ſo make CK to LE; therefore the ratio of CK to LE is given; and CK is given, wherefore LE [a] is given: And becauſe as KD to EF, ſo is CK to LE, and ſo [b] is CD to LF; the ratio of CD to LF is given: But the ratio of CD to HB is given, wherefore [c] the ratio of HB to LF is given: and from HB, LF the given magnitudes HA, LE being taken, the remainders AB, EF ſhall either have a given ratio to one another or the exceſs of one of them above a given magnitude has a given ratio to the other [d].

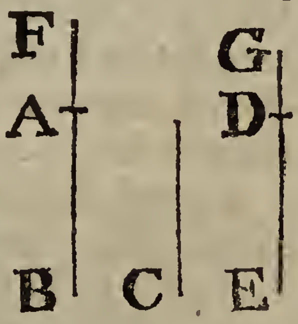


[a] 2. dat.
[b] 12. 5.
[c] 9. dat.
[d] 19. dat.

Another Demonſtration.

Let AB, C, DE be three magnitudes, and let the exceſſes of one of them C above given magnitudes have given ratios to AB and DE; either AB, DE have a given ratio to one another, or the exceſs of one of them above a given magnitude has a given ratio to the other.

Becauſe the exceſs of C above a given magnitude has a given ratio to AB; therefore [a] AB together with a given magnitude has a given ratio to C: Let this given magnitude be AF, wherefore FB has a given ratio to C: Alſo becauſe the exceſs of C above a given magnitude has a given ratio to DE; therefore [a] DE together with a given magnitude has a given ratio to C: Let this given magnitude be DG, wherefore GE has a given ratio to C: And FB has a given ratio to C, therefore [b] the ratio of FB to GE is given: And from FB, GE the given magnitudes AF, DG being taken, the remainders AB, DE either have a given ratio to one another, or the exceſs of one of them above a given magnitude has a given ratio to the other [c].”



[a] 14. dat.
[b] 9. dat.
[c] 19. dat.

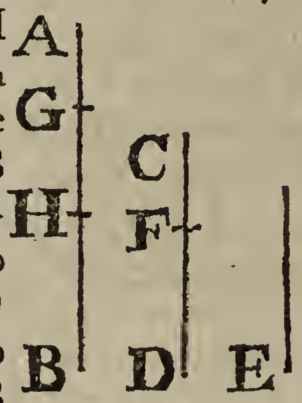
19.

PROP. XXVII.

IF there be three magnitudes: The exceſs of the firſt of which above a given magnitude has a given ratio to the ſecond; and the exceſs of the ſecond above a given magnitude has alſo a given ratio to the third: The exceſs of the firſt above a given magnitude ſhall have a given ratio to the third.

Let AB, CD, E be three magnitudes, the exceſs of the firſt of which AB above the given magnitude AG, viz. GB, has a given ratio to CD; and FD the exceſs of CD above the given magnitude CF, has a given ratio to E: The exceſs of AB above a given magnitude has a given ratio to E.

Becauſe the ratio of GB to CD is given, as GB to CD, ſo make GH to CF; therefore the ratio of GH to CF is given; and CF is given, wherefore [a] GH is given; and AG is given, wherefore the whole AH is given: And becauſe as GB to CD, ſo is GH to CF, and ſo is [b] the remainder HB to the remainder FD; the ratio of HB to FD is given: And the ratio of FD to E is given, wherefore [c] the ratio of HB to E is given: And AH is given; therefore HB the exceſs of AB above a given magnitude AH has a given ratio to E.

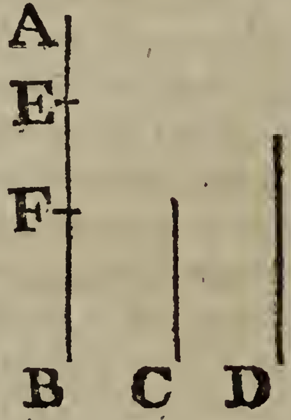


a 2. dat.
b 19. 5.
c 9. dat.

" *Otherwiſe,*

Let AB, C, D be three magnitudes, the exceſs EB of the firſt of which AB above the given magnitude AE has a given ratio to C, and the exceſs of C above a given magnitude has a given ratio to D: The exceſs of AB above a given magnitude has a given ratio to D.

Becauſe EB has a given ratio to C, and the exceſs of C above a given magnitude has a given ratio to D; therefore [d] the exceſs of EB above a given magnitude has a given ratio to D: Let this given magnitude be EF; therefore FB the exceſs of EB above EF has a given ratio to D: And AF is given, Becauſe AE, EF



d 24. dat.

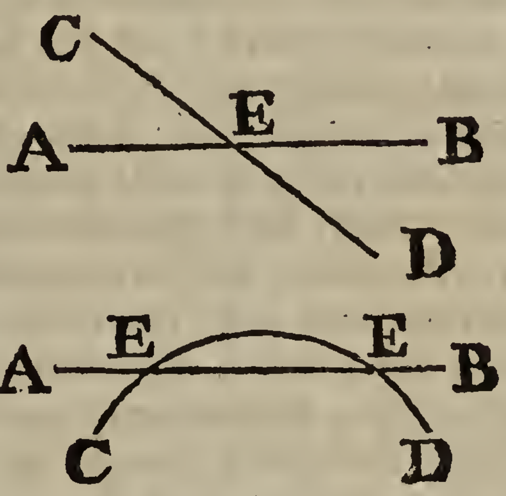
are given: Therefore FB the exceſs of AB above a given magnitude AF has a given ratio to D."

PROP. XXVIII. 25.

IF two lines given in poſition cut one another, the point or points in which they cut one another are given. See N.

Let two lines AB, CD given in poſition cut one another in the point E; the point E is given.

Becaaſe the lines AB, CD are given in poſition, they have always the ſame ſituation [a], and therefore the point, or points in which they cut one another have always the ſame ſituation: And becauſe the lines AB, CD can be found [a], the point, or points, in which they cut one another, are likewiſe found; and therefore are given in poſition [a]. a 4 def.



PROP. XXIX. 26.

IF the extremities of a ſtraight line be given in poſition; the ſtraight line is given in poſition and magnitude.

Becauſe the extremities of the ſtraight line are given, they can be found [a]: Let theſe be the points A, B, between which a ſtraight line AB can be drawn [b]; this has an invariable poſition, becauſe between two given points there can be drawn but one ſtraight line: And when the ſtraight line AB is drawn, its magnitude is at the ſame time exhibited, or given :Therefore the ſtraight line AB is given in poſition and magnitude. a 4 def. b 1. Poſtulate.

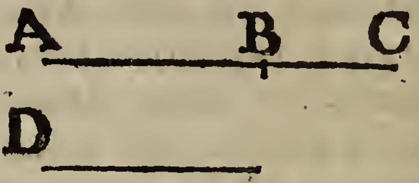
27.

PROP. XXX.

IF one of the extremities of a ſtraight line given in poſition and magnitude be given; the other extremity ſhall alſo be given.

Let the point A be given, to wit, one of the extremities of a ſtraight line given in magnitude, and which lies in the ſtraight line AC given in poſition; the other extremity is alſo given.

a 1. def. Becauſe the ſtraight line is given in magnitude, one equal to it can be found [a]; let this be the ſtraight line D: From the greater ſtraight line AC cut off AB equal to the leſſer D: Therefore the other extremity B of the ſtraight line AB is found: And the point B has always the ſame ſituation; becauſe any other point in AC, upon the ſame ſide of A, cuts off between it and the point A a greater or leſs ſtraight line than AB, that is, than D; Therefore the point B is given [b]: And it is plain another ſuch point can be found in AC produced upon the other ſide of the point A. b 4. def.



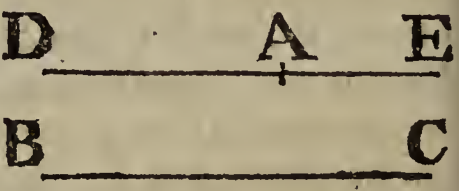
28.

PROP. XXXI.

IF a ſtraight line be drawn through a given point parallel to a ſtraight line given in poſition; that ſtraight line is given in poſition.

Let A be a given point, and BC a ſtraight line given in poſition; the ſtraight line drawn through A parallel to BC is given in poſition.

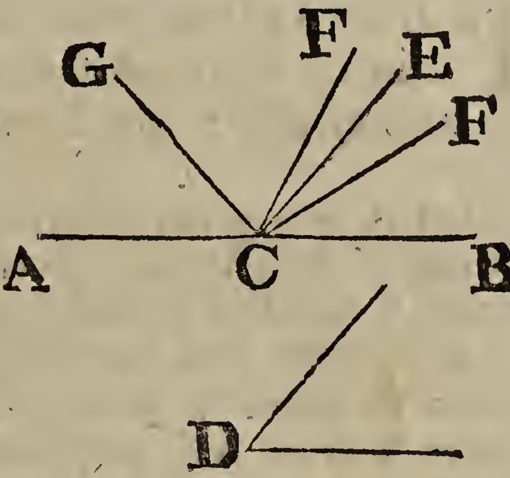
a 31. 1. Through A draw [a] the ſtraight line DAE parallel to BC; the ſtraight line DAE has always the ſame poſition, becauſe no other ſtraight line can be drawn through A parallel to BC: Therefore the ſtraight line DAE which has been found is given [b] in poſition. b 4. def.



PROP.

PROP. XXXII. 29.

IF a ſtraight line be drawn to a given point in a ſtraight line given in poſition, and makes a given angle with it; that ſtraight line is given in poſition.



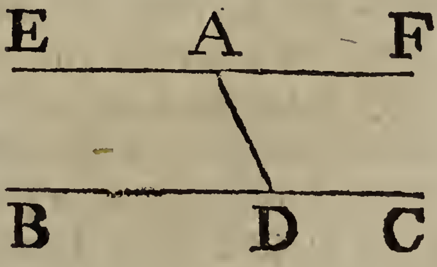
Let AB be a ſtraight line given in poſition, and C a given point in it, the ſtraight line drawn to C, which makes a given angle with CB, is given in poſition.

Becauſe the angle is given, one equal to it can be found[a]; let a 1. def. this be the angle at D, at the given point C, in the given ſtraight line AB, make[b] the angle ECB b 23. 1. equal to the angle at D: Therefore the ſtraight line EC has always the ſame ſituation, becauſe any other ſtraight line FC, drawn to the point C, makes with CB a greater or leſs angle than the angle ECB, or the angle at D: Therefore the ſtraight line EC, which has been found, is given in poſition.

It is to be obſerved, that there are two ſtraight lines EC, GC upon one ſide of AB that make equal angles with it, and which make equal angles with it when produced to the other ſide.

PROB. XXXIII. 30.

IF a ſtraight line be drawn from a given point to a ſtraight line given in poſition, and makes a given angle with it, that ſtraight line is given in poſition.



From the given point A, let the ſtraight line AD be drawn to the ſtraight line BC given in poſition, and make with it a given angle ADC: AD is given in poſition.

Thro' the point A, draw[a] the ſtraight a 31. 1. line EAF parallel to BC; and becauſe thro' the given point A, the ſtraight line EAF is drawn parallel to BC, which is given in poſition, EAF is therefore given in poſition[b]: b 31. dat. And becauſe the ſtraight line AD meets the parallels BC, c 29. 1. EF, the angle EAD is equal c to the angle ADC; and ADC is given, wherefore alſo the angle EAD is given: Therefore, becauſe the ſtraight line DA is drawn to the given point A in the ſtraight line EF given in poſition, and makes with it a d 32. dat. given angle EAD, AD is given d in poſition.

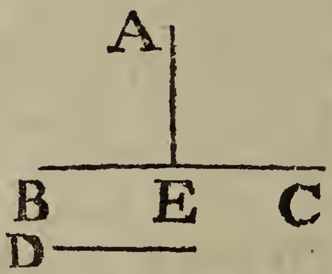
PROP. XXXIV.

31.

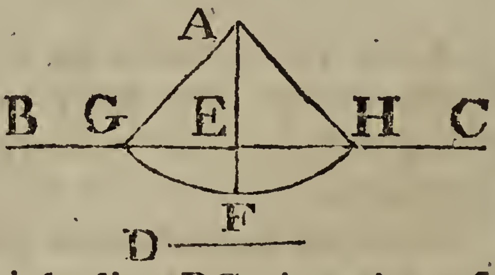
See N. IF from a given point to a ſtraight line given in poſition; a ſtraight line be drawn which is given in magnitude; the ſame is alſo given in poſition.

Let A be a given point, and BC a ſtraight line given in poſition, a ſtraight line given in magnitude drawn from the point A to BC is given in poſition.

Becauſe the ſtraight line is given in magnitude, one equal to a 1. def. it can be found a; let this be the ſtraight line D: From the point A draw AE perpendicular to BC: and becauſe AE is the ſhorteſt of all the ſtraight lines which can be drawn from the point A to BC, the ſtraight line D, to which one equal is to be drawn from the point A to BC, cannot be leſs than AE. If therefore D be equal to AE, AE is the ſtraight line given in magnitude drawn from the given point A to BC: And it b 33. dat. is evident that AE is given in poſition b, becauſe it is drawn from the given point A to BC, which is given in poſition, and makes with BC the given angle AEC.



But if the ſtraight line D be not equal to AE, it muſt be greater than it: Produce AE, and make AF equal to D; and from the centre A, at the diſtance AF, deſcribe the circle GFH, and join AG, AH: Becauſe the circle GFH is gic 6. def. ven in poſition c, and the ſtraight line BC is alſo given in poſition; therefore their interſection d 28. dat. G is given d; and the point A is given; wherefore AG is given in e 29. dat. poſition e, that is, the ſtraight line AG given in magnitude, (for it is equal to D) and drawn from the given point A to the ſtraight line BC given in poſition, is alſo given in poſition: And in like manner AH is given in poſition: Therefore in this caſe there are two ſtraight lines



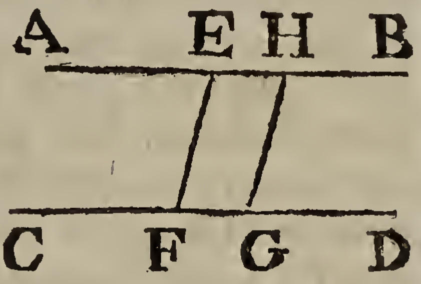
lines AG, AH of the ſame given magnitude which can be drawn from a given point A to a ſtraight line BC given in poſition.

PROP. XXXV. 32.

IF a ſtraight line be drawn between two parallel ſtraight lines given in poſition, and makes given angles with them, the ſtraight line is given in magnitude.

Let the ſtraight line EF be drawn between the parallels AB, CD, which are given in poſition, and make the given angles BEF, EFD : EF is given in magnitude.

In CD take the given point G, and through G draw[a] GH a 31. 1.
parallel to EF : And becauſe CD meets the parallels GH, EF,
the angle EFD is equal[b] to the angle b 29. 1.
HGD : And EFD is a given angle ;
wherefore the angle HGD is given ;
and becauſe HG is drawn to the given
point G, in the ſtraight line CD, given
in poſition, and makes a given angle
HGD : the ſtraight line HG is given
in poſition[c] : And AB is given in poſition : therefore the c 32. dat.
point H is given[d] ; and the point G is alſo given, wherefore d 28. dat.
GH is given in magnitude[e] : And EF is equal to it, there- e 29. dat.
fore EF is given in magnitude.

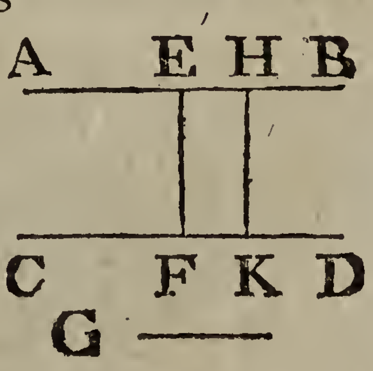


PROP. XXXVI. 33.

IF a ſtraight line given in magnitude be drawn between two parallel ſtraight lines given in poſition, it ſhall make given angles with the parallels. See N.

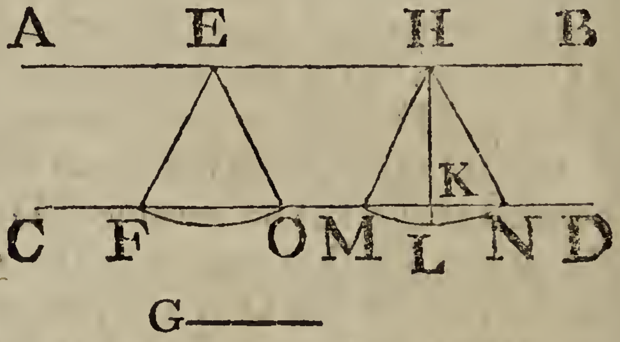
Let the ſtraight line EF given in magnitude be drawn between the parallel ſtraight lines AB, CD, which are given in poſition : the angles AEF, EFC ſhall be given.

Becauſe EF is given in magnitude, a
ſtraight line equal to it can be found[a] : a 1.
let this be G : In AB take a given point
H, and from it draw[b] HK perpendicu- b. 12. 1.
lar to CD : Therefore the ſtraight line G,



that is, EF cannot be leſs than HK: And if G be equal to HK, EF alſo is equal to it; wherefore EF is at right angles to CD; for if it be not, EF would be greater than HK, which is abſurd. Therefore the angle EFD is a right, and conſequently a given angle.

But if the ſtraight line G be not equal to HK, it muſt be greater than it: Produce HK, and take HL, equal to G; and from the centre H, at the diſtance HL, deſcribe the circle
c 6. def. MLN, and join HM, HN: And becauſe the circle [c] MLN, and the ſtraight line CD, are given in poſition, the points M,
d 28. dat. N are [d] given: And the point H is given, wherefore the ſtraight lines HM, HN,
e 29. dat. are given in poſition [e]: And CD is given in poſition; therefore the angles HMN, HNM, are given in poſi-
f A. def. tion [f]: Of the ſtraight lines HM, HN, let HN be that which is not parallel to EF, for EF cannot be parallel to both of them; and draw EO pa-
g 34. 1. rallel to HN: EO therefore is equal [g] to HN, that is, to G; and EF is equal to G; wherefore EO is equal to EF, and
h 29. 1. the angle EFO to the angle ECF, that is [h], to the given angle HNM, and becauſe the angle HNM, which is equal to the angle EFO, or EFD, has been found; therefore the an-
k 1. def. gle EFD, that is the angle AEF, is given in magnitude [k]: and conſequently the angle EFC.

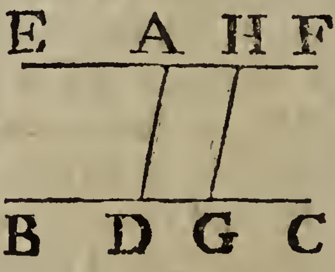


E. ## PROP. XXXVII.

See N. IF a ſtraight line given in magnitude be drawn from a point to a ſtraight line given in poſition, in a given angle; the ſtraight line drawn through that point parallel to the ſtraight line given in poſition, is given in poſition.

Let the ſtraight line AD given in magnitude be drawn from the point A to the ſtraight line BC given in poſition, in the given angle ADC: the ſtraight line EAF drawn through A parallel to BC is given in poſition.

In BC take a given point G, and draw GH parallel to AD: And becauſe HG is drawn to a given point G in the ſtraight line BC gi-



ven in poſition, in a given angle HGC, for it is equal[a] to the given angle ADC; HG is given in poſition[b]: but it is given alſo in magnitude, becauſe it is equal to[c] AD which is given in magnitude; therefore becauſe G one of the extremities of the ſtraight line GH given in poſition and magnitude is given, the other extremity H is given[d]; and the ſtraight line EAF, which is drawn through the given point H parallel to BC given in poſition, is therefore given[e] in poſition.

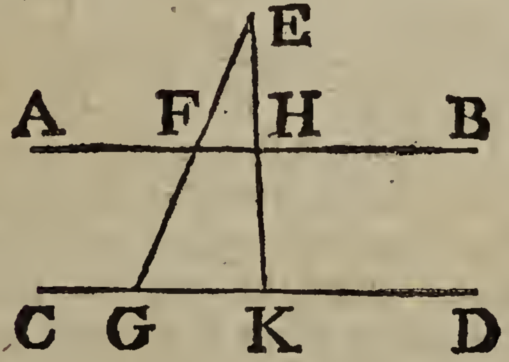
a 29. 1.
b 32. dat.
c 34. 1.
d 30. dat.
e 31. dat.

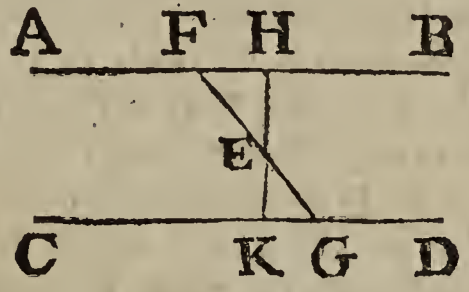
PROP. XXXVIII. 34.

IF a ſtraight line be drawn from a given point to two parallel ſtraight lines given in poſition, the ratio of the ſegments between the given point and the parallels ſhall be given.

Let the ſtraight line EFG be drawn from the given point E to the parallels AB, CD, the ratio of EF to EG is given.

From the point E draw EHK perpendicular to CD; and becauſe from a given point E the ſtraight line EK is drawn to CD which is given in poſition, in a given angle EKC; EK is





given in poſition[a]; and AB, CD are given in poſition; therefore[b] the points H, K are given: And the point E is given; wherefore[c] EH, EK are given in magnitude, and the ratio[d] of them is therefore given. But as EH to EK, ſo is EF to EG, becauſe AB, CD are parallels; therefore the ratio of EF to EG is given.

a 33. dat.
b 28. dat.
c 29. dat.
d 1. dat.

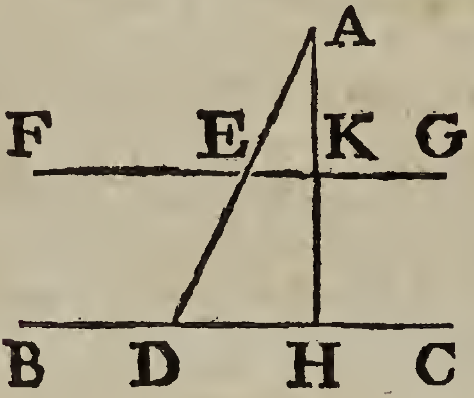
PROP. XXXIX. 35. 36.

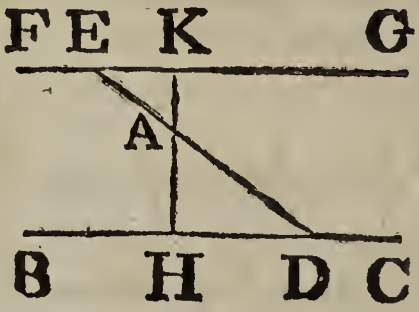
See N.

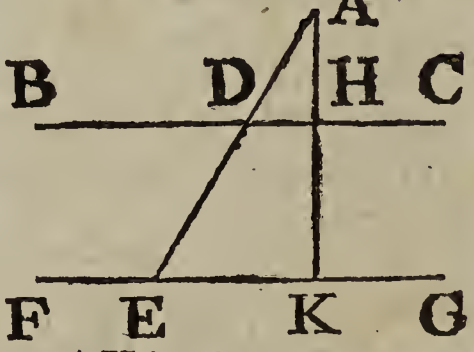
IF the ratio of the ſegments of a ſtraight line between a given point in it and two parallel ſtraight lines, be given, if one of the parallels be given in poſition, the other is alſo given in poſition.

From the given point A, let the ſtraight line AED be drawn to the two parallel ſtraight lines FG, BC, and let the ratio of the ſegments AE, AD be given; if one of the parallels BC be given in poſition, the other FG is alſo given in poſition.

From the point A, draw AH perpendicular to BC, and let it meet FG in K; and becauſe AH is drawn from the given point A to the ſtraight line BC given in poſition, and makes a
a 33. dat. given angle AHD; AH is given [a] in
poſition; and BC is likewiſe given in
poſition, therefore the point H is gi-
b 28. dat. ven [b]: The point A is alſo given;
wherefore AH is given in magni-
c 29. dat. tude [c], and, becauſe FG, BC are pa-
rallels, as AE to AD, ſo is AK to
AH; and the ratio of AE to AD
is given, wherefore the ratio of AK to AH is given; but AH
d 2. dat. is given in magnitude, therefore [d] AK is given in magnitude;
and it is alſo given in poſition, and the point A is given;
e 30. dat. wherefore [e] the point K is given. And becauſe the ſtraight
line FG is drawn through the given point K parallel to BC
f 31. dat. which is given in poſition, therefore [f] FG is given in poſition.







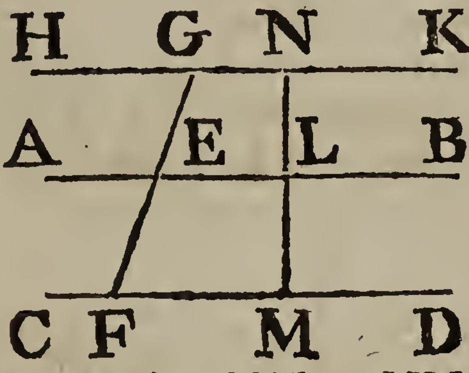
37. 38.

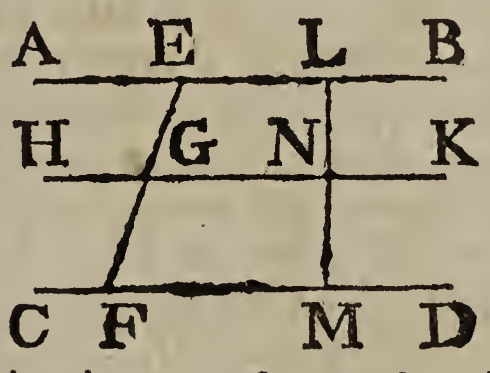
PROP. XL.

See N. IF the ratio of the ſegments of a ſtraight line into which it is cut by three parallel ſtraight lines, be given; if two of the parallels are given in poſition, the third is alſo given in poſition.

Let AB, CD, HK be three parallel ſtraight lines, of which AB, CD are given in poſition; and let the ratio of the ſegments GE, GF into which the ſtraight line GEF is cut by the three parallels, be given; the third parallel HK is given in poſition.

In AB take a given point L, and draw LM perpendicular
to CD, meeting HK in N; becauſe LM is drawn from the gi-
ven point L to CD which is given in poſition and makes a gi-
ven angle LMD; LM is given in poſition [a]; and CD is given a 33. dat.
in poſition, wherefore the point M is given [b]; and the point L b 28. dat.
is given, LM is therefore given in magnitude [c]; and becauſe c. 29. dat.
the ratio of GE to CF is given, and as GE to GF, ſo is NL to





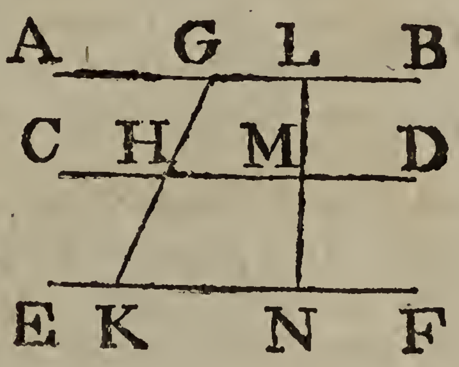
NM; the ratio of NL to NM is given; and therefore [d] the
ratio of ML to LN is given; but LM is given in magnitude [d], d Cor. 6. or 7 dat.
wherefore [e] LN is given in magnitude: and it is alſo given in
poſition, and the point L is given, wherefore [f] the point N is e 2. dat. f 30. dat.
given; and becauſe the ſtraight line HK is drawn through the
given point N parallel to CD which is given in poſition, there-
fore HK is given in poſition [g]. g 31. dat.

PROP. XLI.

F.

IF a ſtraight line meets three parallel ſtraight lines which are given in poſition, the ſegments into which they cut it have a given ratio. See N.

Let the parallel ſtraight lines AB, CD, EF given in poſition, be cut by the ſtraight line GHK; the ratio of GH to HK is given.



In AB take a given point L, and
draw LM perpendicular to CD, meet-
ing EF in N; therefore [a] LM is given a 33. dat.
in poſition; and CD, EF are given
in poſition, wherefore the points M,
N are given: And the point L is given;
therefore [b] the ſtraight lines LM, MN b 29. dat.
are given in magnitude; and the ratio

of

e 1. dat. of LM to MN is therefore given [e]: But as LM to MN, ſo is GH to HK; wherefore the ratio of GH to HK is given.

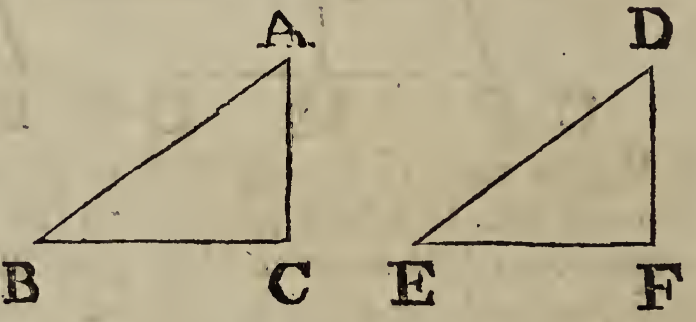
39.

PROP. XLII.

See N. IF each of the ſides of a triangle be given in magnitude, the triangle is given in ſpecies.

Let each of the ſides of the triangle ABC be given in magnitude, the triangle ABC is given in ſpecies.

a 22. 1. Make a triangle [a] DEF the ſides of which are equal, each to each, to the given ſtraight lines AB, BC, CA, which can be done; becauſe any two of them muſt be greater than the third; and let DE be equal to AB, EF to BC, and FD to CA; and becauſe the two ſides ED, DF are equal to the two BA, AC, each to each, and the baſe EF equal to the baſe BC; the angle
b 8. 1. EDF, is equl [b] to the angle BAC; therefore, becauſe the angle EDF, which is equal to the angle BAC, has been found,
c 1. def. the angle BAC is given [c], in like manner the angles at B, C are given. And becauſe the ſides AB, BC, CA are given,
d 1 dat. their ratios to one another are given [d], therefore the triangle
e 3. def. ABC is given [e] in ſpecies.



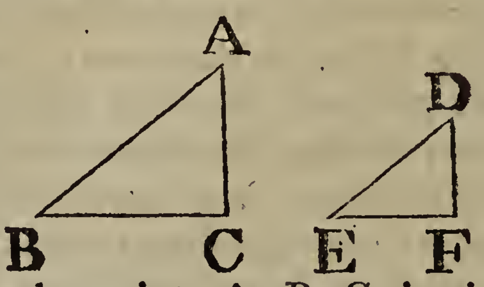
40.

PROP. XLIII.

IF each of the angles of a triangle be given in magnitude, the triangle is given in ſpecies.

Let each of the angles of the triangle ABC be given in magnitude, the triangle ABC is given in ſpecies.

Take a ſtraight line DE given in poſition and magnitude, and at the
a 23. 1. points D, E make [a] the angle EDF equal to the angle BAC, and the angle DEF equal to ABC; therefore the other angles EFD, BCA are equal, and each of the angles at the points A, B, C, is given,



ven, wherefore each of thoſe at the points D, E, F is given :
And becauſe the ſtraight line FD is drawn to the given point
D in DE which is given in poſition, making the given angle
EDF ; therefore DF is given in poſition [b]. In like manner b 32. dat.
EF alſo is given in poſition ; wherefore the point F is given :
And the points D, E are given ; therefore each of the ſtraight
lines DE, EF, FD is given [c] in magnitude ; wherefore the c 29. dat.
triangle DEF is given in ſpecies [d] : and it is ſimilar [e] to the d 42 dat.
triangle ABC : which therefore is given in ſpecies. e 4. 6. 1. def. 6.

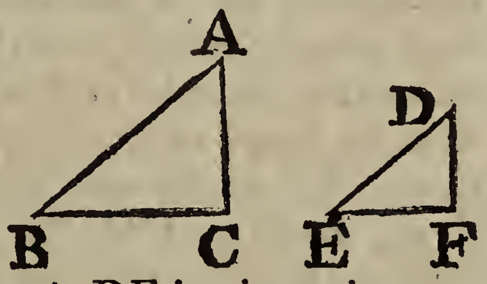
41.

PROP. XLIV.

IF one of the angles of a triangle be given, and if the ſides about it have a given ratio to one another ; the triangle is given in ſpecies.

Let the triangle ABC have one of its angles BAC given, and let the ſides BA, AC about it have a given ratio to one another ; the triangle ABC is given in ſpecies.

Take a ſtraight line DE given in poſition and magnitude,
and at the point D in the given ſtraight line DE, make the
angle EDF equal to the given angle BAC ; wherefore the
angle EDF is given ; and becauſe the ſtraight line FD is drawn
to the given point D in ED which is given in poſition, making
the given angle EDF ; therefore
FD is given in poſition [a]. And a 32. dat.
becauſe the ratio of BA to AC is
given, make the ratio of ED to
DF the ſame with it, and join EF ;
and becauſe the ratio of ED to DF
is given, and ED is given, therefore [b] DF is given in mag- b 2. dat.
nitude : and it is given alſo in poſition, and the point D is
given, wherefore the point F is given [c] : and the points D, c 30. dat.
E are given, wherefore DE, EF, FD are given [d] in magni- d 29. dat.
tude : and the triangle DEF is therefore given [e] in ſpecies ; e 42. dat.
and becauſe the triangles ABC, DEF have one angle BAC
equal to one angle EDF, and the ſides about theſe angles pro-
portionals ; the triangles are [f] ſimilar ; but the triangle DEF f 6. 6.
is given in ſpecies, and therefore alſo the triangle ABC.



PROP.

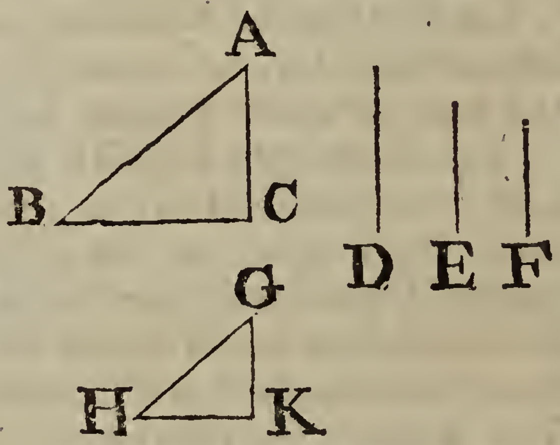
42.

PROP. XLV.

See N. IF the ſides of a triangle have to one another given ratios; the triangle is given in ſpecies.

Let the ſides of the triangle ABC have given ratios to one another, the triangle ABC is given in ſpecies.

Take a ſtraight line D given in magnitude; and becauſe the
ratio of AB to BC is given, make the ratio of D to E the
a 2. dat. ſame with it; and D is given, therefore [a] E is given. And be-
cauſe the ratio of BC to CA is given, to this make the ratio
of E to F the ſame; and E is given, and therefore [a] F. And
becauſe as AB to BC, ſo is D to E; by compoſition AB and
BC together are to BC, as D
and E to E; but as BC to CA,
ſo is E to F; therefore, *ex æ-*
b 22. 5. *quali* [b], as AB and BC are to
CA, ſo are D and E to F, and
c 20. 1. AB and BC are greater [c] than
CA; therefore D and E are
d A. 5. greater [d] than F. In the ſame
manner any two of the three D,
E, F are greater than the third.
e 22. 1. Make [e] the triangle GHK whoſe
ſides are equal to D, E, F, ſo that GH be equal to D, HK to
E, and KG to F; and becauſe D, E, F, are, each of them,
given, therefore GH, HK, KG are each of them given in mag-
f 42. dat. nitude; therefore the triangle GHK is given [f] in ſpecies; But
as AB to BC, ſo is (D to E, that is) GH to HK; and as BC
to CA, ſo is (E to F, that is) HK to KG; therefore, *ex æquali*,
g 5. 6. as AB to AC, ſo is GH to GK. Wherefore [g] the triangle ABC
is equiangular and ſimilar to the triangle GHK; and the tri-
angle GHK is given in ſpecies; therefore alſo the triangle ABC
is given in ſpecies.



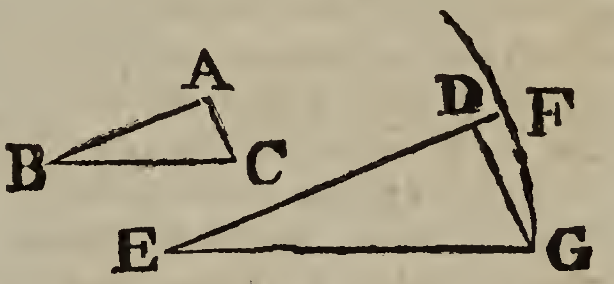
COR. If a triangle is required to be made, the ſides of which ſhall have the ſame ratios which three given ſtraight lines D, E, F have to one another; it it neceſſary that every two of them be greater than the third.

PROP. XLVI. 43.

IF the ſides of a right angled triangle about one of the acute angles have a given ratio to one another; the triangle is given in ſpecies.

Let the ſides AB, BC about the acute angle ABC of the triangle ABC, which has a right angle at A, have a given ratio to one another; the triangle ABC is given in ſpecies.

Take a ſtraight line DE given in poſition and magnitude; and becauſe the ratio of AB to BC is given, make as AB to BC, ſo DE to EF; and becauſe DE has a given ratio to EF, and DE is given, therefore [a] EF is given; and becauſe as AB [a] 2. dat. to BC, ſo is DE to EF; and AB is leſs [b] than BC, therefore [b] 19. 1. DE is leſs [c] than EF. From the point D draw DG at right angles [c] A. 5. to DE, and from the centre E at the diſtance EF, deſcribe a circle which ſhall meet DG in two points; let G be either of them, and join EG; therefore the circumference of the circle is given [d] in poſition; and the ſtraight line DG is given [e] in poſition, becauſe it is drawn to the given point D in DE given in poſition, in a given angle; therefore [f] the point G is given; and the points D, E are given, wherefore DE, EG, GD are given [g] in magnitude, and the triangle DEG in ſpecies [h]. And becauſe the triangles ABC, DEG have the angle BAC equal to the angle EDG, and the ſides about the angles ABC, DEG proportionals, and each of the other angles BCA, EGD leſs than a right angle; the triangle ABC is equiangular [i] and ſimilar to the triangle DEG: But DEG is given in ſpecies; therefore the triangle ABC is given in ſpecies: And in the ſame manner, the triangle made by drawing a ſtraight line from E to the other point in which the circle meets DG is given in ſpecies.



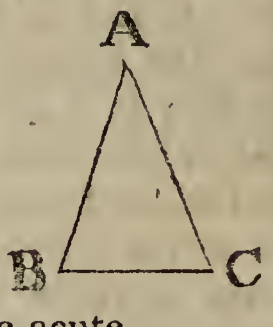
[d] 6. def.
[e] 32. dat.
[f] 28. dat.
[g] 29. dat.
[h] 42. dat.
[i] 7. 6.

44.

PROP. XLVII.

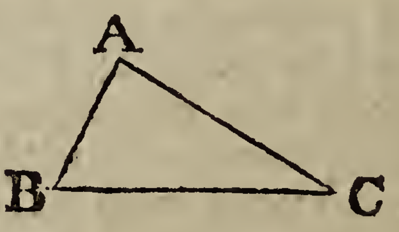
See N. IF a triangle has one of its angles which is not a right angle given, and if the ſides about another angle have a given ratio to one another; the triangle is given in ſpecies.

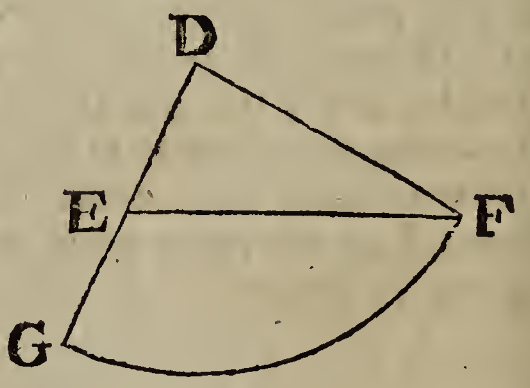
Let the triangle ABC have one of its angles ABC a given, but not a right angle, and let the ſides BA, AC about another angle BAC have a given ratio to one another; the triangle ABC is given in ſpecies.



Firſt, Let the given ratio be the ratio of equality, that is, let the ſides BA, AC, and conſequently the angles ABC, ACB, be equal; and becauſe the angle ABC is given,
a 32. 1. the angle ACB, and alſo the remaining[a] angle BAC is given; therefore the triangle
b. 43. dat. ABC is given[b] in ſpecies; and it is evident that in this caſe the given angle ABC muſt be acute.

Next, Let the given ratio be the ratio of a leſs to a greater, that is, let the ſide AB adjacent to the given angle be leſs than the ſide AC: Take a ſtraight line DE given in poſition and magnitude, and make the angle DEF equal to the given
c. 32. dat. angle ABC; therefore EF is given[c] in poſition; and beeauſe the ratio of BA to AC is given, as BA to AC, ſo make ED to DG; and becauſe the ratio of ED to DG is given, and ED is given, the ſtraight line DG is
d 2. dat. given[d], and BA is leſs than
e A. 5. AC, therefore ED is leſs[e] than DG. From the centre D, at the diſtance DG deſcribe the circle GF meeting EF in F, and join DF; and becauſe the
f 6. def. circle is given[f] in poſition, as alſo the ſtraight line EF, the
g 28. dat. point F is given[g]; and the points D, E are given; where-
h 29. dat. fore the ſtraight lines DE, EF, FD are given[h] in mag-
i 42. dat. nitude, and the triangle DEF in ſpecies[i]. And be-
k 18. 1. cauſe BA is leſs than AC, the angle ACB is leſs[k]
l 17. 1. than the angle ABC, and therefore ACB is leſs[l] than

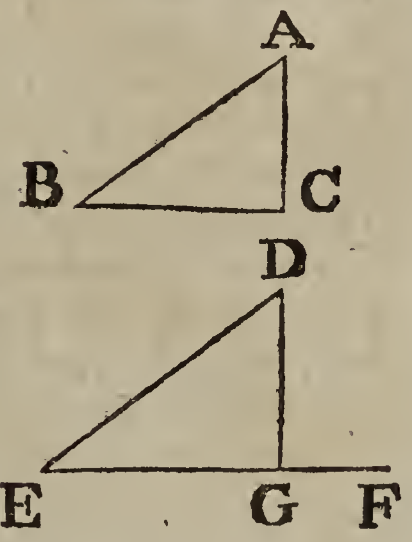




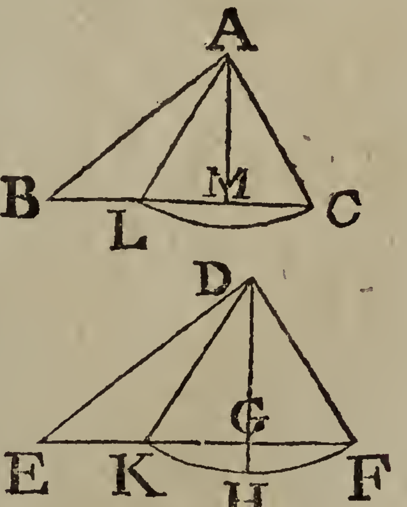
a

a right angle. In the ſame manner, becauſe ED is leſs than DG or DF, the angle DFE is leſs than a right angle: And becauſe the triangles ABC, DEF have the angle ABC equal to the angle DEF, and the ſides about the angles BAC, EDF proportionals, and each of the other angles ACB, DFE leſs than a right angle; the triangles ABC, DEF are [m] ſimilar, m. 7. 6. and DEF is given in ſpecies, wherefore the triangle ABC is alſo given in ſpecies.

Thirdly, Let the given ratio be the ratio of a greater to a leſs, that is, let the ſide AB adjacent to the given angle be greater than AC; and as in the laſt caſe, take a ſtraight line DE given in poſition and magnitude, and make the angle DEF equal to the given angle ABC; therefore EF is given [c] in poſition: c 32. dat. Alſo draw DG perpendicular to EF; therefore if the ratio of BA to AC be the ſame with the ratio of ED to the perpendicular DG, the triangles ABC, DEG are ſimilar [m], becauſe the angles ABC, DEG are equal, and DGE is a right angle: Therefore the angle ACB is a right angle, and the triangle ABC is given in [b] ſpecies. b 43. dat.

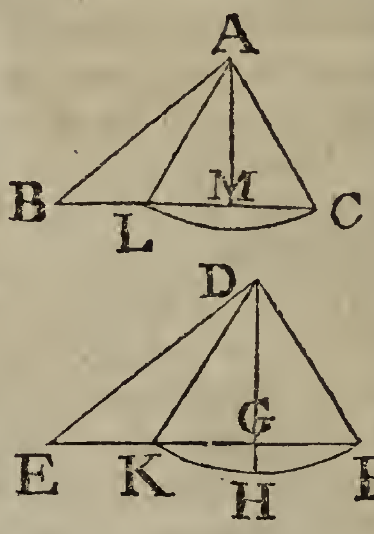


But if, in this laſt caſe, the given ratio of BA to AC be not the ſame with the ratio of ED to DG, that is, with the ratio of BA to the perpendicular AM drawn from A to BC; the ratio of BA to AC muſt be leſs than [o] the ratio of BA o 8. 5. to AM, becauſe AC is greater than AM. Make as BA to AC ſo ED to DH; therefore the ratio of ED to DH is leſs than the ratio of (BA to AM, that is, than the ratio of) ED to DG; and conſequently, DH is greater [p] than DG; and becauſe BA is greater than AC, ED is greater [e] than DH. p 10. 5. e A. 5. From the center D, at the diſtance DH, deſcribe the circle KHF which neceſſarily meets the ſtraight line EF in two points, becauſe DH is greater than DG, and leſs than DE. Let the circle meet EF in the points F, K which are given, as was ſhown in the preceding caſe; and, DF, DK being joined, the triangles DEF, DEK are given in ſpecies, as was there



ſhewn. From the centre, A at the diſtance AC, deſcribe a circle meeting BC again in L: And if the angle ACB be leſs than a right angle, ALB muſt be greater than a right angle: And on the contrary. In the ſame manner, if the angle DFE be leſs than a right angle, DKE muſt be greater than one; and on the contrary. Let each of the angles ACB, DFE be either leſs or greater than a right angle; and becauſe in the triangles ABC, DEF the angles ABC, DEF are equal, and the ſides BA, AC, and ED, DF, about two of the other angles proportionals, the triangle ABC is ſimilar [m] to the triangle DEF. In the ſame manner, the triangle ABL is ſimilar to DEK. And the triangles, DEF, DEK are given in ſpecies; therefore alſo the triangles ABC, ABL are given in ſpecies. And from this it is evident, that, in this third caſe, there are always two triangles of a different ſpecies, to which the things mentioned as given in the propoſition can agree.

m 7. 6.



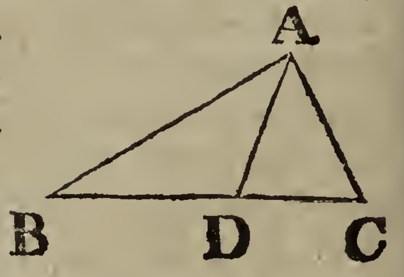
45.

PROP. XLVIII.

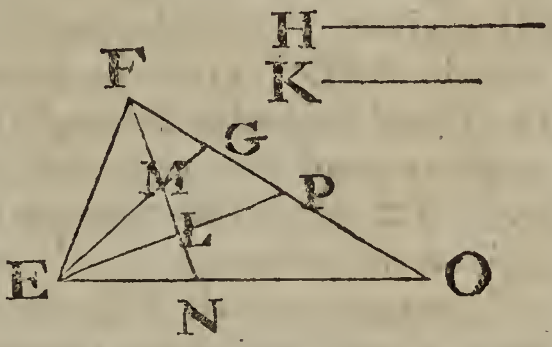
IF a triangle has one angle given, and if both the ſides together about that angle have a given ratio to the remaining ſide; the triangle is given in ſpecies.

Let the triangle ABC have the angle BAC given, and let the ſides BA, AC together about that angle have a given ratio to BC; the triangle ABC is given in ſpecies.

a 9. 1. Biſect [a] the angle BAC by the ſtraight line AD; therefore
b 3. 6. the angle BAD is given. And becauſe as BA to AC, ſo is [b] BD to DC, by permutation, as AB to BD, ſo is AC to CD; and as BA and AC to-
c 12. 5. gether to BC, ſo is [c] AB to BD. But the ratio of BA and AC together to BC is given, wherefore the ratio of AB to BD is given, and the angle BAD is given;
d 47. dat. therefore [d] the triangle ABD is given in ſpecies, and the angle ABD is therefore given; the angle BAC
e 43. dat. is alſo given, wherefore the triangle ABC is given in ſpecies [e].



A triangle which ſhall have the things that are mentioned in the propoſition to be given, can be found in the following manner. Let EFG be the given angle, and let the ratio of H to K be the given ratio which the two ſides about the angle EFG muſt have to the third ſide of the triangle; therefore becauſe two ſides of a triangle are greater than the third ſide, the ratio of H to K muſt be the ratio of a greater to a leſs. Biſect [a] the angle EFG by the ſtraight line FL, and by the a 9. 1. 47th propoſition find a triangle of which EFL is one of the angles, and in which the ratio of the ſides about the angle oppoſite to FL is the ſame with the ratio of H to K: To do which, take FE given in poſition and magnitude, and draw EL perpendicular to FL: Then if the ratio of H to K be the ſame with the ratio of FE to EL, produce EL, and let it meet FG in P; the triangle FEP is that which was to be found: for it has the given angle EFG; and becauſe this angle is biſected by FL, the ſides EF, FP together are to EP, as [b] FE to EL, that b 3. 6. is, as H to K.



But if the ratio of H to K be not the ſame with the ratio of FE to EL, it muſt be leſs than it, as was ſhown in prop. 47. and in this caſe there are two triangles, each of which has the given angle EFL, and the ratio of the ſides about the angle oppoſite to FL the ſame with the ratio of H to K. By prop. 47. find theſe triangles EFM, EFN each of which has the angle EFL for one of its angles, and the ratio of the ſide FE to EM or EN the ſame with the ratio of H to K; and let the angle EMF be greater, and ENF leſs than a right angle. And becauſe H is greater than K, EF is greater than EN, and therefore the angle EFN, that is, the angle NFG, is leſs [f] than the angle ENF. To each of theſe f 18. 1. add the angles NEF, EFN; therefore the angles NEF, EFG are leſs than the angles NEF, EFN, FNE, that is, than two right angles; therefore the ſtraight lines EN, FG muſt meet together when produced; let them meet in O, and produce EM to G. Each of the triangles, EFG, EFO has the things mentioned to be given in the propoſition: For each of them has the given angle EFG; and becauſe this angle is biſected by the ſtraight line FMN, the ſides EF, FG together have to EG the third ſide the ratio of FE to EM, that is, of H to K. In like manner, the ſides EF, FO together have to EO the ratio which H has to K.

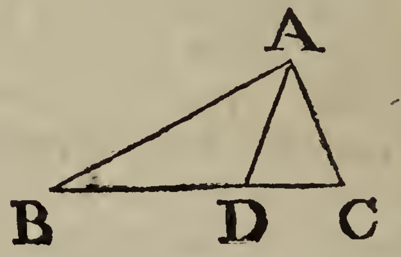
PROP.

46. PROP. XLIX.

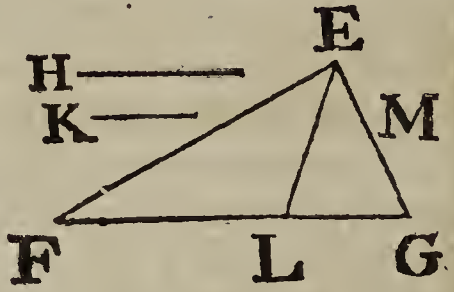
IF a triangle has one angle given, and if the ſides about another angle, both together have a given ratio to the third ſide; the triangle is given in ſpecies.

Let the triangle ABC have one angle ABC given, and let the two ſides BA, AC about another angle BAC have a given ratio to BC; the triangle ABC is given in ſpecies.

Suppoſe the angle BAC to be biſected by the ſtraight line AD; BA and AC together are to BC, as AB to BD, as was ſhown in the preceding propoſition. But the ratio of BA and AC together to BC is given; therefore alſo the ratio of AB to a 44. dat. BD is given. And the angle ABD is given, wherefore [a] the triangle ABD is given in ſpecies; and conſequently the angle BAD; and its double the angle BAC are given; and the angle ABC is given. Therefore the triangle ABC b 43. dat. is given in ſpecies [b].



A triangle which ſhall have the things mentioned in the propoſition to be given, may be thus found. Let EFG be the given angle, and the ratio of H to K the given ratio; and by prop. 44. find the triangle EFL, which has the angle EFG for one of its angles, and the ratio of the ſides EF, FL about this angle the ſame with the ratio of H to K; and make the angle LEM equal to the angle FEL. And becauſe the ratio of H to K is the ratio which two ſides of a triangle have to the third, H muſt be greater than K; and becauſe EF is to FL, as H to K; therefore EF is greater than FL, and the angle FEL, that is, LEM, is therefore leſs than the angle ELF. Wherefore the angles LFE, FEM are leſs than two right angles, as was ſhown in the foregoing propoſition, and the ſtraight lines FL, EM muſt meet if produced; let them meet in G, EFG is the triangle which was to be found; for EFG is one of its angles, and becauſe the angle FEG is biſected by EL, the two ſides FE, EG together have to the third ſide FG the ratio of EF to FL, that is, the given ratio of H to K.

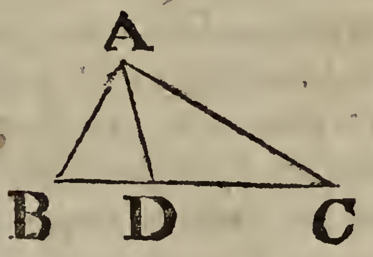


PROP.

PROP. L. 76.

IF from the vertex of a triangle given in ſpecies, a ſtraight line be drawn to the baſe in a given angle; it ſhall have a given ratio to the baſe.

From the vertex A of the triangle ABC which is given in ſpecies, let AD be drawn to the baſe BC in a given angle ADB; the ratio of AD to BC is given.

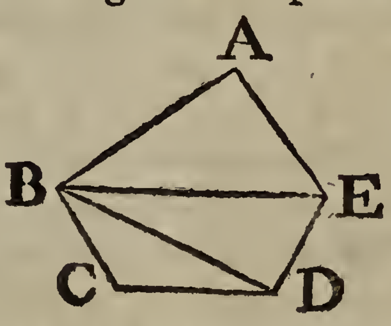


Becauſe the triangle ABC is given in
ſpecies, the angle ABD is given, and the
angle ADB is given, therefore the tri-
angle ABD is given [a] in ſpecies; where- a 43. dat.
fore the ratio of AD to AB is given.
And the ratio of AB to BC is given;
and therefore [b] the ratio of AD to BC is given. b. 9. dat.

PROP. LI. 47.

RECTILINEAL figures given in ſpecies, are divided into triangles which are given in ſpecies.

Let the rectilineal figure ABCDE be given in ſpecies: ABCDE may be divided into triangles given in ſpecies.



Join BE, BD; and becauſe ABCDE is given in ſpecies,
the angle BAE is given [a], and the ra- a 3. def.
tio of BA to AE is given [a]; where-
fore the triangle BAE is given in ſpe-
cies [b], and the angle AEB is there- b 44. dat.
fore given [a]. But the whole angle
AED is given, and therefore the re-
maining angle BED is given, and the
ratio of AE to EB is given, as alſo the ratio AE to ED;
therefore the ratio of BE to ED is given [c]. And the angle c 9. dat.
BED is given, wherefore the triangle BED is given [b] in ſpe-
cies. In the ſame manner, the triangle BDC is given in ſpe-
cies: Therefore rectilineal figures which are given in ſpecies
are divided into triangles given in ſpecies.

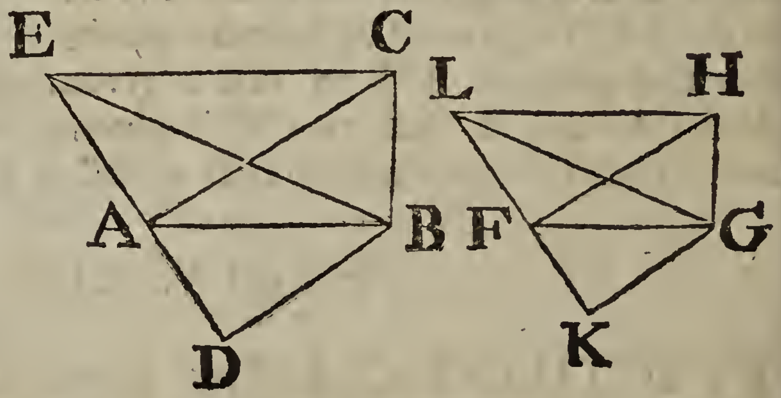
48.

PROP. LII.

If two triangles given in ſpecies be deſcribed upon the ſame ſtraight line; they ſhall have a given ratio to one another.

Let the triangles ABC, ABD given in ſpecies be deſcribed upon the ſame ſtraight line AB; the ratio of the triangle ABC to the triangle ABD is given.

Through the point C, draw CE parallel to AB, and let it meet DA produced in E, and join BE. Becauſe the triangle ABC is given in ſpecies, the angle BAC, that is, the angle ACE, is given; and becauſe the triangle ABD is given in ſpecies, the angle DAB that is, the angle AEC, is given. Therefore the triangle ACE is given in ſpecies; wherefore the ratio of EA to AC is
a 3. def. given [a], and the ratio of CA to AB is given, as alſo the ratio of BA to AD;
b 9 dat. therefore the ratio of [b] EA to AD is given, and the triangle
c 37. 1. ACB is equal [c] to the triangle AEB, and as the triangle
d 1. 6. AEB, or ACB, is to the triangle ADB, ſo is [d] the ſtraight line EA to AD. But the ratio of EA to AD is given; therefore the ratio of the triangle ACB to the triangle ADB is given.



PROBLEM.

To find the ratio of two triangles ABC, ABD given in ſpecies, and which are deſcribed upon the ſame ſtraight line AB.

Take a ſtraight line FG given in poſition and magnitude, and becauſe the angles of the triangles ABC, ABD are given, at the points F, G of the ſtraight line FG, make the angles
e 23. 1. GFH, GFK [e] equal to the angles BAC, BAD; and the angles FGH, FGK equal to the angles ABC, ABD, each to each. Therefore the triangles ABC, ABD are equiangular to the triangles FGH. FGK, each to each. Through the point H draw HL parallel to FG meeting KF produced in L. And becauſe the angles BAC, BAD are equal to the angles GFH, GFK, each to each; therefore the angles ACE, AEC are equal to FHL, FLH, each to each, and the triangle AEC equiangular to the triangle FLH. Therefore as EA to AC, ſo is LF to FH; and

as CA to AB, ſo HF to FG: and as BA to AD, ſo is GF to FK; wherefore, ex æquali, as EA to AD, ſo is LF to FK. But as was ſhown, the triangle ABC is to the triangle ABD, as the ſtraight line EA to AD, that is, as LF to FK. The ratio therefore of LF to FK has been found, which is the ſame with the ratio of the triangle ABC to the triangle ABD.

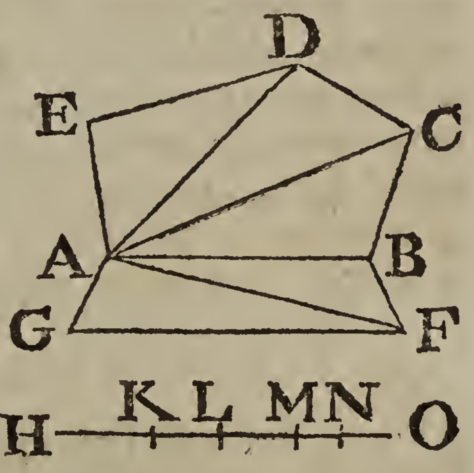
PROP. LIII.

49.

IF two rectilineal figures given in ſpecies be deſcribed upon the ſame ſtraight line; they ſhall have a given ratio to one another. See N.

Let any two rectilineal figures ABCDE, ABFG which are given in ſpecies, be deſcribed upon the ſame ſtraight line AB; the ratio of them to one another is given.

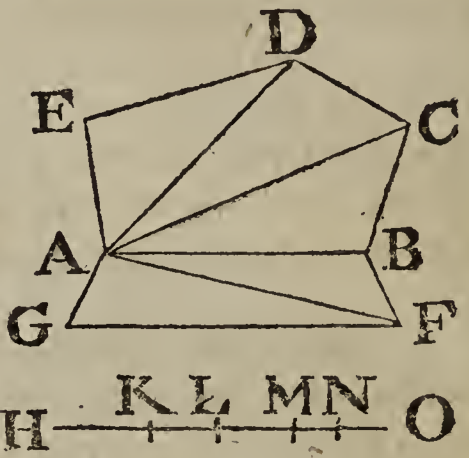
Join AC, AD, AF; each of the triangles AED, ADC, ACB, AGF, ABF, is given [a] in ſpecies. a 51. dat. And becauſe the triangles ADE, ADC given in ſpecies are deſcribed upon the ſame ſtraight line AD, the ratio of EAD to DAC is given [b]; b 52. dat. and, by compoſition, the ratio of EACD to DAC is given [c]. c 7. dat. And the ratio DAC to CAB is given [b], becauſe they are deſcribed upon the ſame ſtraight line AC; therefore the ratio of EACD to ACB is given [d]; d 9. dat. and, by compoſition, the ratio of ABCDE to ABC is given. In the ſame manner, the ratio, of ABFG to ABF is given. But the ratio of the triangle ABC to the triangle ABF is given; wherefore [b], becauſe the ratio of ABCDE to ABC is given, as alſo the ratio of ABC to ABF, and the ratio of ABF to ABFG; the ratio of the rectilineal ABCDE to the rectilineal ABFG is given [d].



PROBLEM.

To find the ratio of two rectilineal figures given in ſpecies, and deſcribed upon the ſame ſtraight line.

Let ABCDE, ABFG be two rectilineal figures given in ſpecies, and deſcribed upon the ſame ſtraight line AB, and join AC, AD, AF. Take a ſtraight line HK given in poſition and magnitude, and by the 52d dat. find the ratio of the triangle ADE to the triangle ADC, and make the ratio of HK to KL the ſame with it. Find alſo the ratio of the triangle ACD to the triangle ACB And make the ratio of KL to LM the ſame. Alſo, find the ratio of the triangle ABC to the triangle ABF, and make the ratio of LM to MN the ſame. And laſtly, find the ratio of the triangle AFB to the triangle AFG, and make the ratio of MN to NO the ſame. Then the ratio of ABCDE to ABFG is the ſame with the ratio of HM to MO.

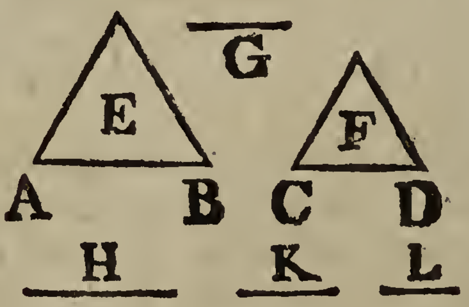


Becauſe the triangle EAD is to the triangle DAC, as the ſtraight line HK to KL; and as the triangle DAC to CAB, ſo is the ſtraight line KL to LM; therefore by uſing compoſition as often as the number of triangles requires, the rectilineal ABCDE is to the triangle ABC, as the ſtraight line HM to ML. In like manner, becauſe the triangle GAF is to FAB, as ON to NM. by compoſition, the rectilineal ABFG is to the triangle ABF as MO to NM; and by inverſion, as ABF to ABFG, ſo is NM to MO. And the triangle ABC is to ABF, as LM to MN. Wherefore, becauſe as ABCDE to ABC, ſo is HM to ML; and as ABC to ABF, ſo is LM to MN; and as ABF to ABFG, ſo is MN to MO; ex æquali, as the rectilineal ABCDE to ABFG, ſo is the ſtraight line HM to MO.

50. PROP. LIV.

IF two ſtraight lines have a given ratio to one another; the ſimilar rectilineal figures deſcribed upon them ſimilarly, ſhall have a given ratio to one another.

Let the ſtraight lines AB, CD, have a given ratio to one another, and let the ſimilar and ſimilarly placed rectilineal figures E, F be deſcribed upon them; the ratio of E to F is given.



To AB, CD, let G be a third proportional; therefore as AB to CD, ſo is CD to G. And the ratio of AB to CD is given; wherefore the ratio of CD to G is given; and conſequently the ratio of AB to G
a 9. dat. is alſo given [a]. But as AB to G, ſo
b 2.Cor.20. is the figure E to the figure [b] F. Therefore the ratio of E
6. to F is given.

PROBLEM.

PROBLEM.

To find the ratio of two ſimilar rectilineal figures, E, F, ſimilarly deſcribed upon ſtraight lines AB, CD which have a given ratio to one another: Let G be a third proportional to AB, CD.

Take a ſtraight line H given in magnitude; and becauſe the ratio of AB to CD is given, make the ratio of H to K the ſame with it; and becauſe H is given, K is given. As H is to K, ſo make K to L; then the ratio of E to F is the ſame with the ratio of H to L; for AB is to CD, as H to K, wherefore CD is to G, as K to L; and, ex æquali, as AB to G, ſo is H to L: But the figure E is to [b] the figure F, as AB to G, that is, as H to L.

b 2. cor. 20. 6.

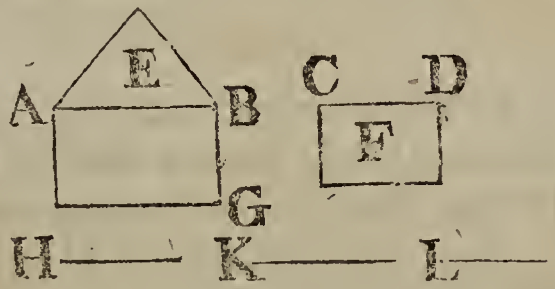
PROP. LV.

51.

IF two ſtraight lines have a given ratio to one another; the rectilineal figures given in ſpecies deſcribed upon them, ſhall have to one another a given ratio.

Let AB, CD be two ſtraight lines which have a given ratio to one another; the rectilineal figures E, F given in ſpecies and deſcribed upon them, have a given ratio to one another.

Upon the ſtraight line AB, deſcribe the figure AG ſimilar and ſimilarly placed to the figure F; and becauſe F is given in ſpecies, AG is alſo given in ſpecies: Therefore, ſince the figures E, AG which are given in ſpecies, are deſcribed upon the ſame ſtraight line AB, the ratio of E to AG is given [a], and becauſe the ratio of AB to CD is given and upon them are deſcribed the ſimilar and ſimiliarly placed rectilineal figures AG, F, the ratio of AG to F is given [b]; and the ratio of AG to E is given; thererfore the ratio of E to F is given [c].



a 53. dat.
b 54. dat.
c 9. dat.

PROBLEM.

To find the ratio of two rectilineal figures E, F given in ſpecies and deſcribed upon the ſtraight lines AB, CD which have a given ratio to one another.

Take a ſtraight line H given in magnitude; and becauſe the rectilineal figures E, AG given in ſpecies are deſcribed upon the ſame ſtraight line AB, find their ratio by the 53d dat. and make the ratio of H to K the ſame, K is therefore given. And becauſe the ſimilar rectilineal figures AG, F are deſcribed upon the ſtraight lines AB, CD which have a given ratio, find their ratio by the 54th dat. and make the ratio of K to L the ſame: The figure E has to F the ſame ratio which H has to L; For, by the conſtruction, as E is to AG, ſo is H to K; and as AG to F, ſo is K to L; therefore, ex æquali, as E to F; ſo is H to L.

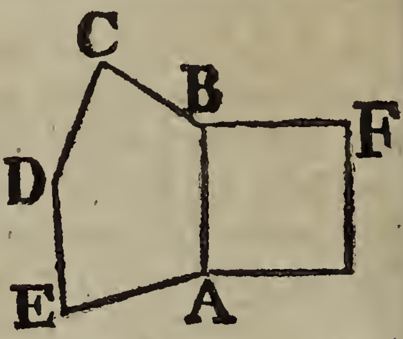
52.

PROP. LVI.

IF a rectilineal figure given in ſpecies be deſcribed upon a ſtraight line given in magnitude; the figure is given in magnitude.

Let the rectilineal figure ABCDE given in ſpecies be deſcribed upon the ſtraight line AB given in magnitude; the figure ABCDE is given in magnitude.

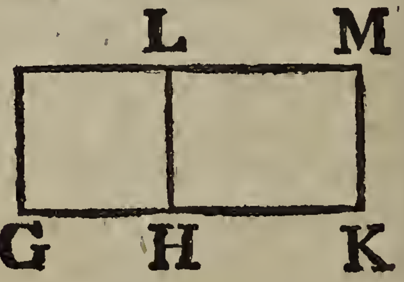
Upon AB let the ſquare AF be deſcribed; therefore AF is
given in ſpecies and magnitude, and becauſe the rectilineal figures ABCDE, AF given in ſpecies are
deſcribed upon the ſame ſtraight line
AB, the ratio of ABCDE to AF is
a 53. dat. given[a]: But the ſquare AF is given in
b 2. dat. magnitude, therefore [b] alſo the figure
ABCDE is given in magnitude.



PROB.

To find the magnitude of a rectilineal figure given in ſpecies deſcribed upon a ſtraight line given in magnitude.

Take the ſtraight line GH equal to the given ſtraight line AB, and by the 53d dat. find the ratio which the ſquare AF upon AB has to the figure ABCDE; and make the ratio
of GH to HK the ſame; and upon GH deſcribe the ſquare GL,
and complete the parallelogram LHKM; the figure ABCDE
is equal to LHKM; becauſe AF is to ABCDE, as the ſtraight
line GH to HK, that is, as the figure GL to HM; and AF is
c 14. 5. equal to GL; therefore ABCDE is equal to HM[c].



53.

PROP. LVII.

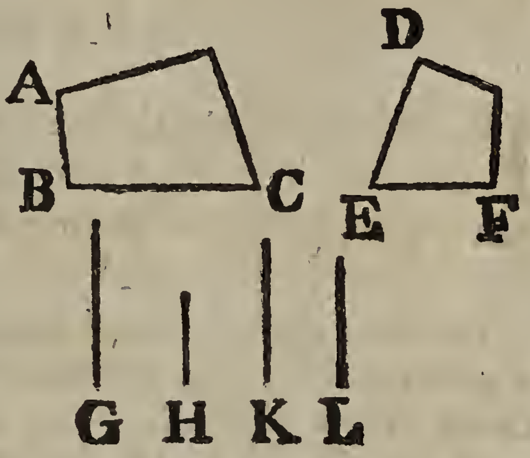
IF two rectilineal figures are given in ſpecies, and if a ſide of one of them has a given ratio to a ſide of the other; the ratios of the remaining ſides to the remaining ſides ſhall be given.

Let AC, DF be two rectilineal figures given in ſpecies, and let the ratio of the ſide AB to the ſide DE be given, the ratios of the remaining ſides to the remaining ſides are alſo given.

Becauſe the ratio of AB to DE is given, as alſo [a] the ratios of AB to BC, and of DE to EF, the ratio of BC to EF is given [b]. In the ſame manner, the ratios of the other ſides to the other ſides are given.

a 3 def.

b 10. dat.



The ratio which BC has to EF may be found thus: Take a ſtraight line G given in magnitude, and becauſe the ratio of BC to BA is given, make the ratio of G to H the ſame; and becauſe the ratio of AB to DE is given, make the ratio of H to K the ſame; and make the ratio of K to L the ſame with the given ratio of DE to EF. Since therefore as BC to BA, ſo is G to H; and as BA to DE, ſo is H to K; and as DE to EF, ſo is K to L; *ex æquali*, BC is to EF, as G to L; therefore the ratio of G to L has been found, which is the ſame with the ratio of BC to EF.

PROP. LVIII.

G.

See N.

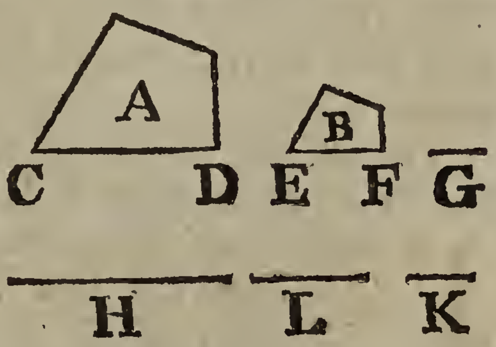
IF two ſimilar rectilineal figures have a given ratio to one another, their homologous ſides have alſo a given ratio to one another.

Let the two ſimilar rectilineal figures A, B, have a given ratio to one another, their homologous ſides have alſo a given ratio.

Let the ſide CD be homologous to EF, and to CD, EF let the ſtraight line G be a third proportional. As therefore [a] CD to G, ſo is the figure A to B; and the ratio of A to B is given, therefore the ratio of CD to G is given; and CD, EF, G, are proportionals; wherefore [b] the ratio of CD to EF is given.

a 2 Cor. 20. 6.

b 13. dat.



The ratio of CD to EF may be found thus: Take a ſtraight line H given in magnitude; and becauſe the ratio of the figure A to B is given, make the ratio of H to K the ſame with it; And, as the 13th dat. directs to be done, find a mean proportional L between H and K; the ratio of CD to EF is the ſame with that of H to L. Let G be a third proportional to CD, EF; therefore as CD to G, ſo is (A to B, and ſo is) H to K; and as CD to EF, ſo is H to L, as is ſhown in the 13th dat.

54. PROP. LIX.

See N. IF two rectilineal figures given in ſpecies have a given ratio to one another, their ſides ſhall likewiſe have given ratios to one another.

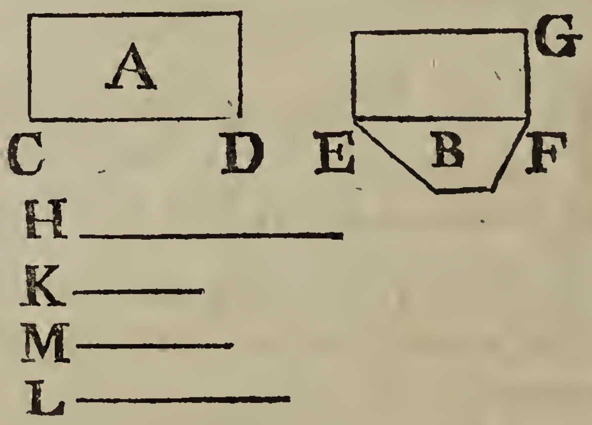
Let the two rectilineal figures A, B, given in ſpecies, have a given ratio to one another, their ſides ſhall alſo have given ratios to one another.

If the figure A be ſimilar to B, their homologous ſides ſhall have a given ratio to one another, by the preceding propoſition; and becauſe the figures are given in ſpecies, the ſides of each of them have given ratios [a] to one another; therefore each ſide of one of them has [b] to each ſide of the other a given ratio.

a 3. def.
b 9. dat.

But if the figure A be not ſimilar to B, let CD, EF be any two of their ſides; and upon EF conceive the figure EG to be deſcribed ſimilar and ſimilarly placed to the figure A, ſo that CD, EF be homologous ſides; therefore EG is given in ſpecies; and the figure B is given in ſpecies; wherefore [c] the ratio of B to EG is given; and the ratio of A to B is given, therefore [b] the ratio of the figure A to EG is given; and A is ſimilar to EG; therefore [d] the ratio of the ſide CD to EF is given; and conſequently [b] the ratios of the remaining ſides to the remaining ſides are given.

c 53. dat.
d 58. dat.



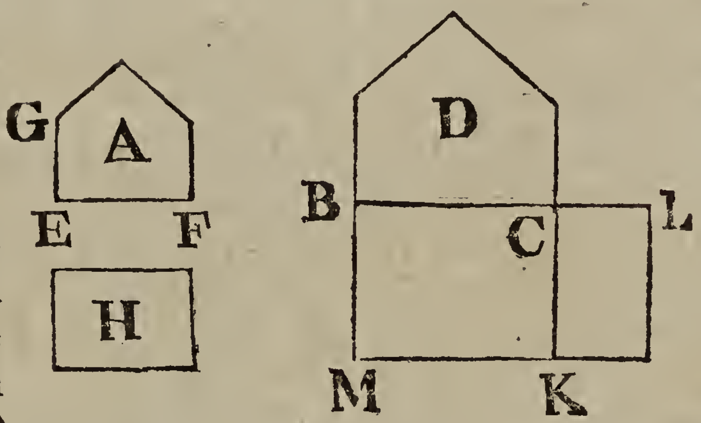
The ratio of CD to EF may be found thus: Take a ſtraight line H given in magnitude, and becauſe the ratio of the figure A to B is given, make the ratio of H to K the ſame with it. And by the 53d dat. find the ratio of the figure B to EG, and make the ratio of K to L the ſame: Between H and L find a mean proportional M, the ratio of CD to EF is the ſame with the ratio of H to M; becauſe the figure A is to B as H to K; and as B to EG, ſo is K to L; *ex æquali*, as A to EG, ſo is H to L: And the figures A, EG are ſimilar, and M is a mean proportional between H and L; therefore, as was ſhewn in the preceding propoſition, CD is to EF as H to M.

PROP. LX. 55.

IF a rectilineal figure be given in ſpecies and magnitude, the ſides of it ſhall be given in magnitude.

Let the rectilineal figure A be given in ſpecies and magnitude, its ſides are given in magnitude.

Take a ſtraight line BC given in poſition and magnitude, and upon BC deſcribe [a] the figure D ſimilar, and ſimilarly placed, to the figure A, and let EF be the ſide of the figure A homologous to BC the ſide of D; therefore the figure D is given in ſpecies. And becauſe upon the given ſtraight line BC the figure D given in ſpecies is deſcribed, D is given [b] in magnitude, and the figure A is given in magnitude, therefore the ratio of A to D is given: And the figure A is ſimilar to D; therefore the ratio of the ſide EF to the homologous ſide BC is given [c]; and BC is given, wherefore [d] EF is given: And the ratio of EF to EG is given [e], therefore EG is given. And, in the ſame manner, each of the other ſides of the figure A can be ſhewn to be given.



[a] 18. 6.
[b] 56. dat.
[c] 58. dat.
[d] 2. dat.
[e] 3. def.

PROBLEM.

To deſcribe a rectilinial figure A ſimilar to a given figure D and equal to another given figure H. It is prop. 25. b. 6. Elem.

Becauſe each of the figures D, H is given, their ratio is given, which may be found by making [f] upon the given ſtraight line BC the parallelogram BK equal to D, and upon its ſide CK making [f] the parallelogram KL equal to H in the angle KCL equal to the angle MBC; therefore the ratio of D to H, that is, of BK to KL, is the ſame with the ratio of BC to CL: And becauſe the figures D, A are ſimilar, and that the ratio of D to A, or H is the ſame with the ratio of BC to CL; by the 58th dat. the ratio of the homologous ſides BC, EF is the ſame with the ratio of BC to the mean proportional between BC and CL. Find EF the mean proportional; then EF is the ſide

[f] cor. 45. 1.

ſide of the figure to be deſcribed, homologous to BC the ſide of D, and the figure itſelf can be deſcribed by the 18th prop. g 2. Cor. 20. 6. B. 6. which, by the conſtruction, is ſimilar to D; and becauſe D is to A, as [g] BC to CL, that is as the figure BK to KL; and h 14. 5. that D is equal to BK, therefore A [h] is equal to KL, that is, to H.

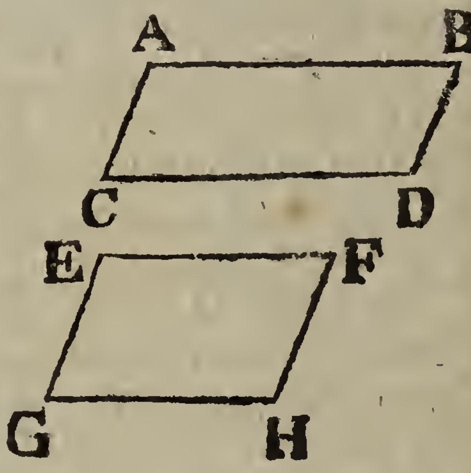
57.

PROP. LXI.

See N. IF a parallelogram given in magnitude has one of its ſides and one of its angles given in magnitude, the other ſide alſo is given.

Let the parallelogram ABDC given in magnitude, have the ſide AB and the angle BAC given in magnitude, the other ſide AC is given.

Take a ſtraight line EF given in poſition and magnitude; and becauſe the parallelogram AD is given in magnitude, a rectilineal a 1. def. figure equal to it can be found [a]. And a parallelogram equal to this b Cor. 45. 1. figure can be applied [b] to the given ſtraight line EF in an angle equal to the given angle BAC. Let this be the parallelogram EFHG having the angle FEG equal to the angle BAC. And becauſe the parallelograms AD, EH are equal, and have the angles at A and E equal; the ſides about them c 14. 6. are reciprocally proportional [c]; therefore as AB to EF, ſo is EG to AC: and AB, EF, EG are given, therefore alſo AC d 12. 6. is given [d]. Whence the way of finding AC is manifeſt.



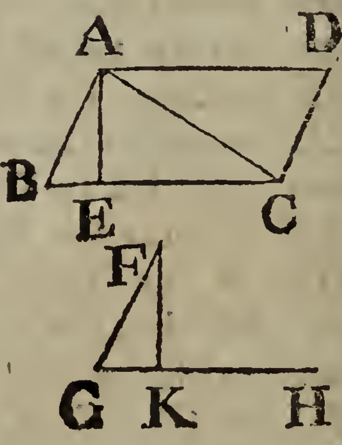
H.

PROP. LXII.

See N. IF a parallelogram has a given angle, the rectangle contained by the ſides about that angle has a given ratio to the parallelogram.

Let the parallelogram ABCD have the given angle ABC, the rectangle AB, BC has a given ratio to the parallelogram AC.

From the point A draw AE perpendicular to BC; becauſe the angle ABC is given, as alſo the angle AEB, the triangle a 43. dat. ABE is given [a] in ſpecies; therefore the ratio of BA to AE is given. But as BA to b 1. 6. AE, ſo is [b] the rectangle AB, BC to the rectangle AE, BC; therefore the ratio of



the rectangle AB, BC to AE, BC that is[c], to the parallelogram AC is given. c 35. 1.

And it is evident how the ratio of the rectangle to the parallelogram may be found by making the angle FGH equal to the given angle ABC, and drawing, from any point F in one of its ſides, FK perpendicular to the other GH; for GF is to FK, as BA to AE, that is, as the rectangle AB, BC, to the parallelogram AC.

Cor. And if a triangle ABC has a given angle ABC, the rectangle AB, BC contained by the ſides about that angle, ſhall have a given ratio to the triangle ABC. 66.

Complete the parallelogram ABCD; therefore, by this propoſition, the rectangle AB, BC has a given ratio to the parallelogram AC; and AC has a given ratio to its half the triangle[d] ABC; therefore the rectangle AB, BC has a given ratio[e] to triangle ABC. d 41. 1. e 9. dat.

And the ratio of the rectangle to the triangle is found thus: Make the triangle FGK, as was ſhown in the propoſition; the ratio of GF to the half of the perpendicular FK is the ſame with the ratio of the rectangle AB, BC to the triangle ABC. Becauſe, as was ſhown, GF is to FK, as AB, BC to the parallelogram AC; and FK is to its half, as AC is to its half, which is the triangle ABC; therefore, ex æquali, GF is to the half of FK, as AB, BC rectangle is to the triangle ABC.

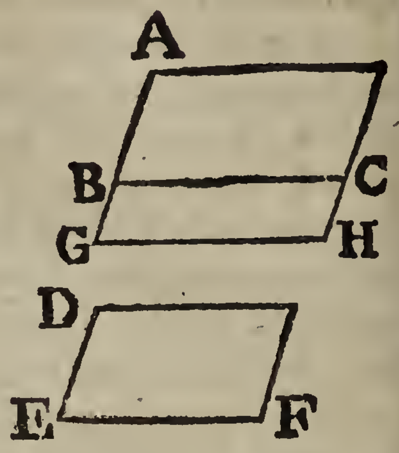
PROP. LXIII. 56.

IF two parallelograms be equiangular, as a ſide of the firſt to a ſide of the ſecond, ſo is the other ſide of the ſecond to the ſtraight line to which the other ſide of the firſt has the ſame ratio which the firſt parallelogram has to the ſecond. And conſequently, if the ratio of the firſt parallelogram to the ſecond be given, the ratio of the other ſide of the firſt to that ſtraight line is given; and if the ratio of the other ſide of the firſt to that ſtraight line be given, the ratio of the firſt parallelogram to the ſecond is given.

Let AC, DF be two equiangular parallelograms, as BC, a ſide of the firſt, is to EF, a ſide of the ſecond, ſo is DE, the other ſide of the ſecond, to the ſtraight line to which AB, the other ſide of the firſt has the ſame ratio which AC has to DF.

Produce the ſtraight line AB, and make as BC to EF, ſo DE to BG, and complete the parallelogram BGHC; therefore, becauſe BC or GH, is to EF, as DE to BG, the ſides about the equal angles BGH, DEF are reciprocally proportional; wherefore [a] the parallelogram BH is equal to DF; and AB is to BG, as the parallelogram AC is to BH, that is, to DF; as therefore BC is to EF, ſo is DE to BG, which is the ſtraight line to which AB has the ſame ratio that AC has to DF.

a 14. 6.



And if the ratio of the parallelogram AC to DF be given, then the ratio of the ſtraight line AB to BG is given; and if the ratio of AB to the ſtraight line BG be given, the ratio of the parallelogram AC to DF is given.

74. 73.
See N.

PROP. LXIV.

IF two parallelograms have unequal, but given angles, and if as a ſide of the firſt to a ſide of the ſecond, ſo the other ſide of the ſecond be made to a certain ſtraight line; if the ratio of the firſt parallelogram to the ſecond be given, the ratio of the other ſide of the firſt to that ſtraight line ſhall be given. And if the ratio of the other ſide of the firſt to that ſtraight line be given, the ratio of the firſt parallelogram to the ſecond ſhall be given.

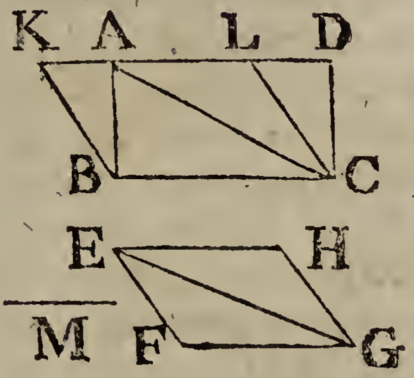
Let ABCD, EFGH be two parallelograms which have the unequal, but given, angles ABC, EFG; and as BC to FG, ſo make EF to the ſtraight line M. If the ratio of the parallelogram AC to EG be given the ratio of AB to M is given.

At the point B of the ſtraight line BC make the angle CBK equal to the angle EFG, and complete the parallelogram KBCL. And becauſe the ratio of AC to EG is given, and that AC is equal [a] to the parallelogram KC, therefore the ratio of KC to EG is given; and KC, EG are equiangular; therefore as BC to FG, ſo is [b] EF to the ſtraight line to which KB has a given ratio, viz. the ſame which the parallelogram KC has to EG; but as BC to FG, ſo is EF to the ſtraight line M; therefore KB has a given ratio to M; and the ratio of

a 35. 1.
b 63. dat.

of AB to BK is given, becauſe the triangle ABK is given in ſpecies[c]; therefore the ratio of AB to M is given[d]. c 43 dat.

d 9. dat. And if the ratio of AB to M be given, the ratio of the parallelogram AC to EG is given; for ſince the ratio of KB to BA is given, as alſo the ratio of AB to M, the ratio of KB to M is given[d]; and becauſe the parallelograms KC, EG are equiangular, as BC to FG, ſo is [b] EF to the ſtraight line to which KB b 63. dat. has the ſame ratio which the parallelogram KC has to EG; but as BC to FG, ſo is EF to M; therefore KB is to M, as the parallelogram KC is to EG; and the ratio of KB to M is given, therefore the ratio of the parallelogram KC, that is, of AC to EG, is given



Cor. And if two triangles ABC, EFG have two equal angles, or two unequal, but given, angles ABC, EFG, and if as BC a ſide of the firſt to FG a ſide of the ſecond, ſo the other ſide of the ſecond EF be made to a ſtraight line M; if the ratio of the triangles be given, the ratio of the other ſide of the firſt to the ſtraight line M is given. 75.

Complete the parallelograms ABCD, EFGH; and becauſe the ratio of the triangle ABC to the triangle EFG is given, the ratio of the parallelogram AC to EG is given[e], becauſe the parallelograms are double[f] of the triangles; and becauſe BC is to FG, as EF to M, the ratio of AB to M is given by the 63d dat. if the angles ABC, EFG are equal; but if they be unequal, but given angles, the ratio of AB to M is given by this propoſition. e 15. 5. f 41. 1.

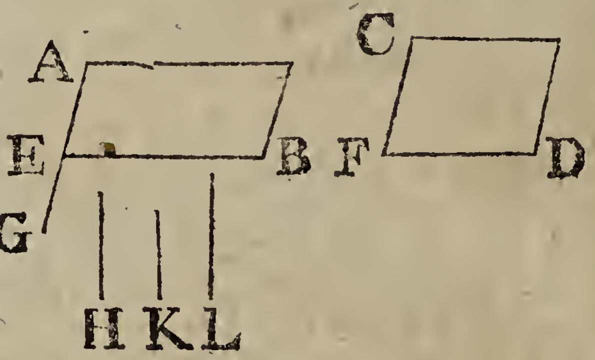
And if the ratio of AB to M be given, the ratio of the parallelogram AC to EG is given by the ſame propoſition; and therefore the ratio of the triangle ABC to EFG is given.

PROP. LXV. 68.

If two equiangular parallelograms have a given ratio to one another, and if one ſide has to one ſide a given ratio; the other ſide ſhall alſo have to the other ſide a given ratio.

Let the two equiangular parallelograms AB, CD have a given ratio to one another, and let the ſide EB have a given ratio to the ſide FD; the other ſide AE has alſo a given ratio to the other ſide CF.

Becauſe the two equiangular parallelograms AB, CD have a given ratio to one another; as EB, a ſide of the firſt, is to FD,
a 63. dat. a ſide of the ſecond, ſo is [a] FC, the other ſide of the ſecond, to the ſtraight line to which AE, the other ſide of the firſt, has the ſame given ratio which the firſt parallelogram AB has to the other CD. Let this ſtraight line be EG; therefore the ratio of AE to EG is given; and EB is to FD, as FC to EG, therefore the ratio of FC to EG is given, becauſe the ratio of EB to FD is given; and becauſe the ratio of AE to EG, as alſo the ratio of FC to EG is given; the
b 9. dat. ratio of AE to CF is given [b].



The ratio of AE to CF may be found thus: Take a ſtraight line H given in magnitude; and becauſe the ratio of the parallelogram AB to CD is given, make the ratio of H to K the ſame with it. And becauſe the ratio of FD to EB is given, make the ratio of K to L the ſame: The ratio of AE to CF is the ſame with the ratio of H to L. Make as EB to FD, ſo FC to EG, therefore, by inverſion, as FD to EB, ſo is EG to FC; and as AE to EG, ſo is [a] (the parallelogram AB to CD, and ſo is) H to K; but as EG to FC, ſo is (FD to EB, and ſo is) K to L; therefore, ex æquali, as AE to FC, ſo is H to L.

69.

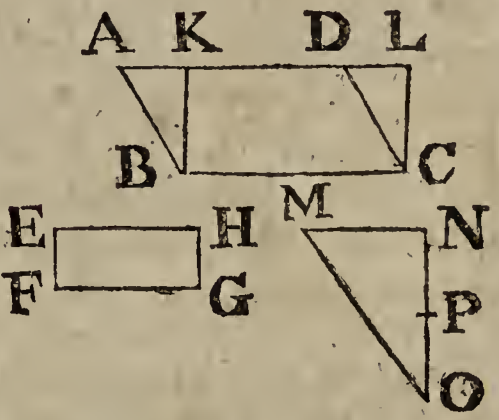
PROP. LXVI.

IF two parallelograms have unequal, but given angles, and a given ratio to one another; if one ſide has to one ſide a given ratio, the other ſide has alſo a given ratio to the other ſide.

Let the two parallelograms ABCD, EFGH which have the given unequal angles ABC, EFG have a given ratio to one another, and let the ratio of BC to FG be given; the ratio alſo of AB to EF is given.

At the point B of the ſtraight line BC make the angle CBK equal to the given angle EFG, and complete the parallelogram BKLC; and becauſe each of the angles BAK, AKB is
a 43. dat. given, the triangle ABK is given [a] in ſpecies; therefore the ratio of AB to BK is given; and becauſe, by the hypotheſis, the

the ratio of the parallelogram AC to EG is given, and that AC is equal[b] to BL; therefore the ratio of BL to EG is given: b 35. 1. And becauſe BL is equiangular to EG, and by the hypotheſis, the ratio of BC to FG is given; therefore[c] the ratio of KB to c 65. dat. EF is given, and the ratio of KB to BA is given; the ratio therefore[d] of AB to EF is given. d 9. dat.



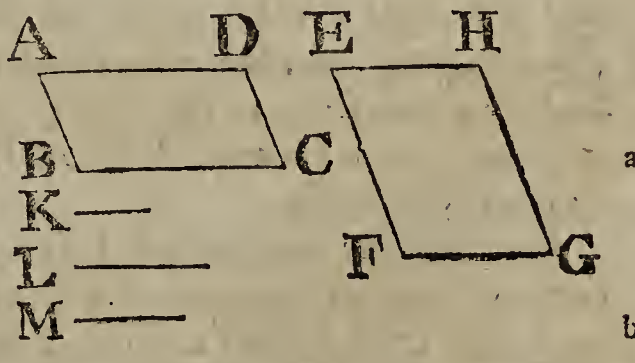
The ratio of AB to EF may be found thus: Take the ſtraight line MN given in poſition and magnitude; and make the angle NMO equal to the given angle BAK, and the angle MNO equal to the given angle EFG or AKB: And becauſe the parallelogram BL is equiangular to EG, and has a given ratio to it, and that the ratio of BC to FG is given; find by the 65th dat. the ratio of KB to EF; and make the ratio of NO to OP the ſame with it: Then the ratio of AB to EF is the ſame with the ratio of MO to OP: For ſince the triangle ABK is equiangular to MON, as AB to BK, ſo is MO to ON: And as KB to EF, ſo is NO to OP; therefore, ex æquali, as AB to EF, ſo is MO to OP.

PROP. LXVII. 70.

IF the ſides of two equiangular parallelograms have See N. given ratios to one another; the parallelograms ſhall have a given ratio to one another.

Let ABCD, EFGH be two equiangular parallelograms, and let the ratio of AB to EF, as alſo the ratio of BC to FG, be given; the ratio of the parallelogram AC to EG is given.

Take a ſtraight line K given in magnitude, and becauſe the ratio of AB to EF is given, make the ratio of K to L the ſame with it; therefore L is given[a]: And becauſe the ratio a 2. dat. of BC to FG is given, make the ratio of L to M the ſame: Therefore M is given[a]: and K is given, wherefore[b]: the b 1. dat. ratio of K to M is given: But the parallelogram AC is to the parallelogram EG, as the ſtraight line K to the ſtraight line M,



as is demonſtrated in the 23d prop. of B. 6. Elem. therefore the ratio of AC to EG is given.

From this it is plain how the ratio of two equiangular parallelograms may be found when the ratios of their ſides are given.

70.

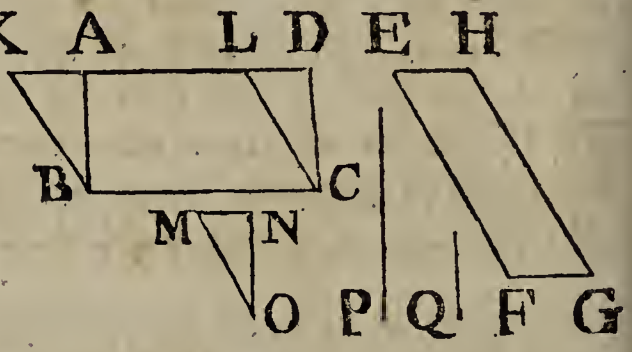
PROP. LXVIII.

See N.

IF the ſides of two parallelograms which have unequal, but given angles, have given ratios to one another; the parallelograms ſhall have a given ratio to one another.

Let two parallelograms ABCD, EFGH which have the given unequal angles ABC, EFG have the ratios of their ſides, viz. of AB to EF, and of BC to FG, given; the ratio of the parallelogram AC to EG is given.

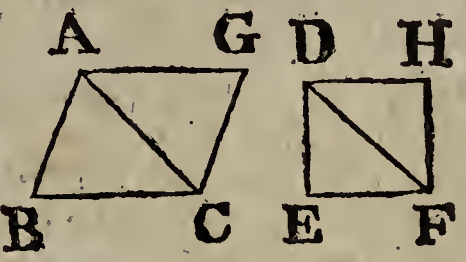
At the point B of the ſtraight line BC make the angle CBK
equal to the given angle EFG, and complete the parallelogram KBCL: And becauſe each of the angles BAK, BKA is
a 43. dat. given, the triangle ABK is given[a] in ſpecies: Therefore the
ratio of AB to BK is given; and the ratio of AB to EF is gi-
b 9. dat. ven, wherefore[b] the ratio of BK to EF is given: And the
ratio of BC to FG is given;
and the angle KBC is equal
to the angle EFG; there-
c 67. dat. fore[c] the ratio of the paral-
lelogram KC to EG is gi-
d 35. 1. ven: But KC is equal[d] to
AC; therefore the ratio of
AC to EG is given.



The ratio of the parallelogram AC to EG may be found thus: Take the ſtraight line MN given in poſition and magnitude, and make the angle MNO equal to the given angle KAB, and the angle NMO equal to the given angle AKB or FEH: And becauſe the ratio of AB to EF is given, make the ratio of NO to P the ſame; alſo make the ratio of P to Q the ſame with the given ratio of BC to FG, the parallelogram AC is to EG, as MO to Q.

Becauſe the angle KAB is equal to the angle MNO, and the angle AKB equal to the angle NMO; the triangle AKB is equiangular to NMO: Therefore as KB to BA ſo is MO to ON; and as BA to EF, ſo is NO to P; wherefore, ex æquali, as KB to EF, ſo is MO to P: And BC is to FG, as P to Q, and the parallelograms KC, EG are equiangular; therefore, as was ſhown in prop. 67. the parallelogram KC, that is, AC, is to EG, as MO to Q.

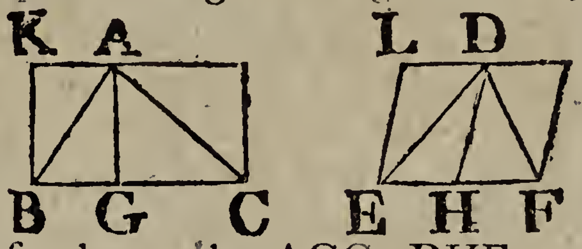
Cor. 1. If two triangles ABC, DEF have two equal angles, 71.
or two unequal, but given angles ABC, DEF, and if the ratios of the ſides about theſe angles, viz. the ratios of AB to DE, and of BC to EF be given; the triangles ſhall have a given ratio to one another.



Complete the parallelograms BG, EH; the ratio of BG to EH is given[a]; and therefore the triangles which are the halves[b] of them have a given[c] ratio to one another.

[a] 67. or 68. dat.
[b] 34. 1.
[c] 15. 5.

Cor. 2. If the baſes BC, EF of two triangles ABC, DEF have a given ratio to one another, and if alſo the ſtraight lines AG, 72.
DH which are drawn to the baſes from the oppoſite angles, either in equal angles, or unequal, but given angles AGC, DHF have a given ratio to one another; the triangles ſhall have a given ratio to one another.



Draw BK, EL parallel to AG, DH, and complete the parallelograms KC, LF. And becauſe the angles AGC, DHF, or their equals, the angles KBC, LEF are either equal, or unequal, but given; and that the ratio of AG to DH, that is, of KB to LE, is given, as alſo the ratio of BC to EF; therefore[a] the ratio of the parallelogram KC to LF is given; wherefore alſo the ratio of the triangle ABC to DEF is given[b].

[a] 67. or 68. dat.
[b] 41. 1. 15. 5.

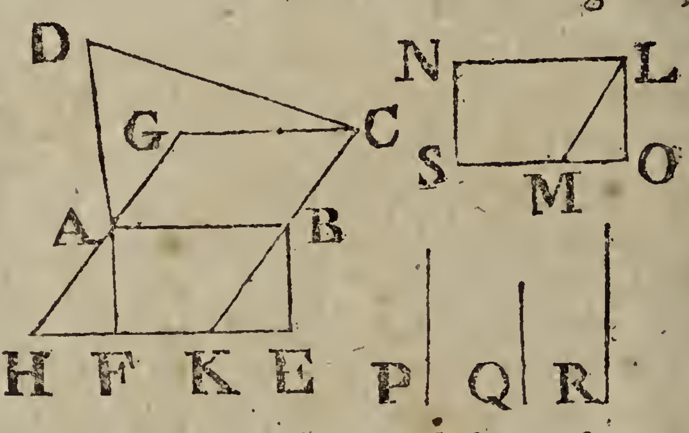
PROP. LXIX. 61.

IF a parallelogram which has a given angle be applied to one ſide of a rectilineal figure given in ſpecies; if the figure have a given ratio to the parallelogram, the parallelogram is given in ſpecies.

Let ABCD be a rectilineal figure given in ſpecies, and to one ſide of it AB, let the parallelogram ABEF having the given angle ABE be applied; if the figure ABCD has a given ratio to the parallelogram BF, the parallelogram BF is given in ſpecies.

Through the point A draw AG parallel to BC, and through the point C draw CG parallel to AB, and produce GA, CB to

a 3. def. the points H, K; becauſe the angle ABC is given [a], and the
ratio of AB to BC is given, the figure ABCD being given in
ſpecies; therefore, the parallelogram BG is given [a] in ſpecies.
And becauſe upon the ſame ſtraight line AB the two rectilineal
figures BD, BG given in ſpecies are deſcribed, the ratio of
b 53. dat BD to BG is given [b]; and, by hypotheſis, the ratio of
c 9 dat BD to the parallelogram BF is given; wherefore [c] the ratio of
d 35. 1. BF, that is [d], of the parallelogram BH, to BG is given, and
e 1. 6. therefore [e] the ratio of the ſtraight line KB to BC is given;
and the ratio of BC to BA is given, wherefore the ratio of
KB to BA is given [c]: And becauſe the angle ABC is given, the
adjacent angle ABK is given; and the angle ABE is given,
therefore the remaining angle KBE is given. The angle EKB
is alſo given, becauſe it is equal to the angle ABK; therefore
the triangle BKE is given in ſpecies, and conſequently the ratio of EB to BK is given; and the ratio of KB to BA is given,
wherefore [c] the ratio of EB to BA is given; and the angle ABE is given, therefore the parallelogram BF is given in ſpecies.



A parallelogram ſimilar to BF may be found thus: Take a ſtraight line LM given in poſition and magnitude; and becauſe the angles ABK, ABE are given, make the angle NLM equal to ABK, and the angle NLO equal to ABE. And becauſe the ratio of BF to BD is given, make the ratio of LM to P the ſame with it; and becauſe the ratio of the figure BD to BG is given, find this ratio by the 53d dat. and make the ratio of P to Q the ſame. Alſo, becauſe the ratio of CB to BA is given, make the ratio of Q to R the ſame; and take LN equal to R; through the point M draw OM parallel to LN and complete the parallelogram NLOS; then this is ſimilar to the parallelogram BF.

Becauſe the angle ABK is equal to NLM, and the angle ABE to NLO, the angle KBE is equal to MLO; and the angles BKE, LMO are equal, becauſe the angle ABK is equal to NLM; therefore the triangles BKE, LMO are equiangular to one another; wherefore as BE to BK, ſo is LO to LM; and becauſe as the figure BF to BD, ſo is the ſtraight line LM to P; and as BD to BG, ſo is P to Q; ex æquali, as BF, that is [d] BH to BG, ſo is LM to Q: But BH is to [e]

BG, as KB to BC; as therefore KB to BC, ſo is LM to Q; and becauſe BE is to BK as LO to LM; and as BK to BC, ſo is LM to Q: And as BC to BA, ſo Q was made to R; therefore, ex æquali, as BE to BA, ſo is LO to R, that is to LN; and the angles ABE, NLO are equal; therefore the parallelogram BF is ſimilar to LS.

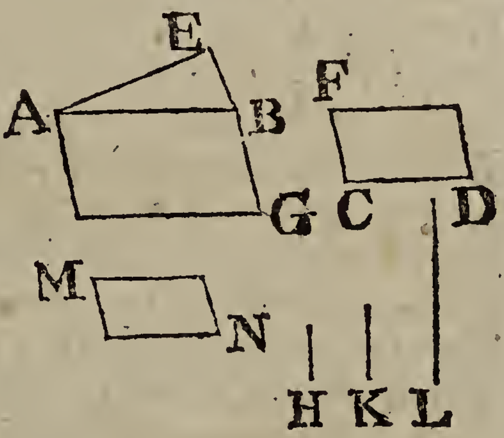
PROP. LXX.

62. 78.

IF two ſtraight lines have a given ratio to one another, and upon one of them be deſcribed a rectilineal figure given in ſpecies, and upon the other a parallelogram having a given angle; if the figure have a given ratio to the parallelogram, the parallelogram is given in ſpecies. See N.

Let the two ſtraight lines AB, CD have a given ratio to one another, and upon AB let the figure AEB given in ſpecies be deſcribed, and upon CD the parallelogram DF having the given angle FCD; if the ratio of AEB to DF be given, the parallelogram DF is given in ſpecies.

Upon the ſtraight line AB, conceive the parallelogram AG to be deſcribed ſimilar, and ſimilarly placed to FD; and becauſe the ratio of AB to CD is given, and upon them are deſcribed the ſimilar rectilineal figures AG, FD; the ratio of AG to FD is given[a]; and the ratio of FD to AEB is given; therefore[b] the ratio of AEB to AG is given; and the angle ABG is given, becauſe it is equal to the angle FCD; becauſe therefore the parallelogram AG which has a given angle ABG is applied to a ſide AB of the figure AEB given in ſpecies, and the ratio of AEB to AG is given, the parallelogram AG is given[c] in ſpecies; but FD is ſimilar to AG; therefore FD is given in ſpecies.



a 54. dat.
b 9. dat.
c 69. dat.

A parallelogram ſimilar to FD may be found thus: Take a ſtraight line H given in magnitude; and becauſe the ratio of the figure AEB to FD is given, make the ratio of H to K the ſame with it: Alſo, becauſe the ratio of the ſtraight line CD to AB is given, find by the 54th dat. the ratio which the figure FD deſcribed upon CD has to the figure AG deſcribed upon AB ſimilar to FD; and make the ratio of K to L the ſame with this ratio: And becauſe the ratios of H to K, and of K to b 9. dat. to L are given, the ratio of H to L is given[b]; becauſe, therefore, as AEB to FD, ſo is H to K; and as FD to AG, ſo is K to L; ex æquali, as AEB to AG ſo is H to L; therefore the ratio of AEB to AG, is given; and the figure AEB is given in ſpecies, and to its ſide AB the parallelogram AG is applied in the given angle ABG; therefore by the 69th dat. a parallelogram may be found ſimilar to AG: Let this be the parallelogram MN; MN alſo is ſimilar to FD; for, by the conſtruction, MN is ſimilar to AG, and AG is ſimilar to FD; therefore the parallelogram FD is ſimilar to MN.

81.

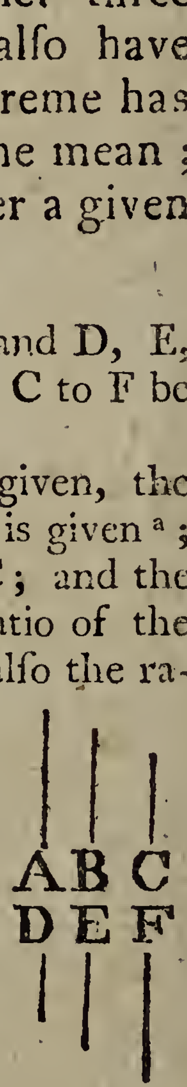
PROP. LXXI.

IF the extremes of three proportional ſtraight lines have given ratios to the extremes of other three proportional ſtraight lines; the means ſhall alſo have a given ratio to one another: And if one extreme has a given ratio to one extreme, and the mean to the mean; likewiſe the other extreme ſhall have to the other a given ratio.

Let A, B, C be three proportional ſtraight lines, and D, E, F, three other; and let the ratios of A to D, and of C to F be given; then the ratio of B to E is alſo given.

Becauſe the ratio of A to D, as alſo of C to F is given, the ratio of the rectangle A, C to the rectangle D, F is given[a]; but the ſquare of B is equal[b] to the rectangle A, C; and the ſquare of E to the rectangle[b] D, F; therefore the ratio of the ſquare of B to the ſquare of E is given; wherefore[c] alſo the ratio of the ſtraight line B to E is given.

a 67. dat.
b 17. 6.
c 58. dat.



Next, let the ratio of A to D, and of B to E be given; then the ratio of C to F is alſo given.

Becauſe the ratio of B to E is given, the ratio of the ſquare of B to the ſquare of E is given[d]; therefore[b] the ratio of the rectangle A, C to the rectangle D, F is given; and the ratio of the ſide A to the ſide D is given; therefore the ratio of the other ſide C to the other F is given[e].

d 54. dat.
e 65. dat.

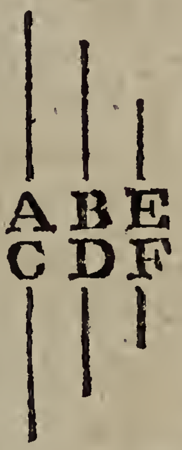
COR. And if the extremes of four proportionals have to the extremes of four other proportionals given ratios, and one of the means a given ratio to one of the means; the other mean ſhall have a given ratio to the other mean, as may be ſhown in the ſame manner as in the foregoing propoſition.

PROP. LXXII. 82.

IF four ſtraight lines be proportionals; as the firſt is to the ſtraight line to which the ſecond has a given ratio, ſo is the third to a ſtraight line to which the fourth has a given ratio.

Let A, B, C, D be four proportional ſtraight lines, viz. as A to B, ſo C to D; as A is to the ſtraight line to which B has a given ratio, ſo is C to a ſtraight line to which D has a given ratio.

Let E be the ſtraight line to which B has a given ratio, and as B to E, ſo make D to F: The ratio of B to E is given[a], and therefore the ratio of D to F; and becauſe as A to B, ſo is C to D; and as B to E ſo D to F; therefore, ex æquali, as A to E, ſo is C to F; and E is the ſtraight line to which B has a given ratio, and F that to which D has a given ratio; therefore as A is to the ſtraight line to which B has a given ratio, ſo is C to a line to which D has a given ratio.



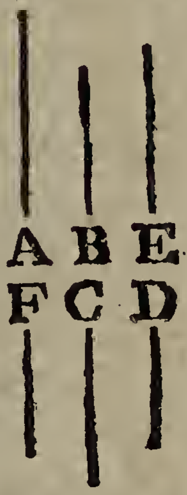
[a] Hyp.

PROP. LXXIII. 83.

IF four ſtraight lines be proportionals; as the firſt is to the ſtraight line to which the ſecond has a given ratio, ſo is a ſtraight line to which the third has a given ratio to the fourth. See N.

Let the ſtraight line A be to B, as C to D; as A to the ſtraight line to which B has a given ratio, ſo is a ſtraight line to which C has a given ratio to D.

Let E be the ſtraight line to which B has a given ratio, and as B to E, ſo make F to C; becauſe the ratio of B to E is given, the ratio of C to F is given: And becauſe A is to B, as C to D; and as B to E, ſo F to C; therefore, ex æquali in proportione perturbata[a], A is to E, as F to D; that is, A is to E to which B has a given ratio, as F, to which C has a given ratio, is to D.



[a] 23. 5.

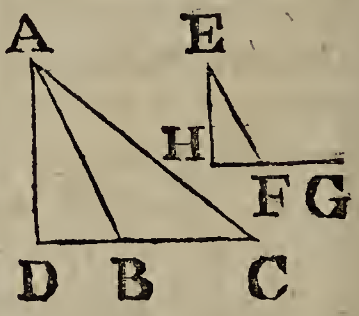
PROP.

64. PROP. LXXIV.

IF a triangle has a given obtuſe angle; the exceſs of the ſquare of the ſide which ſubtends the obtuſe angle, above the ſquares of the ſides which contain it, ſhall have a given ratio to the triangle.

Let the triangle ABC have a given obtuſe angle ABC; and
produce the ſtraight line CB, and from the point A draw AD
perpendicular to BC: The exceſs of the ſquare of AC above
a 12. 2. the ſquares of AB, BC, that is [a], the double of the rectangle
contained by DB, BC, has a given ratio to the triangle
ABC.

Becauſe the angle ABC is given, the angle ABD is alſo given;
and the angle ADB is given; wherefore the triangle ABD
b 43. dat. is given [b] in ſpecies; and therefore the ratio of AD to DB
c 1. 6. is given: And as AD to DB, ſo is [c] the rectangle AD, BC
to the rectangle DB, BC; wherefore the ratio of the rectangle AD, BC to the rectangle DB, BC is given, as alſo the
ratio of twice the rectangle DB, BC to
the rectangle AD, BC: But the ratio of
the rectangle AD, BC to the triangle ABC
d 41. 1. is given, becauſe it is double [d] of the triangle; therefore the ratio of twice the
rectangle DB, BC to the triangle ABC is
e 9. dat. given [e]; and twice the rectangle DB, BC
is the exceſs [a] of the ſquare of AC above the ſquares of AB,
BC; therefore this exceſs has a given ratio to the triangle ABC,



And the ratio of this exceſs to the triangle ABC may be found thus: Take a ſtraight line EF given in poſition and magnitude; and becauſe the angle ABC is given, at the point F of the ſtraight line EF, make the angle EFG equal to the angle ABC; produce GF, and draw EH perpendicular to FG; then the ratio of the exceſs of the ſquare of AC above the ſquares of AB, BC to the triangle ABC, is the ſame with the ratio of quadruple the ſtraight line HF to HE.

Becauſe the angle ABD is equal to the angle EFH, and
the angle ADB to EHF, each being a right angle; the tri-
f 4. 6. angle ADB is equiangular to EHF; therefore [f] as BD to DA,
g Cor. 4. 5. ſo FH to HE; and as quadruple of BD to DA, ſo is [g] qua-
druple of FH to HE: But as twice BD is to DA, ſo is [c] twice
the rectangle DB, BC to the rectangle AD, BC; and as DA
h C. 5. to the half of it, ſo is [h] the rectangle AD, BC to its half the
triangle

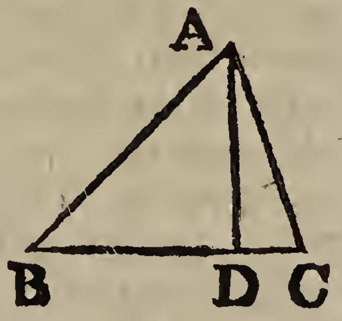
triangle ABC; therefore, ex æquali, as twice BD is to the half of DA, that is, as quadruple of BD is to DA, that is, as quadruple of FH to HE, ſo is twice the rectangle DB, BC to the triangle ABC.

PROP. LXXV.

65.

IF a triangle has a given acute angle, the ſpace by which the ſquare of the ſide ſubtending the acute angle is leſs than the ſquares of the ſides which contain it, ſhall have a given ratio to the triangle.

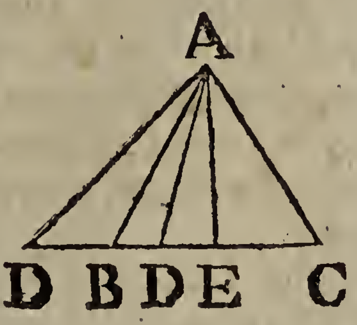
Let the triangle ABC have a given acute angle ABC, and draw AD perpendicular to BC, the ſpace by which the ſquare of AC is leſs than the ſquares of AB, BC, that is [a], the double of the rectangle contained by CB, BD, has a given ratio to the triangle ABC. a 13. 2.



Becauſe the angles ABD, ADB are each of them given, the triangle ABD is given in ſpecies; and therefore the ratio of BD DA is given: And as BD to DA, ſo is the rectangle CB, BD to the rectangle CB, AD: therefore the ratio of theſe rectangles is given, as alſo the ratio of twice the rectangle CB, BD to the rectangle CB, AD, but the rectangle CB, AD has a given ratio to its half the triangle ABC: therefore [b] the ratio of twice the rectangle CB, BD to the triangle ABC is given; and twice the rectangle CB, BD is [a] the ſpace by which the ſquare of AC is leſs than the ſquares of AB, BC; therefore the ratio of this ſpace to the triangle ABC is given: And the ratio may be found as in the preceding propoſition. b 9. dat.

LEMMA.

IF from the vertex A of an iſoſceles triangle ABC, any ſtraight line AD be drawn to the baſe BC, the ſquare of the ſide AB is equal to the rectangle BD, DC of the ſegments of the baſe together with the ſquare of AD; but if AD be drawn to the baſe produced, the ſquare of AD is equal to the rectangle BD, DC together with the ſquare of AB.



Cas. 1. Biſect the baſe BC in E, and join AE which will be perpendicular [a] to BC; wherefore the ſquare of AB is equal [b] to the ſquares of AE, EB; but the ſquare of EB is equal [c] to the rectangle BD, DC together with the ſquare of DE; therefore the ſquare of AB is equal to the

a 8. 1.

b 47. 1.

c 5. 2.

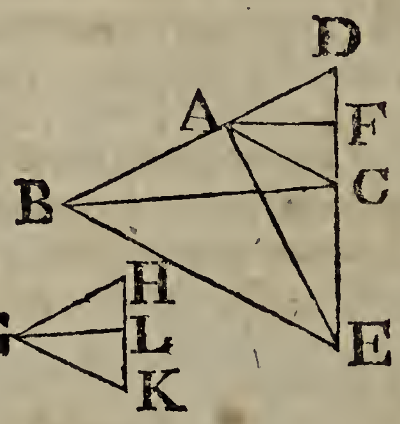
ſquares

b 47. 1. ſquares of AE, ED, that is, to[b] the ſquare of AD, together with the rectangle BD, DC; the other caſe is ſhown in the ſame way by 6. 2. Elem.

67. PROP. LXXVI.

IF a triangle have a given angle, the exceſs of the ſquare of the ſtraight line which is equal to the two ſides that contain the given angle, above the ſquare of the third ſide, ſhall have a given ratio to the triangle.

Let the triangle ABC have the given angle BAC, the exceſs of the ſquare of the ſtraight line which is equal to BA, AC together above the ſquare of BC, ſhall have a given ratio to the triangle ABC.



Produce BA, and take AD equal to AC, join DC and produce it to E, and through the point B draw BE parallel to AC; join AE, and draw AF perpendicular to DC; and becauſe AD is equal to AC, BD is equal to BE; and BC is drawn from the vertex B of the iſoſceles triangle DBE, therefore, by the Lemma, the ſquare of BD, that is, of BA and AC together, is equal to the rectangle DC, CE together with the ſquare of BC; and, therefore, the ſquare of BA, AC together, that is, of BD, is greater than the ſquare of BC by the rectangle DC, CE; and this rectangle has a given ratio to the triangle ABC; becauſe the angle BAC is given, the adjacent angle CAD is given; and each of the angles ADC, DCA is given, for
a 5. & 32. 1. each of them is the half[a] of the given angle BAC; therefore the triangle
b 43. dat. ADC is given[b] in ſpecies; and AF is drawn from its vertex to the baſe in a given angle; wherefore the ratio of AF to the baſe CD is
c 50. dat. given[c]; and as CD to AF, ſo is[d] the rectangle DC, CE to
d 1. 6. the rectangle AF, CE; and the ratio of the rectangle AF,
e 41. 1. CE to its half[e]; the triangle ACE is given; therefore the ratio
f 37. 1. of the rectangle DC, CE to the triangle ACE, that is[f], to
g 9. dat. the triangle ABC, is given[g]: and the rectangle DC, CE is the exceſs of the ſquare of BA, AC together above the ſquare of BC: therefore the ratio of this exceſs to the triangle ABC is given.

The ratio which the rectangle DC, CE has to the triangle ABC is found thus: Take the ſtraight line GH given in poſition

tion and magnitude, and at the point G in GH make the angle HGK equal to the given angle CAD, and take GK equal to GH, join KH, and draw GL perpendicular to it: Then the ratio of HK to the half of GL is the ſame with the ratio of the rectangle DC, CE to the triangle ABC: Becauſe the angles HGK, DAC at the vertices of the iſoſceles triangles GHK, ADC are equal to one another, theſe triangles are ſimilar; and becauſe GL, AF are perpendicular to the baſes HK, DC, as HK to GL, ſo is[h] (DC to AF, and ſo is) the rectangle DC, CE to the rectangle AF, CE; but as GL to its half, ſo is the rectangle AF, CE to its half, which is the triangle ACE, or the triangle ABC; therefore, ex æquali, HK is to the half of the ſtraight line GL, as the rectangle DC, CE is to the triangle ABC.

h 4. 6. 22. 5.

Cor. And if a triangle have a given angle, the ſpace by which the ſquare of the ſtraight line which is the difference of the ſides which contain the given angle is leſs than the ſquare of the third ſide, ſhall have a given ratio to the triangle. This is demonſtrated the ſame way as the preceding propoſition, by help of the ſecond caſe of the Lemma.

PROP. LXXVII.

L.

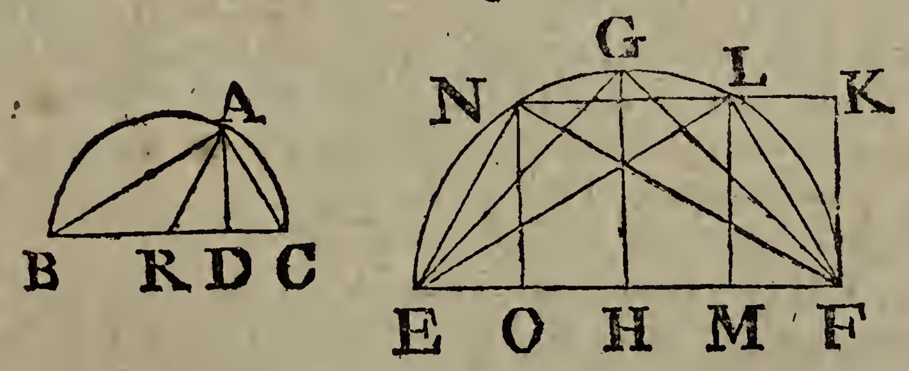
IF the perpendicular drawn from a given angle of a triangle to the oppoſite ſide, or baſe, has a given ratio to the baſe, the triangle is given in ſpecies.

See N.

Let the triangle ABC have the given angle BAC, and let the perpendicular AD drawn to the baſe BC, have a given ratio to it, the triangle ABC is given in ſpecies.

If ABC be an iſoſceles triangle, it is evident[a] that if any

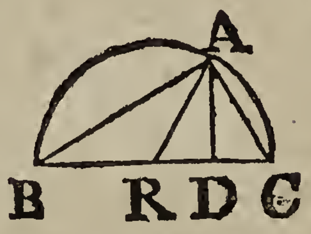
a 5. & 32, 1.

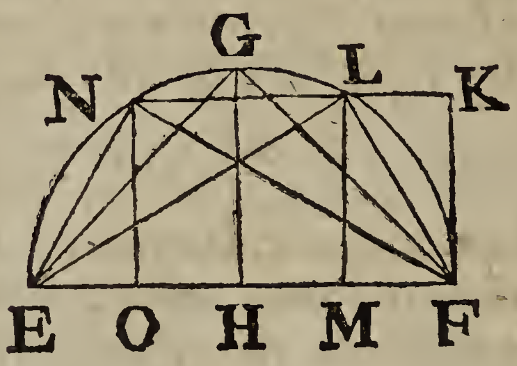


one of its angles be given, the reſt are alſo given; and therefore the triangle is given in ſpecies, without the conſideration of the ratio of the perpendicular to the baſe, which in this caſe is given by prop. 50.

But when ABC is not an iſoſceles triangle, take any ſtraight line EF given in poſition and magnitude, and upon it deſcribe

the ſegment of a circle EGF containing an angle equal to the given angle BAC, draw GH biſecting EF at right angles, and join EG, GF: Then, ſince the angle EGF is equal to the angle BAC, and that EGF is an iſoſceles triangle, and ABC is not, the angle FEG is not equal to the angle CBA: Draw EL making the angle FEL equal to the angle CBA; join FL, and draw LM perpendicular to EF; then, becauſe the triangles ELF BAC are equiangular, as alſo are the triangles MLE, DAB, as ML to LE, ſo is DA to AB; and as LE to EF, ſo is AB to BC; wherefore, ex æquali, as LM to EF, ſo is AD to BC; and becauſe the ratio of AD to BC is given, therefore the ratio
[b] 2. dat. of LM to EF is given; and EF is given, wherefore [b] LM alſo is given. Complete the parallelogram LMFK; and becauſe LM is given, FK is given in magnitude; it is alſo given in poſition,
[c] 30. dat. and the point F is given, and conſequently [c] the point K; and becauſe through K the ſtraight line KL is drawn parallel to EF
[d] 31. dat. which is given in poſition, therefore [d] KL is given in poſition:





and the circumference ELF is given in poſition; therefore the
[e] 28. dat. point L is given [e]. And becauſe the points L, E, F, are given,
[f] 29. dat. the ſtraight lines LE, EF, FL, are given [f] in magnitude; there-
[g] 42. dat. fore the triangle LEF is given in ſpecies [g]; and the triangle ABC is ſimilar to LEF, wherefore alſo ABC is given in ſpecies.

Becauſe LM is leſs than GH, the ratio of LM to EF, that is, the given ratio of AD to BC, muſt be leſs than the ratio of GH to EF, which the ſtraight line, in a ſegment of a circle containing an angle equal to the given angle, that biſects the baſe of the ſegment at right angles, has unto the baſe.

Cor. 1. If two triangles, ABC, LEF, have one angle BAC equal to one angle ELF, and if the perpendicular AD be to the baſe BC, as the perpendicular LM to the baſe EF, the triangles ABC, LEF are ſimilar.

Deſcribe the circle EGF about the triangle ELF, and draw LN parallel to EF, join EN, NF, and draw NO perpendicular to EF; becauſe the angles ENF, ELF are equal, and that the angle EFN is equal to the alternate angle FNL, that is, to the angle FEL in the ſame ſegment; therefore the triangle NEF is ſimilar to LEF; and in the ſegment EGF there can be no other triangle upon the baſe EF, which has the ratio of its perpendicular to that baſe the ſame with the ratio of LM or NO to EF, becauſe the perpendicular muſt be greater or leſs than LM or NO; but, as has been ſhewn in the preceding demonſtration, a triangle, ſimilar to ABC, can be deſcribed in the ſegment EGF upon the baſe EF, and the ratio of its perpendicular to the baſe is the ſame, as was there ſhewn, with the ratio of AD to BC, that is, of LM to EF; therefore that triangle muſt be either LEF, or NEF, which therefore are ſimilar to the triangle ABC.

Cor. 2. If a triangle ABC has a given angle BAC, and if the ſtraight line AR drawn from the given angle to the oppoſite ſide BC, in a given angle ARC, has a given ratio to BC, the triangle ABC is given in ſpecies.

Draw AD perpendicular to BC; therefore the triangle ARD is given in ſpecies; wherefore the ratio of AD to AR is given: and the ratio of AR to BC is given, and conſequently[h] the ratio of AD to BC is given; and the triangle ABC is therefore given in ſpecies[i].

Cor. 3. If two triangles ABC, LEF have one angle BAC equal to one angle ELF, and if ſtraight lines drawn from theſe angles to the baſes, making with them given and equal angles, have the ſame ratio to the baſes, each to each; then the triangles are ſimilar; for having drawn perpendiculars to the baſes from the equal angles, as one perpendicular is to its baſe, ſo is the other to its baſe[k]; wherefore, by Cor. 1. the triangles are ſimilar.

A triangle ſimilar to ABC may be found thus: Having deſcribed the ſegment EGF and drawn the ſtraight line GH as was directed in the propoſition, find FK which has to EF the given ratio of AD to BC; and place FK at right angles to EF from the point F; then becauſe, as has been ſhewn, the ratio of AD to BC, that is of FK to EF, muſt be leſs than the ratio of GH to EF; therefore FK is leſs than GH; and conſequently the parallel to EF drawn through the point K, muſt meet the circumference of the ſegment in two points: Let L be either of them, and join EL, LF, and draw LM perpendicular to EF: then, becauſe the angle BAC is equal to the angle ELF, and that AD is to BC, as KF, that is, LM to EF, the triangle ABC is ſimilar to the triangle LEF, by Cor. 1.

PROP.

[h] 9. dat.

[i] 77. dat.

[k] 4. 6.
22. 5.

80.

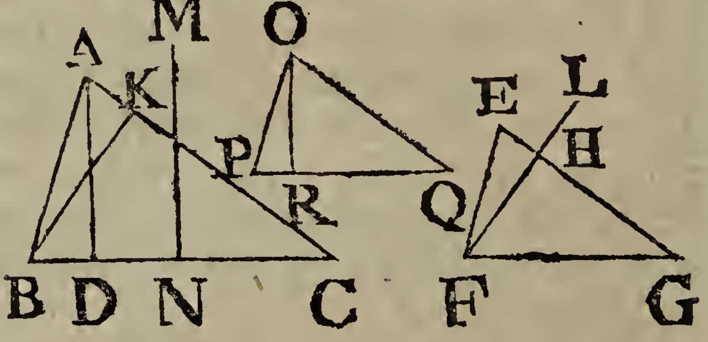
PROP. LXXVIII.

IF a triangle have one angle given, and if the ratio of the rectangle of the ſides which contain the given angle to the ſquare of the third ſide be given, the triangle is given in ſpecies.

Let the triangle ABC have the given angle BAC, and let the ratio of the rectangle BA, AC to the ſquare of BC be given; the triangle ABC is given in ſpecies.

From the point A, draw AD perpendicular to BC, the rect-
a 41. 1. angle AD, BC has a given ratio to its half[a] the triangle ABC;
and becauſe the angle BAC is given, the ratio of the triangle
b Cor. 62. dat. BAC to the rectangle BA, AC is given[b]; and by the hypo-
theſis, the ratio of the rectangle BA, AC to the ſquare of BC is
c 9. dat. given; therefore[c] the ratio of the rectangle AD, BC to the
d 1. 6. ſquare of BC, that is[d], the ratio of the ſtraight line AD to BC,
e 77. dat. is given; wherefore the triangle ABC is given in ſpecies[e].

A triangle ſimilar to ABC may be found thus: Take a ſtraight line EF given in poſition and magnitude, and make the angle FEG equal to the given angle BAC, and draw FH perpendicular to EG, and BK perpendicular to AC; therefore the triangles ABK, EFH are ſimilar, and the rectangle AD, BC, or the rectangle BK, AC which is equal to it, is to the rectangle BA, BC as the ſtraight line BK to BA, that is, as FH to FE. Let the given ratio of the rectangle BA, AC to the ſquare of BC be the ſame with the ratio of the ſtraight line EF to FL; therefore, ex æquali, the ratio of the rectangle AD, BC to the ſquare of BC, that is, the ratio of the ſtraight line AD to BC, is the ſame with the ratio of HF to FL; and becauſe AD is not greater than the ſtraight line MN in the ſegment of the circle deſcribed about the triangle ABC, which biſects BC at right angles; the ratio of AD to BC, that is, of HF to FL, muſt not be greater than the ratio of MN to BC: Let it be ſo, and, by the 77th dat. find a triangle OPQ which has one of its angles POQ equal to the given angle BAC, and the ratio of the perpendicular OR, drawn from that angle to the baſe PQ the ſame with the ratio of HF to FL; then the triangle ABC is ſimilar to



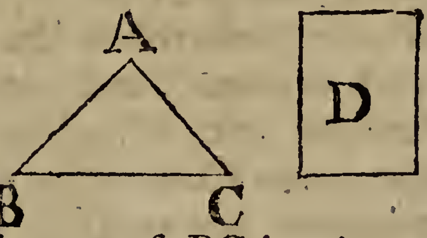
OPQ: Becauſe, as has been ſhown, the ratio of AD to BC is the ſame with the ratio of (HF to FL, that is, by the conſtruction, with the ratio of) OR to PQ; and the angle BAC is equal to the angle POQ. Therefore the triangle ABC is ſimilar[f] to the triangle POQ.

f 1. Cor. 77. dat.

Otherwiſe,

Let the triangle ABC have the given angle BAC, and let the ratio of the rectangle BA, AC to the ſquare of BC be given; the triangle ABC is given in ſpecies.

Becauſe the angle BAC is given, the exceſs of the ſquare of both the ſides BA, AC together above the ſquare of the third ſide BC has a given[a] ratio to the triangle ABC. Let the figure D be equal to this exceſs; therefore the ratio of D to the triangle ABC is given; and the ratio of the triangle ABC to the rectangle BA, AC is given[b], becauſe BAC is a given angle; and the rectangle BA, AC has a given ratio to the ſquare of BC; wherefore[c] the ratio of D to the ſquare of BC is given; and, by compoſition[d], the ratio of the ſpace D together with the ſquare of BC to the ſquare of BC is given; but D together with the ſquare of BC is equal to the ſquare of both BA and AC together; therefore the ratio of the ſquare of BA, AC together to the ſquare of BC is given; and the ratio of BA, AC together to BC is therefore given[e]; and the angle BAC is given, wherefore[f] the triangle ABC is given in ſpecies.



a 76. dat.
b Cor. 62. dat.
c 10. dat.
d 7. dat.
e 59. dat.
f 48. dat.

The compoſition of this, which depends upon thoſe of the 76th and 48th propoſitions, is more complex than the preceding compoſition, which depends upon that of prop. 77. which is eaſy.

PROP. LXXIX.

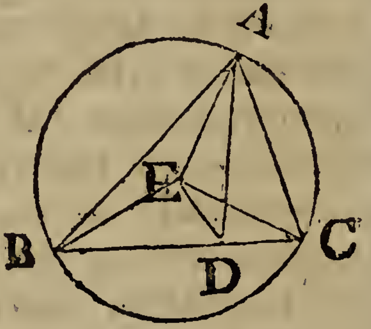
K.

IF a triangle have a given angle, and if the ſtraight line drawn from that angle to the baſe, making a given angle with it, divides the baſe into ſegments which have a given ratio to one another; the triangle is given in ſpecies.

See N.

Let the triangle ABC have the given angle BAC, and let the ſtraight line AD drawn to the baſe BC making the given angle ADB, divide CB into the ſegments BD, DC which have

a given ratio to one another; the triangle ABC is given in ſpecies.
a 5. 4. Deſcribe[a] the circle BAC about the triangle, and from its
centre E, draw EA, EB, EC, ED; becauſe the angle BAC is
b 20. 3. given, the angle BEC at the centre, which is the double[b] of it,
is given. And the ratio of BE to EC is given, becauſe they
c 44. dat. are equal to one another; therefore[c] the triangle BEC is
given in ſpecies, and the ratio of EB to BC is given; alſo the
d 7. dat. ratio of CB to BD is given[d], becauſe the ratio of BD to DC
e 9. dat. is given; therefore the ratio of EB to BD is given[e], and the
angle EBC is given, wherefore the triangle EBD is given[e]
in ſpecies, and the ratio of EB, that is, of EA, to ED, is therefore
given; and the angle EDA is given, becauſe each of the
angles BDE, BDA is given; therefore the triangle AED is
f 47. dat. given[f] in ſpecies, and the angle AED given:
alſo the angle DEC is given, becauſe
each of the angles BED, BEC is
given; therefore the angle AEC is given,
and the ratio of EA to EC, which are
equal, is given; and the triangle AEC is
therefore given[c] in ſpecies, and the angle
ECA given; and the angle ECB is given,
wherefore the angle ACB is given, and the angle BAC is alſo
g 43. dat. given; therefore[g] the triangle ABC is given in ſpecies.



A triangle ſimilar to ABC may be found, by taking a ſtraight line given in poſition and magnitude, and dividing it in the given ratio which the ſegments BD, DC are required to have to one another; then, if upon that ſtraight line a ſegment of a circle be deſcribed containing an angle equal to the given angle BAC, and a ſtraight line be drawn from the point of diviſion in an angle equal to the given angle ADB, and from the point where it meets the circumference, ſtraight lines be drawn to the extremity of the firſt line, theſe, together with the firſt line, ſhall contain a triangle ſimilar to ABC, as may eaſily be ſhown.

The demonſtration may be alſo made in the manner of that of the 77th prop. and that of the 77th may be made in the manner of this.

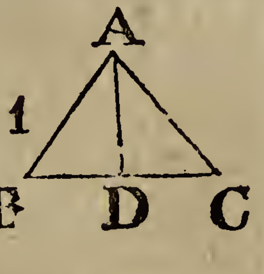
L.

PROP. LXXX.

IF the ſides about an angle of a triangle have a given ratio to one another, and if the perpendicular drawn from that angle to the baſe has a given ratio to the baſe; the triangle is given in ſpecies.

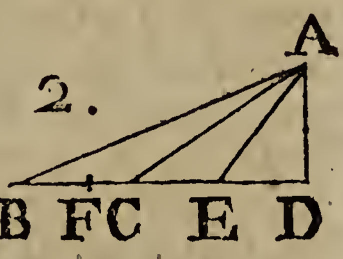
Let the ſides BA, AC, about the angle BAC of the triangle ABC have a given ratio to one another, and let the perpendicular AD have a given ratio to the baſe BC; the triangle ABC is given in ſpecies.

Firſt, let the ſides AB, AC be equal to one another, therefore the perpendicular AD biſects[a] the baſe BC; and the ratio of AD to BC, and therefore to its half DB, is given; and the angle ADB is given; wherefore the triangle * ABD, and conſequently the triangle ABC, is given[b] in ſpecies.



[a] 26. 1.
* 43. dat.
[b] 44. dat.

But let the ſides be unequal, and BA be greater than AC; and make the angle CAE equal to the angle ABC; becauſe the angle AEB is common to the triangles AEB, CEA, they are ſimilar; therefore as AB to BE, ſo is CA to AE, and, by permutation, as BA to AC, ſo is BE to EA, and ſo is EA to EC; and the ratio of BA to AC is given, therefore the ratio of BE to EA, and the ratio of EA to EC, as alſo the ratio of BE to EC is given[c]; wherefore the ratio of EB to BC is given[d]; and the ratio of AD to BC is given by the hypotheſis, therefore[c] the ratio of AD to BE is given; and the ratio of BE to EA was ſhown to be given; wherefore the ratio of AD to AE is given, and ADE is a right angle, therefore the triangle ADE is given[e] in ſpecies, and the angle AEB given; the ratio of BE to EA is likewiſe given, therefore[b] the triangle ABE is given in ſpecies, and conſequently the angle EAB, as alſo the angle ABE, that is, the angle CAE, is given; therefore the angle BAC is given, and the angle ABC being alſo given, the triangle ABC is given[f] in ſpecies.



[c] 9. dat.
[d] 6. dat.
[e] 46. dat.
[f] 43. dat.

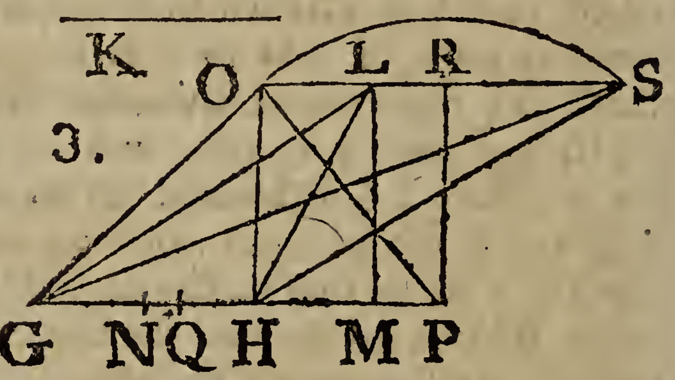
How to find a triangle which ſhall have the things which are mentioned to be given in the propoſition, is evident in the firſt caſe; and to find it the more eaſily in the other caſe, it is to be obſerved that, if the ſtraight line EF equal to EA be placed in EB towards B, the point F divides the baſe BC into the ſegments BF, FC which have to one another the ratio of the ſides BA, AC, becauſe BE, EA, or EF, and EC were ſhown to be proportionals, therefore * BF is to FC, as BE to EF, or EA, that is, as BA to AC; and AE cannot be leſs than the altitude of the triangle ABC, but it may be

* 19. 5.

equal to it, which, if it be, the triangle, in this caſe, as alſo the ratio of the ſides, may be thus found: Having given the ratio of the perpendicular to the baſe, take the ſtraight line GH, given in poſition and magnitude, for the baſe of the triangle to be found; and let the given ratio of the perpendicular to the baſe be that of the ſtraight line K to GH, that is, let K be equal to the perpendicular; and ſuppoſe GLH to be the triangle which is to be found, therefore having made the angle HLM equal to LGH, it is required that LM be perpendicular to GM, and equal to K; and becauſe GM, ML, MH are proportionals, as was ſhown of BE, EA, EC, the rectangle GMH is equal to the ſquare of ML. Add the common ſquare of NH, (having biſected GH in N), and the ſquare of NM is equal[g] to the ſquares of the given ſtraight lines NH and ML or K; therefore the ſquare of NM and its ſide NM, is given, as alſo the point M, viz. by taking the ſtraight line NM, the ſquare of which is equal to the ſquares of NH, ML. Draw ML equal to K, at right angles to GM; and becauſe ML is given in poſition and magnitude, therefore the point L is given, join LG, LH; then the triangle LGH is that which was to be found, for the ſquare of NM is equal to the ſquares of NH and ML, and taking away the common ſquare of NH, the rectangle GMH is equal[g] to the ſquare of ML: therefore as GM to ML, ſo is ML to MH, and the triangle LGM is[h] therefore equiangular to HLM, and the angle HLM equal to the angle LGM, and the ſtraight line LM, drawn from the vertex of the triangle making the angle HLM equal to LGH, is perpendicular to the baſe and equal to the given ſtraight line K, as was required; and the ratio of the ſides GL, LH is the ſame with the ratio of GM to ML, that is, with the ratio of the ſtraight line which is made up of GN the half of the given baſe and of NM, the ſquare of which is equal to the ſquares of GN and K, to the ſtraight line K.

g 6. 2.

h 6. 6.



And whether this ratio of GM to ML is greater or leſs than the ratio of the ſides of any other triangle upon the baſe GH, and of which the altitude is equal to the ſtraight line K, that is, the vertex of which is in the parallel to GH drawn through the point L, may be thus found. Let OGH be any ſuch triangle, and draw OP, making the angle HOP equal to the angle OGH; therefore as before, GP, PO, PH are proportionals, and PO cannot be equal to LM, becauſe the rectangle GPH would be equal to the rectangle GMH, which is impoſſible; for the point P cannot fall upon M, becauſe O would then fall on L; nor can PO be leſs than LM, therefore it is greater; and conſequently the rectangle GPH is greater than the rectangle GMH, and the ſtraight line GP greater than GM: Therefore the ratio of GM to MH is greater than the ratio of GP to PH, and the ratio of the ſquare of GM to the ſquare of ML is therefore [i] greater than the ratio of the ſquare of GP to the ſquare of PO, and the ratio of the ſtraight line GM to ML greater than the ratio of GP to PO. But as GM to ML, ſo is GL to LH; and as GP to PO, ſo is GO to OH; therefore the ratio of GL to LH is greater than the ratio of GO to OH; wherefore the ratio of GL to LH is the greateſt of all others; and conſequently the given ratio of the greater ſide to the leſs muſt not be greater than this ratio.

But if the ratio of the ſides be not the ſame with this greateſt ratio of GM to LM, it muſt neceſſarily be leſs than it: Let any leſs ratio be given, and the ſame things being ſuppoſed, viz. that GH is the baſe, and K equal to the altitude of the triangle, it may be found as follows. Divide GH in the point Q, ſo that the ratio of GQ to QH may be the ſame with the given ratio of the ſides; and as GQ to QH, ſo make GP to PQ, and ſo will [f] PQ be to PH; wherefore the ſquare of GP is to the ſquare of PQ, as [i] the ſtraight line GP to PH: And becauſe GM, ML MH are proportionals, the ſquare of GM is to the ſquare of ML, as [i] the ſtraight line GM to MH: But the ratio of GQ to QH, that is, the ratio of GP to PQ, is leſs than the ratio of GM to ML; and therefore the ratio of the ſquare of GP to the ſquare of PQ is leſs than the ratio of the ſquare of GM to that of ML; and conſequently the ratio of the ſtraight line GP to PH is leſs than the ratio of GM to MH; and, by diviſion, the ratio of GH to HP is leſs than that of GH to HM; wherefore [k] the ſtraight line HP is greater than HM, and the rectangle GPH, that is, the ſquare of PQ, greater than the rectangle GMH, that is, than the

[i] 2. Cor. 20. 6.

[f] 19. 5.

[k] 10. 5.

ſquare of ML, and the ſtraight line PQ is therefore greater than ML. Draw LR parallel to GP, and from P draw PR at right angles to GP: Becauſe PQ is greater than ML, or PR, the circle deſcribed from the centre P, at the diſtance PQ, muſt neceſſarily cut LR in two points; let theſe be O, S, and join OG, OH; SG, SH: each of the triangles OGH, SGH have the things mentioned to be given in the propoſition: Join OP, SP; and becauſe as GP to PQ, or PO, ſo is PO to PH, the triangle OGP is equiangular to HOP; as, therefore, OG to GP, ſo is HO to OP; and, by permutation, as GO to OH, ſo is GP to PO, or PQ: and ſo is GQ to QH: Therefore the triangle OGH has the ratio of its ſides GO, OH the ſame with the given ratio of GQ to QH: and the perpendicular has to the baſe the given ratio of K to GH, becauſe the perpendicular is equal to LM, or K: The like may be ſhewn in the ſame way of the triangle SGH.

This conſtruction by which the triangle OGH is found, is ſhorter than that which would be deduced from the demonſtration of the datum, by reaſon that the baſe GH is given in poſition and magnitude, which was not ſuppoſed in the demonſtration: The ſame thing is to be obſerved in the next propoſition.

M.

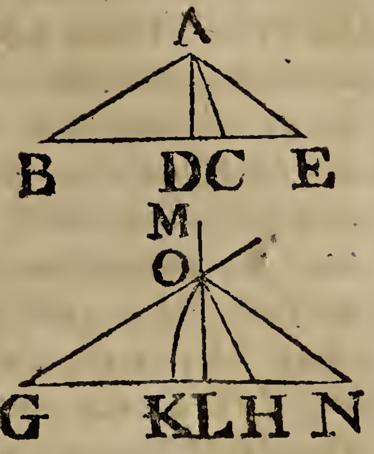
PROP. LXXXI.

IF the ſides about an angle of a triangle be unequal and have a given ratio to one another, and if the perpendicular from that angle to the baſe divides it into ſegments that have a given ratio to one another, the triangle is given in ſpecies.

Let ABC be a triangle, the ſides of which about the angle BAC are unequal and have a given ratio to one another, and let the perpendicular AD to the baſe BC divide it into the ſegments BD, DC which have a given ratio to one another, the triangle ABC is given in ſpecies.

Let AB be greater than AC, and make the angle CAE equal to the angle ABC; and becauſe the angle AEB is common to the triangles ABE, CAE, they are [a] equiangular to one another: Therefore as AB to BE, ſo is CA to AE, and, by

a 4. 6.



by permutation, as AB to AC, ſo BE to EA, and ſo is EA to EC: But the ratio of BA to AC is given, therefore the ratio of BE to EA, as alſo the ratio of EA to EC is given; wherefore[b] the ratio of BE to EC, as alſo[c] the ratio of EC to CB is given: And the ratio of BC to CD is given[d], becauſe the ratio of BD to DC is given; therefore[b] the ratio of EC to CD is given, and conſequently[d] the ratio of DE to EC: And the ratio of EC to EA was ſhown to be given, therefore[b] the ratio of DE to EA is given: And ADE is a right angle, wherefore[e] the triangle ADE is given in ſpecies, and the angle AED given: And the ratio of CE to EA is given, therefore[f] the triangle AEC is given in ſpecies, and conſequently the angle ACE is given, as alſo the adjacent angle ACB. In the ſame manner, becauſe the ratio of BE to EA is given, the triangle BEA is given in ſpecies, and the angle ABE is therefore given: And the angle ACB is given; wherefore the triangle ABC is given[g] in ſpecies.

b 9. dat.
c Cor. 6. dat.
d 7. dat.
e 46. dat.
f 44. dat.
g 43. dat.

But the ratio of the greater ſide BA to the other AC muſt be leſs than the ratio of the greater ſegment BD to DC: Becauſe the ſquare of BA is to the ſquare of AC, as the ſquares of BD, DA to the ſquares of DC, DA; and the ſquares of BD, DA have to the ſquares of DC, DA a leſs ratio than the ſquare of BD has to the ſquare of DC †, becauſe the ſquare of BD is greater than the ſquare of DC; therefore the ſquare of BA has to the ſquare of AC a leſs ratio than the ſquare of BD has to that of DC: And conſequently the ratio of BA to AC is leſs than the ratio of BD to DC.

This being premiſed, a triangle which ſhall have the things mentioned to be given in the propoſition, and to which the triangle ABC is ſimilar, may be found thus: Take a ſtraight line GH given in poſition and magnitude, and divide it in K, ſo that the ratio of GK to KH may be the ſame with the given ratio of BA to AC: Divide alſo GH in L, ſo that the ratio

---

† If A be greater than B, and C any third magnitude; then A and C together have to B and C together a leſs ratio than A has to B.

Let A be to B as C to D, and becauſe A is greater than B, C is greater than D: But as A is to B, ſo A and C to B and D; and A and C have to B and C a leſs ratio than A and C have to B and D, becauſe C is greater than D, therefore A and C have to B and C a leſs ratio than A to B.

of GL to LH may be the ſame with the given ratio of BD to DC, and draw LM at right angles to GH: And becauſe the ratio of the ſides of a triangle is leſs than the ratio of the ſegments of the baſe, as has been ſhewn, the ratio of GK to KH is leſs than the ratio of GL to LH; wherefore the point L muſt fall betwixt K and H: Alſo make as GK to KH, ſo GN
h 19. 5. to NK, and ſo ſhall [h] NK be to NH. And from the centre N, at the diſtance NK, deſcribe a circle, and let its circumference meet LM in O, and join OG, OH; then OGH is the triangle which was to be deſcribed: Becauſe GN is to NK, or NO, as NO to NH, the triangle OGN is equiangular to HON; therefore as OG to GN, ſo is HO to ON, and, by permutation, as GO to OH, ſo is GN to NO, or NK, that is, as GK to KH, that is, in the given ratio of the ſides, and by the conſtruction, GL, LH have to one another the given ratio of the ſegments of the baſe.

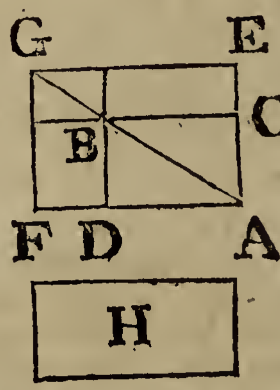
69.

PROP. LXXXII.

IF a parallelogram given in ſpecies and magnitude be increaſed or diminiſhed by a gnomon given in magnitude, the ſides of the gnomon are given in magnitude.

Firſt, let the parallelogram AB given in ſpecies and magnitude be increaſed by the given gnomon ECBDFG, each of the ſtraight lines CE, DF is given.

Becauſe AB is given in ſpecies and magnitude, and that the gnomon ECBDFG is given, therefore the whole ſpace AG is given in magnitude: But AG is alſo given in ſpecies, be-
a { 2. def. 2. and 24. 6. cauſe it is ſimilar [a] to AB; therefore the ſides of AG are given [b]: Each of the ſtraight lines AE, AF is therefore given; and each of the ſtraight
b 60. dat. lines CA, AD is given [b], therefore each of
c 4 dat. the remainders EC, DF is given [c].



Next let the parallelogram AG given in ſpecies and magnitude, be diminiſhed by the given gnomon ECBDFG, each of the ſtraight lines CE, DF is given.

Becauſe the parallelogram AG is given, as alſo its gnomon ECBDFG, the remaining ſpace AB is given in magnitude;

magnitude: But it is alſo given in ſpecies; becauſe it is ſimilar[a] to AG; therefore[b] its ſides CA, AD are given, and each of the ſtraight lines EA AF is given; therefore EC, DF are each of them given.

a { 2. def. 2 and 24 6.
b 60. dat.

The gnomon and its ſides CE, DF may be found thus in the firſt caſe. Let H be the given ſpace to which the gnomon muſt be made equal, and find[d] a parallelogram ſimilar to AB and equal to the figures AB and H together, and place its ſides AE, AF from the point A, upon the ſtraight lines AC, AD, and complete the parallelogram AG which is about the ſame diameter[c] with AB; becauſe therefore AG is equal to both AB and H, take away the common part AB, the remaining gnomon ECBDFG is equal to the remaining figure H; therefore a gnomon equal to H, and its ſides CE, DF are found: And in like manner they may be found in the other caſe, in which the given figure H muſt be leſs than the figure FE from which it is to be taken.

d 25. 6.
c 26. 6.

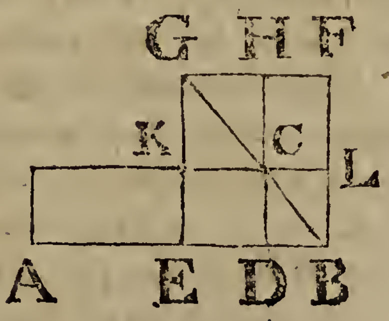
PROP. LXXXIII.

58.

If a parallelogram equal to a given ſpace be applied to a given ſtraight line, deficient by a parallelogram given in ſpecies, the ſides of the defect are given.

Let the parallelogram AC equal to a given ſpace be applied to the given ſtraight line AB, deficient by the parallelogram BDCL given in ſpecies, each of the ſtraight lines CD, DB are given.

Biſect AB in E; therefore EB is given in magnitude, upon EB deſcribe[a] the parallelogram EF ſimilar to DL and ſimilarly placed; therefore EF is given in ſpecies, and is about the ſame diameter[b] with DL; let BCG be the diameter, and conſtruct the figure; therefore, becauſe the figure EF given in ſpecies is deſcribed upon the given ſtraight line EB, EF is given[c] in magnitude, and the gnomon ELH is equal[d] to the given figure AC: therefore[e] ſince EF is diminiſhed by the given gnomon ELH, the ſides EK, FH of the gnomon are given; but EK is equal to DC, and FH to DB; wherefore CD, DB are each of them given.



a 18. 6.
b 26. 6.
c 56. dat.
d 36. and 43. 1.
e 82. dat.

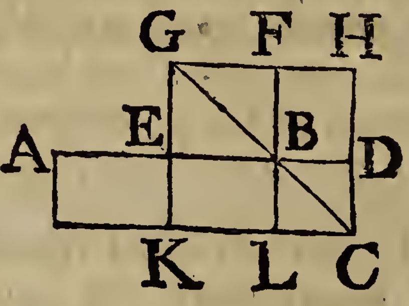
This demonſtration is the analyſis of the problem in the 28th prop. of book 6. the conſtruction and demonſtration of which propoſition is the compoſition of the analyſis; and becauſe the given ſpace AC or its equal the gnomon ELH is to be taken from the figure EF deſcribed upon the half of AB ſimilar to BC, therefore AC muſt not be greater than EF, as is ſhewn in the 27th prop. B. 6.

59. PROP. LXXXIV.

IF a parallelogram equal to a given ſpace be applied to a given ſtraight line, exceeding by a parallelogram given in ſpecies; the ſides of the exceſs are given.

Let the parallelogram AC equal to a given ſpace be applied to the given ſtraight line AB, exceeding by the parallelogram BDCL given in ſpecies; each of the ſtraight lines CD, DB are given.

Biſect AB in E; therefore EB is given in magnitude: Upon
a 18. 6. BE deſcribe[a] the parallelogram EF ſimilar to LD, and ſimilarly placed; therefore EF is given in ſpecies, and is about the
b 26. 6. ſame diameter[b] with LD. Let CBG be the diameter and conſtruct the figure: Therefore, becauſe the figure EF given in ſpecies is deſcribed upon the given ſtraight line EB, EF is given in magni-
c 56. dat. tude[c], and the gnomon ELH is equal
d 36. and 43. 1. to the given figure[d] AC; wherefore, ſince EF is increaſed by the given gnomon ELH, its ſides EK,
e 82. dat. FH are given[e]; but EK is equal to CD, and FH to BD; therefore CD, DB are each of them given.

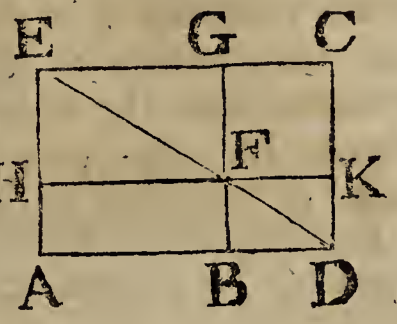


This demonſtration is the analyſis of the problem in the 29th prop. book 6. the conſtruction and demonſtration of which is the compoſition of the analyſis.

Cor. If a parallelogram given in ſpecies be applied to a given ſtraight line, exceeding by a parallelogram equal to a given ſpace; the ſides of the parallelogram are given.

Let the parallelogram ADCE given in ſpecies be applied to the given ſtraight line AB exceeding by the parallelogram BDCG equal to a given ſpace; the ſides AD, DC of the parallelogram are given.

Draw the diameter DE of the parallelogram AC, and conſtruct the figure. Becauſe the parallelogram AK is equal [a] to a 43. 1. BC which is given, therefore AK is given; and BK is ſimilar [b] to AC, there- b 24. 6. fore BK is given in ſpecies. And ſince the parallelogram AK given in magnitude is applied to the given ſtraight line AB, exceeding by the parallelogram BK given in ſpecies, therefore by this propoſition, BD, DK the ſides of the exceſs are given, and the ſtraight line AB is given; therefore the whole AD, as alſo DC, to which it has a given ratio is given.



P R O B.

To apply a parallelogram ſimilar to a given one to a given ſtraight line AB, exceeding by a parallelogram equal to a given ſpace.

To the given ſtraight line AB apply [c] the parallelogram AK c 29. 6. equal to the given ſpace, exceeding by the parallelogram BK ſimilar to the one given. Draw DF the diameter of BK, and through the point A draw AE parallel to BF meeting DF produced in E, and complete the parallelogram AC.

The parallelogram BC is equal [a] to AK, that is, to the given ſpace; and the parallelogram AC is ſimilar [b] to BK; therefore the parallelogram AC is applied to the ſtraight line AB ſimilar to the one given and exceeding by the parallelogram BC which is equal to the given ſpace.

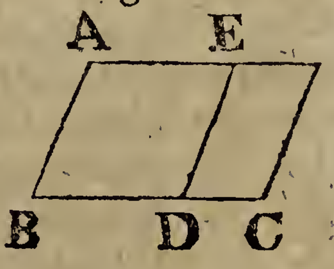
P R O P. LXXXV.

84.

IF two ſtraight lines contain a parallelogram given in magnitude, in a given angle; if the difference of the ſtraight lines be given, they ſhall each of them be given.

Let AB, BC contain the parallelogram AC given in magnitude, in the given angle ABC, and let the exceſs of BC above AB be given; each of the ſtraight lines AB, BC is given.

Let DC be the given exceſs of BC above BA, therefore the remainder BD is equal to BA. Complete the parallelogram AD; and becauſe AB is equal to BD, the ratio of AB to BD is given; and the angle ABD is given, therefore the parallelogram AD is given in ſpecies; and becauſe the given parallelogram AC is applied to the given ſtraight line DC, exceeding by the parallelogram AD given in ſpecies, the ſides of the exceſs are given [a]: a 84. dat. therefore



therefore BD is given; and DC is given, wherefore the whole BC is given: And AB is given, therefore AB, BC are each of them given.

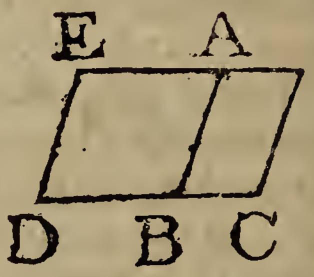
85.

PROP. LXXXVI.

IF two ſtraight lines contain a parallelogram given in magnitude, in a given angle; if both of them together be given, they ſhall each of them be given.

Let the two ſtraight lines AB, BC contain the parallelogram AC given in magnitude, in the given angle ABC, and let AB, BC together be given; each of the ſtraight lines AB, BC is given.

Produce CB, and make DB equal to BA, and complete the parallelogram ABDE. Becauſe DB is equal to BA, and the angle ABD given, becauſe the adjacent angle ABC is given, the parallelogram AD is given in ſpecies: And becauſe AB, BC together are given, and AB is equal to BD; therefore DC is given: And becauſe the given parallelogram AC is applied to the given ſtraight line DC, deficient by the parallelogram AD given in ſpecies, the ſides AB, BD of the defect are given[a]; and DC is given, wherefore the remainder BC is given; and each of the ſtraight lines AB, BC is therefore given.



a 83. dat.

87.

PROP. LXXXVII.

IF two ſtraight lines contain a parallelogram given in magnitude, in a given angle; if the exceſs of the ſquare of the greater above the ſquare of the leſſer be given, each of the ſtraight lines ſhall be given.

Let the two ſtraight lines AB, BC contain the given parallelogram AC in the given angle ABC; if the exceſs of the ſquare of BC above the ſquare of BA be given; AB and BC are each of them given.

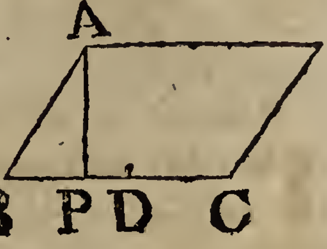
Let the given exceſs of the ſquare of BC above the ſquare of BA be the rectangle CB, BD; take this from the ſquare of BC, the remainder, which is[a] the rectangle BC, CD is equal to the ſquare of AB; and becauſe the angle ABC of the parallelogram AC is given, the ratio of the rectangle of the ſides AB, BC to the parallelogram AC is given[b]; and AC is given, therefore the rectangle AB, BC is given; and the rectangle CB, BD is given; therefore the ratio of the rect-

angle

a 2. 2.

b 62. dat.

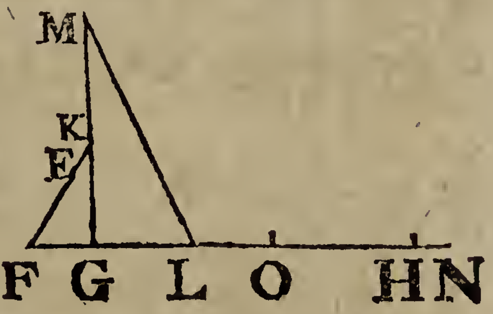
angle CB, BD to the rectangle AB, BC, that is [c], the ratio of the c 1. 6.
ſtraight line DB to BA is given; therefore [d] the ratio of the d 54. dat.
ſquare of DB to the ſquare of BA is given: And the ſquare of BA is equal to
the rectangle BC, CD: wherefore the ratio of the rectangle BC, CD to the ſquare
of BD is given, as alſo the ratio of four times the rectangle BC, CD to the ſquare
of BD; and, by compoſition [e], the ratio of four times the rect- e 7. dat.
angle BC, CD together with the ſquare of BC to the ſquare
of BD is given: But four times the rectangle BC, CD toge-
ther with the ſquare of BD is equal [f] to the ſquare of the ſtraight f 8. 2.
lines BC, CD taken together: therefore the ratio of the ſquare
of BC, CD together to the ſquare of BD is given; wherefore
[g] the ratio of the ſtraight line BC together with CD to BD is g 58. dat.
given: And, by compoſition, the ratio of BC together with
CD and DB, that is, the ratio of twice BC to BD, is given;
therefore the ratio of BC to BD is given, as alſo [c] the ratio of
the ſquare of BC to the rectangle CB, BD: But the rectangle
CB, BD is given, being the given exceſs of the ſquares of BC,
BA; therefore the ſquare of BC, and the ſtraight line BC is
given: And the ratio of BC to BD, as alſo of BD to BA has
been ſhewn to be given; therefore [h] the ratio of BC to BA is h 9. dat.
given; and BC is given, wherefore BA is given.



The preceding demonſtration is the analyſis of this problem, viz.

A parallelogram AC which has a given angle ABC being given in magnitude, and the exceſs of the ſquare of BC one of its ſides above the ſquare of the other BA being given; to find the ſides: And the compoſition is as follows.

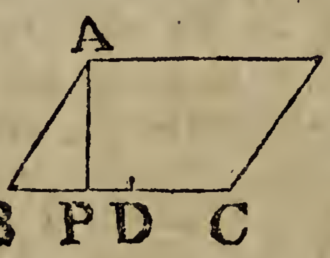
Let EFG be the given angle to which the angle ABC is required to be equal, and from any point E in FE, draw EG perpendicular to FG; let the rectangle EG, GH be the given ſpace to which the parallelogram AC is to be made equal; and the rectangle HG, GL, be the given exceſs of the ſquares of BC, BA.

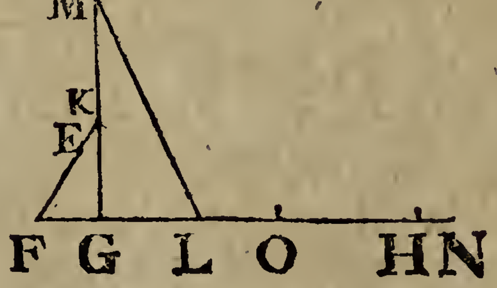


Take, in the ſtraight line GE, GK equal to FE, and make GM double of GK; join ML, and in GL produced, take LN equal to LM: Biſect GN in O, and between GH, GO find a mean proportional BC: As OG to GL, ſo make CB to BD; and make the angle CBA equal to GFE, and as LG to GK ſo make DB to BA; and complete the parallelogram AC: AC is equal to the rectangle EG, GH, and the exceſs of the ſquares of CB, BA is equal to the rectangle HG, GL.

Becauſe as CB to BD, ſo is OG to GL, the ſquare of CB is to the rectangle CB, BD as[a] the rectangle HG, GO to the rectangle HG, GL: And the ſquare of CB is equal to the rectangle, HG, GO, becauſe GO, BC, GH are proportionals; therefore the rectangle CB, BD is equal[b] to HG, GL. And becauſe as CB to BD, ſo is OG to GL; twice CB is to BD, as twice OG, that is, GN to GL; and, by diviſion, as BC together with CD is to BD, ſo is NL, that is, LM, to LG: Therefore[c] the ſquare of BC together with CD is to the ſquare of BD, as the ſquare of ML to the ſquare of LG: But the ſquare of BC and CD together is equal[d] to four times the rectangle BC, CD together with the ſquare of BD; therefore four times the rectangle BC, CD together with the ſquare of BD is to the ſquare of BD, as the ſquare of ML to the ſquare of LG: And, by diviſion, four times the rectangle BC, CD is to the ſquare of BD, as the ſquare of MG to the ſquare of GL; wherefore the rectangle BC, CD is to the ſquare of BD as (the ſquare of KG the half of MG to the ſquare of GL, that is, as) the ſquare of AB to the ſquare of BD, becauſe as LG to GK, ſo DB was made to BA: Therefore[b] the rectangle BC, CD is equal to the ſquare of AB. To each of theſe add the rectangle CB, BD, and the ſquare of BC becomes equal to the ſquare of AB together with the rectangle CB, BD; therefore this rectangle, that is, the given rectangle HG, GL is the exceſs of the ſquares of BC, AB. From the point A, draw AP perpendicular to BC, and becauſe the angle ABP is equal to the angle EFG, the triangle ABP is equiangular to EFG: And DB was made to BA, as LG to GK; therefore as the rectangle CB, BD to CB, BA, ſo is the rectangle HG,

[a] 1. 6.
[b] 14. 5.
[c] 22. 6.
[d] 8. 2.





GL to HG, GK; and as the rectangle CB, BA to AP, BC, ſo is (the ſtraight line BA to AP, and ſo is FE or GK to

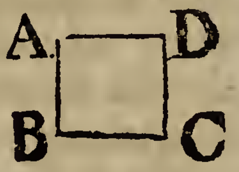
EG, and ſo is) the rectangle HG, GK to HG, GE; therefore, ex æquali, as the rectangle CB, BD to AP, BC, ſo is the rectangle HG, GL to EG, GH: And the rectangle CB, BD is equal to HG, GL; therefore the rectangle AP, BC, that is, the parallelogram AC, is equal to the given rectangle EG, GH.

PROP. LXXXVIII.

N.

IF two ſtraight lines contain a parallelogram given in magnitude, in a given angle; if the ſum of the ſquares of its ſides be given, the ſides ſhall each of them be given.

Let the two ſtraight lines AB, BC contain the parallelogram ABCD given in magnitude in the given angle ABC, and let the ſum of the ſquares of AB, BC be given; AB, BC are each of them given.

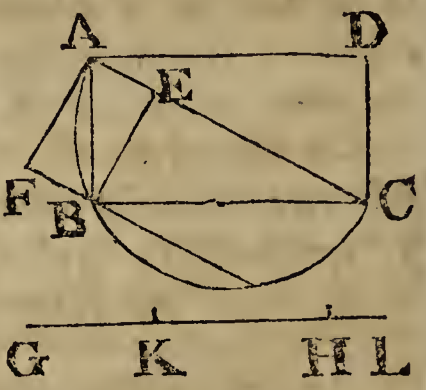


Firſt, let ABC be a right angle; and becauſe twice the rectangle contained by two equal ſtraight lines is equal to both their ſquares; but if two ſtraight lines are unequal, twice the rectangle contained by them is leſs than the ſum of their ſquares, as is evident from the 7th prop. B. 2. Elem; therefore twice the given ſpace, to which ſpace the rectangle of which the ſides are to be found is equal, muſt not be greater than the given ſum of the ſquares of the ſides: And if twice that ſpace be equal to the given ſum of the ſquares, the ſides of the rectangle muſt neceſſarily be equal to one another: Therefore in this caſe deſcribe a ſquare ABCD equal to the given rectangle, and its ſides AB, BC are thoſe which were to be found: For the rectangle AC is equal to the given ſpace, and the ſum of the ſquares of its ſides AB, BC is equal to twice the rectangle AC, that is by the hypotheſis, to the given ſpace to which the ſum of the ſquares was required to be equal.

But if twice the given rectangle be not equal to the given ſum of the ſquares of the ſides, it muſt be leſs than it, as has been ſhown. Let ABCD be the rectangle, join AC and draw BE perpendicular to it, and complete the rectangle AEBF, and deſcribe the circle ABC about the triangle ABC; AC is its diameter[a]: And becauſe the triangle ABC is ſimilar[b] to AEB, as AC to CB ſo is AB to BE; therefore the rectangle AC, BE is equal to AB, BC; and the rectangle AB, BC

[a] Cor. 5. 4.

[b] 8. 6.



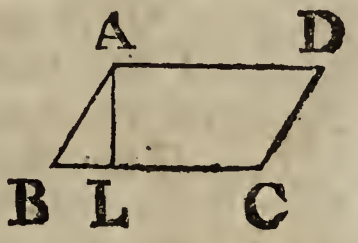
BC is given, wherefore AC, BE is given: And becauſe the ſum
of the ſquares of AB, BC is given, the ſquare of AC which is
c 47. 1. equal[c] to that ſum is given; and AC itſelf is therefore given
in magnitude: Let AC be likewiſe given in poſition, and the
d 32. dat. point A; therefore AF is given[d] in poſition:
And the rectangle AC, BE is
given, as has been ſhewn, and AC is
e 61. dat. given, wherefore[e] BE is given in magnitude,
as alſo AF which is equal to it;
and AF is alſo given in poſition, and
f 30. dat. the point A is given, wherefore[f] the
point F is given, and the ſtraight line
g 31. dat. FB in poſition[g]: And the circumference
h 28. dat. ABC is given in poſition, wherefore[h] the point B is given:
And the points A, C are given; therefore the ſtraight lines
i 29. dat. AB, BC are given[i] in poſition and magnitude.

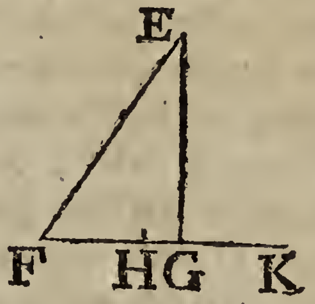
The ſides AB, BC of the rectangle may be found thus: Let
the rectangle GH, GK be the given ſpace to which the rectangle
AB, BC is equal; and let GH, GL be the given rectangle
to which the ſum of the ſquares of AB, BC is equal:
k 14. 2. Find[k] a ſquare equal to the rectangle GH, GL: And let its
ſide AC be given in poſition; upon AC as a diameter deſcribe
the ſemicircle ABC, and as AC to GH, ſo make GK to AF,
and from the point A place AF at right angles to AC: Therefore
l 16. 6. the rectangle CA, AF is equal[l] to GH, GK; and, by
the hypotheſis, twice the rectangle GH, GK is leſs than GH,
GL, that is, than the ſquare of AC; wherefore twice the
rectangle CA, AF is leſs than the ſquare of AC, and the
rectangle CA, AF itſelf leſs than half the ſquare of AC, that
is, than the rectangle contained by the diameter AC and its
half; wherefore AF is leſs than the ſemidiameter of the circle,
and conſequently the ſtraight line drawn through the point F
parallel to AC muſt meet the circumference in two points: Let
B be either of them, and join AB, BC, and complete the rectangle
ABCD, ABCD is the rectangle which was to be found:
m 34. 1. Draw BE perpendicular to AC; therefore BE is equal[m] to AF,
and becauſe the angle ABC in a ſemicircle is a right angle, the
b 8. 6. rectangle AB, BC is equal[b] to AC, BE, that is, to the rectangle
CA, AF which is equal to the given rectangle GH, GK: And
the ſquares of AB, BC are together equal[c] to the ſquare of AC,
that is, to the given rectangle GH, GL.

But if the given angle ABC of the parallelogram AC be not a right angle, in this caſe, becauſe ABC is a given angle, the ratio of the rectangle contained by the ſides AB, BC to the parallelogram AC is given[n]; and AC is given, therefore the rectangle AB, BC is given; and the ſum of the ſquares of AB, BC is given; therefore the ſides AB, BC are given by the preceding caſe.

[n] 62. dat.

The ſides AB, BC and the parallelogram AC may be found thus: Let EFG be the given angle of the parallelogram, and from any point E in FE draw EG perpendicular to FG; and let the rectangle EG, FH be the given ſpace to which the parallelogram is to be made equal, and let EF, FK be the given rectangle to which the ſum of the ſquares of the ſides is to be equal. And, by the preceding caſe, find the ſides of a rectangle which is equal to the given rectangle EF, FH, and the ſquares of the ſides of which are together equal to the given rectangle EF, FK; therefore, as was ſhewn in that caſe, twice the rectangle EF, FH muſt not be greater than the rectangle EF, FK; let it be ſo, and let AB, BC be the ſides of the rectangle joined in the angle ABC equal to the given angle EFG, and complete the parallelogram ABCD, which will be that which was to be found: Draw AL perpendicular to BC, and becauſe the angle ABL is equal to EFG, the triangle ABL is equiangular to EFG; and the parallelogram AC, that is, the rectangle AL, BC, is to the rectangle AB, BC as (the ſtraight line AL to AB, that is, as EG to EF, that is as) the rectangle EG, FH to EF, FH; and, by the conſtruction, the rectangle AB, BC is equal to EF, FH, therefore the rectangle AL, BC or, its equal, the parallelogram AC, is equal to the given rectangle EG, FH; and the ſquares of AB, BC are together equal, by conſtruction, to the given rectangle EF, FK.



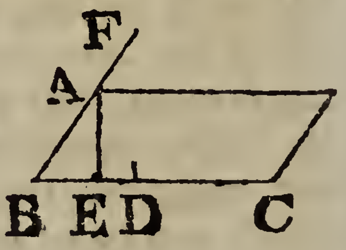


86. PROP. LXXXIX.

IF two ſtraight lines contain a given parallelogram in a given angle, and if the exceſs of the ſquare of one of them above a given ſpace, has a given ratio to the ſquare of the other; each of the ſtraight lines ſhall be given.

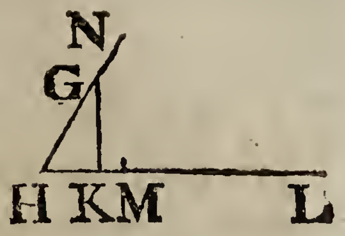
Let the two ſtraight lines AB, BC contain the given parallelogram AC in the given angle ABC, and let the exceſs of the ſquare of BC above a given ſpace have a given ratio to the ſquare of AB, each of the ſtraight lines AB, BC is given.

Becauſe the exceſs of the ſquare of BC above a given ſpace has a given ratio to the ſquare of BA, let the rectangle CB, BD be the given ſpace; take this from the ſquare of BC, the
a 2. 2. remainder, to wit, the rectangle[a] BC, CD has a given ratio to the ſquare of BA: Draw AE perpendicular to BC, and let the ſquare of BF be equal to the rectangle BC, CD, then, becauſe the angle ABC, as alſo BEA is given, the
b 43. dat. triangle ABE is given[b] in ſpecies, and the ratio of AE to AB given: And becauſe the ratio of the rectangle BC, CD, that is, of the ſquare of BF to the ſquare of BA, is given, the ratio of the ſtraight line BF to BA
c 58. dat. is given[c]; and the ratio of AE to AB is given, wherefore[d] the
d 9. dat. ratio of AE to BF is given; as alſo the ratio of the rectangle
e 35. 1. AE, BC, that is[e] of the parallelogram AC to the rectangle FB, BC; and AC is given, wherefore the rectangle FB, BC is given. The exceſs of the ſquare of BC above the ſquare of BF, that is, above the rectangle BC, CD, is given, for it is equal[a] to the given rectangle CB, BD; therefore, becauſe the rectangle contained by the ſtraight lines FB, BC is given, and alſo the exceſs of the ſquare of BC above the ſquare of BF; FB, BC
f 87. dat. are each of them given[f]; and the ratio of FB to BA is given; therefore, AB, BC are given.

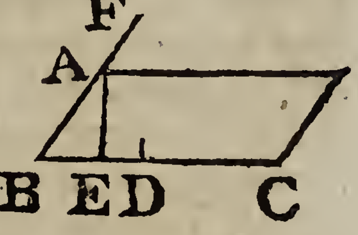


The compoſition is as follows:

Let GHK be the given angle to which the angle of the parallelogram is to be made equal, and from any point G in HG, draw GK perpendicular to HK; let GK, HL be the rectangle



angle to which the parallelogram is to be made equal, and let LH, HM be the rectangle equal to the given ſpace which is to be taken from the ſquare of one of the ſides; and let the ratio of the remainder to the ſquare of the other ſide be the ſame with the ratio of the ſquare of the given ſtraight line NH to the ſquare of the given ſtraight line HG.



By help of the 87th dat. find two ſtraight lines BC, BF which contain a rectangle equal to the given rectangle NH, HL, and ſuch that the exceſs of the ſquare of BC above the ſquare of BF be equal to the given rectangle LH, HM; and join CB, BF in the angle FBC equal to the given angle GHK: And as NH to HG, ſo make FB to BA, and complete the parallelogram AC, and draw AE perpendicular to BC; then AC is equal to the rectangle GK, HL; and if from the ſquare of BC, the given rectangle LH, HM be taken, the remainder ſhall have to the ſquare of BA the ſame ratio which the ſquare of NH has to the ſquare of HG.

Becauſe, by the conſtruction, the ſquare of BC is equal to the ſquare of BF together with the rectangle LH, HM; if from the ſquare of BC there be taken the rectangle LH, HM, there remains the ſquare of BF which has[g] to the ſquare of g 22. 6. BA the ſame ratio which the ſquare of NH has to the ſquare of HG, becauſe, as NH to HG, ſo FB was made to BA; but as HG to GK, ſo is BA to AE, becauſe the triangle GHK is equiangular to ABE; therefore, ex æquali, as NH to GK ſo is FB to AE; wherefore[h] the rectangle NH, HL is to the rect- h 1. 6. angle GK, HL, as the rectangle FB, BC to AE, BC; but by the conſtruction, the rectangle NH, HL is equal to FB, BC; therefore[i] the rectangle GK, HL is equal to the rectangle AE, i 14. 5. BC, that is, to the parallelogram AC.

The analyſis of this problem might have been made as in the 86th prop. in the Greek, and the compoſition of it may be made as that which is in prop. 87th of this edition.

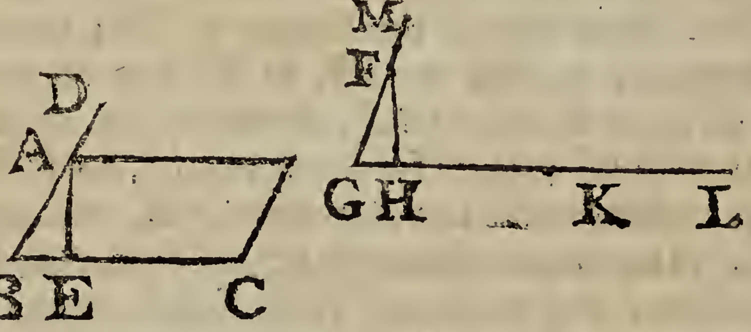
O.

PROP. XC.

IF two ſtraight lines contain a given parallelogram in a given angle, and if the ſquare of one of them together with the ſpace which has a given ratio to the ſquare of the other be given, each of the ſtraight lines ſhall be given.

Let the two ſtraight lines AB, BC contain the given parallelogram AC in the given angle ABC, and let the ſquare of BC together with the ſpace which has a given ratio to the ſquare of AB be given, AB, BC are each of them given.

Let the ſquare of BD be the ſpace which has the given ratio to the ſquare of AB; therefore, by the hypotheſis, the ſquare of BC together with the ſquare of BD is given. From the point A, draw AE perpendicular to BC; and becauſe the angles ABE, BEA are given, the triangle ABE is given [a] in ſpecies; therefore the ratio of BA to AE is given: And becauſe the ratio of the ſquare of BD to the ſquare of BA is given, the ratio of the ſtraight line BD to BA is given [b]; and the ratio of BA to AE is given; therefore [c] the ratio of AE to BD is given, as alſo the ratio of the rectangle AE, BC, that is, of the parallelogram AC to the rectangle DB, BC; and AC is given, therefore the rectangle DB, BC is given; and the ſquare of BC together with the ſquare of BD is given; therefore [d] becauſe the rectangle contained by the two ſtraight lines DB, BC is given, and the ſum of their ſquares is given: The ſtraight lines DB, BC are each of them given; and the ratio of DB to BA is given; therefore AB, BC are given.



The compoſition is as follows:

Let FGH be the given angle to which the angle of the parallelogram is to be made equal, and from any point F in GF draw FH perpendicular to GH; and let the rectangle FH, GK be that to which the parallelogram is to be made equal; and let the rectangle KG, GL be the ſpace to which the ſquare of

[a] 43. dat.

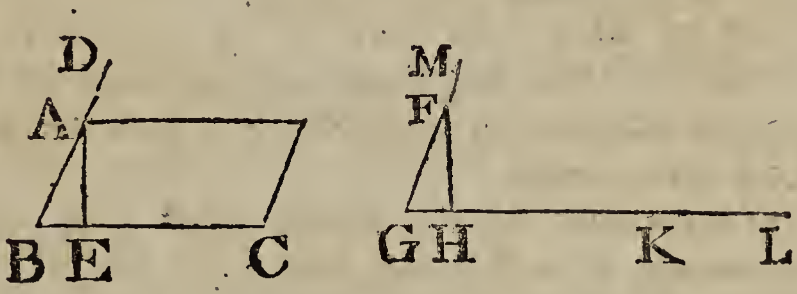
[b] 58. dat.

[c] 9. dat.

[d] 88. dat.

of one of the ſides of the parallelogram together with the ſpace which has a given ratio to the ſquare of the other ſide, is to be made equal; and let this given ratio be the ſame which the ſquare of the given ſtraight line MG has to the ſquare of GF.

By the 88th dat. find two ſtraight lines DB, BC which contain a rectangle equal to the given rectangle MG, GK, and ſuch that the ſum of their ſquares is equal to the given rectangle KG, GL; therefore, by the determination of the problem in that propoſition, twice the rectangle MG, GK muſt not be greater than the rectangle KG, GL. Let it be ſo, and join the ſtraight lines DB, BC in the angle DBC equal to the given angle FGH; and, as MG to GF, ſo make DB to BA, and complete the parallelogram AC: AC is equal to the rectangle FH, GK; and the ſquare of BC together with the ſquare of BD, which, by the conſtruction, has to the ſquare of BA the given ratio which the ſquare of MG has to the ſquare of GF, is equal, by the conſtruction, to the given rectangle KG, GL. Draw AE perpendicular to BC.



Becauſe, as DB to BA, ſo is MG to GF; and as BA to AE, ſo GF to FH; ex æquali, as DB to AE, ſo is MG to FH; therefore, as the rectangle DB, BC to AE, BC, ſo is the rectangle MG, GK to FH, GK and the rectangle DB, BC is equal to the rectangle MG, GK; therefore the rectangle AE, BC, that is, the parallelogram AC, is equal to the rectangle FH, GK.

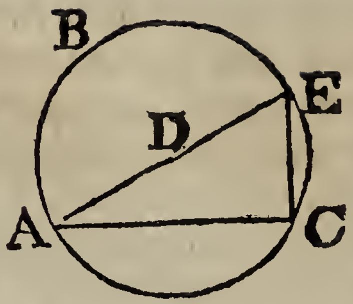
PROP. XCI.

88.

IF a ſtraight line drawn within a circle given in magnitude cuts off a ſegment which contains a given angle; the ſtraight line is given in magnitude.

In the circle ABC given in magnitude, let the ſtraight line AC be drawn, cutting off the ſegment AEC which contains the given angle AEC; the ſtraight line AC is given in magnitude.

Take D the centre of the circle[a], join AD and produce it [a] 1. 3.

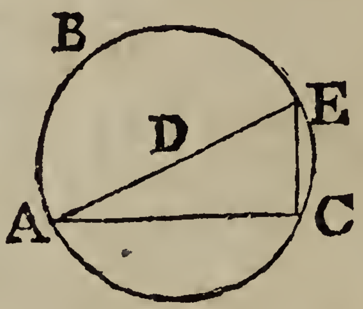


to E, and join EC: The angle ACE being b 31. 3. a right[b] angle is given; and the angle c 43. dat. AEC is given; therefore[c] the triangle ACE is given in ſpecies, and the ratio of EA to AC is therefore given; and EA is given in magnitude, becauſe the circle is d 5. def. given[d] in magnitude; AC is therefore gi- e 2. dat. ven[e] in magnitude.

89. PROP. XCII.

IF a ſtraight line given in magnitude be drawn within a circle given in magnitude, it ſhall cut off a ſegment containing a given angle.

Let the ſtraight line AC given in magnitude be drawn within the circle ABC given in magnitude; it ſhall cut off a ſegment containing a given angle.

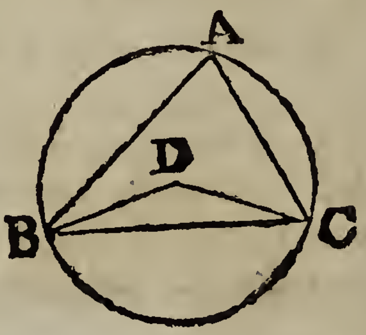


Take D the centre of the circle, join AD and produce it to E, and join EC: And becauſe each of the ſtraight lines EA, a 1. dat. and AC is given, their ratio is given[a]; and the angle ACE is a right angle, therefore b 46. dat. the triangle ACE is given[b] in ſpecies, and conſequently, the angle AEC is given.

90. PROP. XCIII.

IF from any point in the circumference of a circle given in poſition two ſtraight lines be drawn, meeting the circumference and containing a given angle; if the point in which one of them meets the circumference again be given, the point in which the other meets it is alſo given.

From any point A in the circumference of a circle ABC given in poſition, let AB, AC be drawn to the circumference making the given angle BAC; if the point B be given, the point C is alſo given.



Take D the centre of the circle, and join BD, DC; and becauſe each of the a 29. dat. points B, D is given, BD is given[a] in poſition; and becauſe the angle BAC is gi- b 20. 3. ven, the angle BDC is given[b], therefore

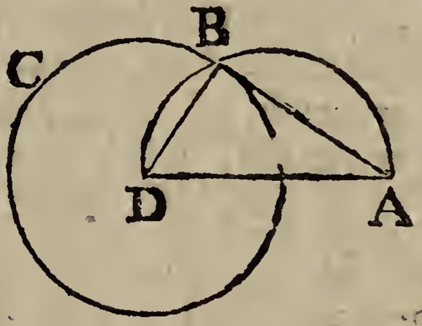
becauſe the ſtraight line DC is drawn to the given point D in the ſtraight line BD given in poſition in the given angle BDC, DC is given[c] in poſition: And the circumference ABC is gi- c 32. dat.
ven in poſition, therefore[d] the point C is given. d 28. dat.

PROP. XCIV.

91.

IF from a given point a ſtraight line be drawn touching a circle given in poſition; the ſtraight line is given in poſition and magnitude.

Let the ſtraight line AB be drawn from the given point A touching the circle BC given in poſition; AB is given in poſition and magnitude.



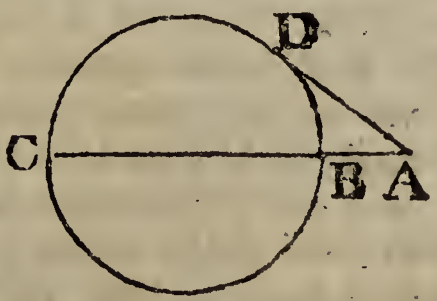
Take D the centre of the circle, and join DA, DB: Becauſe each of the points D, A is given, the
ſtraight line AD is given[a] in poſition a 29. dat.
and magnitude: And DBA is a right[b] b 18. 3.
angle, wherefore DA is a diameter[c] of c cor. 5. 4.
the circle DBA, deſcribed about the triangle DBA; and that circle is therefore
given[d] in poſition: And the circle BC is d 6. def.
given in poſition, therefore the point B
is given[e]. The point A is alſo given: therefore the ſtraight e 28. dat.
line AB is given[a] in poſition and magnitude.

PROP. XCV.

92.

IF a ſtraight line be drawn from a given point without a circle given in poſition; the rectangle contained by the ſegments betwixt the point and the circumference of the circle is given.

Let the ſtraight line ABC be drawn from the given point A without the circle BCD given in poſition, cutting it in B, C; the rectangle BA, AC is given.

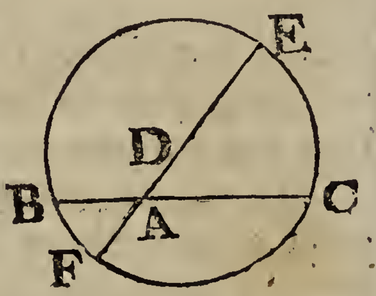


From the point A, draw[a] AD touching the circle; therefore AD is given a 17. 3.
[b] in poſition and magnitude: And be- b 94. dat.
cauſe AD is given, the ſquare of AD is
given[c] which is equal[d] to the rectangle BA, AC: Therefore c 56. dat.
the rectangle BA, AC is given. d 36. 3.

93. PROP. XCVI.

IF a ſtraight line be drawn through a given point within a circle given in poſition, the rectangle contained by the ſegments betwixt the point and the circumference of the circle is given.

Let the ſtraight line BAC be drawn through the given point A within the circle BCE given in poſition; the rectangle BA, AC is given.



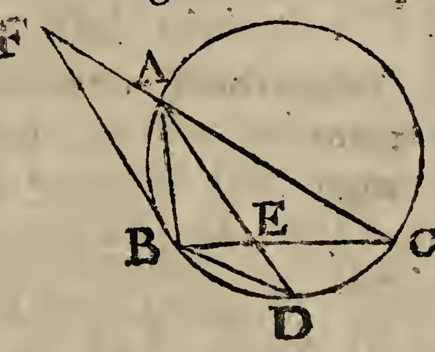
Take D the centre of the circle, join AD and produce it to the points E, F: Becauſe the points A, D are given, the a 29. dat. ſtraight line AD is given[a] in poſition; and the circle BEC is given in poſition; b 28. dat. therefore the points E, F are given[b]; and the point A is given, therefore EA, AF are each of them given[a]; and the rectangle c 35. 3. EA, AF is therefore given; and it is equal[c] to the rectangle BA, AC, which conſequently is given.

94. PROP. XCVII.

IF a ſtraight line be drawn within a circle given in magnitude cutting off a ſegment containing a given angle; if the angle in the ſegment be biſected by a ſtraight line produced till it meets the circumference, the ſtraight lines which contain the given angle ſhall both of them together have a given ratio to the ſtraight line which biſects the angle: And the rectangle contained by both theſe lines together which contain the given angle, and the part of the biſecting line cut off below the baſe of the ſegment, ſhall be given.

Let the ſtraight line BC be drawn within the circle ABC given in magnitude cutting off a ſegment containing the given angle BAC, and let the angle BAC be biſected by the ſtraight line AD; BA together with AC has a given ratio to AD; and the rectangle contained by BA and AC together, and the ſtraight line ED cut off from AD below BC the baſe of the ſegment, is given.

Join BD; and becauſe BC is drawn within the circle ABC given in magnitude cutting off the ſegment BAC, containing the given angle BAC; BC is given[a] in magnitude: By the ſame reaſon BD is given; therefore[b] the ratio of BC to BD is given: And becauſe the angle BAC is biſected by AD, as BA to AC, ſo is[c] BE to EC; and, by permutation, as AB to BE, ſo is AC to CE; wherefore[d] as BA and AC together to BC, ſo is AC to CE: And becauſe the angle BAE is equal to EAC, and the angle ACE to[e] ADB, the triangle ACE is equiangular to the triangle ADB; therefore as AC to CE, ſo is AD to DB: But as AC to CE, ſo is BA together with AC to BC; as therefore BA and AC to BC, ſo is AD to DB; and, by permutation, as BA and AC to AD, ſo is BC to BD: And the ratio of BC to BD is given, therefore the ratio of BA together with AC to AD is given.



[a] 91. dat.
[b] 1. dat.
[c] 3. 6.
[d] 12. 5.
[e] 21. 3.

Alſo the rectangle contained by BA and AC together, and DE is given.

Becauſe the triangle BDE is equiangular to the triangle ACE, as BD to DE, ſo is AC to CE; and as AC to CE, ſo is BA and AC to BC; therefore as BA and AC to BC, ſo is BD to DE; wherefore the rectangle contained by BA and AC together, and DE, is equal to the rectangle CB, BD: But CB, BD is given; therefore the rectangle contained by BA and AC together, and DE, is given.

Otherwiſe.

Produce CA, and make AF equal to AB, and join BF; and becauſe the angle BAC is double[a] of each of the angles BFA, BAD, the angle BFA is equal to BAD; and the angle BCA is equal to BDA, therefore the triangle FCB is equiangular to ABD: As therefore FC to CB, ſo is AD to DB; and, by permutation, as FC, that is, BA and AC together, to AD, ſo is CB to BD: And the ratio of CB to BD is given, therefore the ratio of BA and AC to AD is given.

[a] 5. & 32. 1.

And becauſe the angle BFC is equal to the angle DAC, that is, to the angle DBC, and the angle ACB equal to the angle ADB; the triangle FCB is equiangular to BDE, as therefore FC to CB, ſo is BD to DE; therefore the rectangle contained by FC, that is, BA and AC together, and DE is equal

qual to the rectangle CB, BD, which is given, and therefore the rectangle contained by BA, AC together, and DE is given.

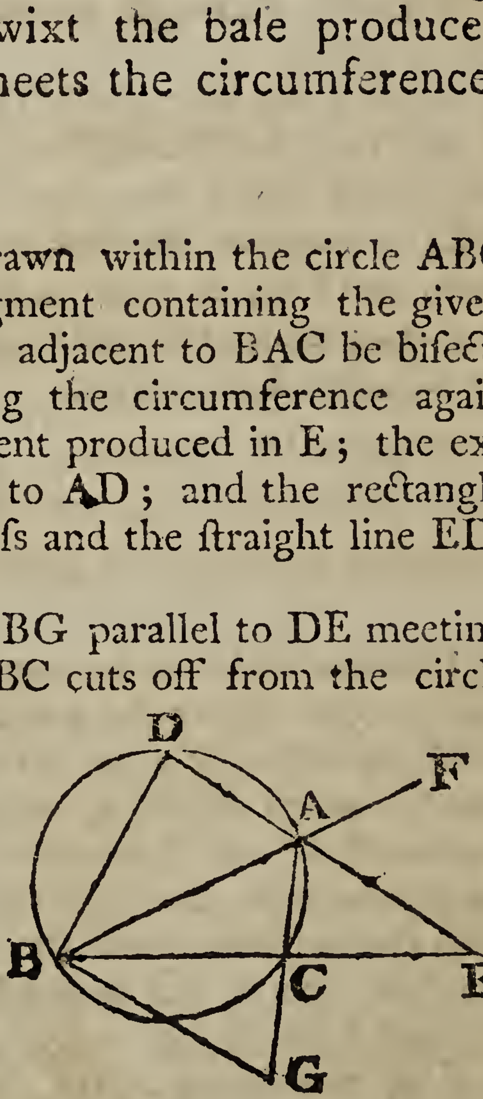
P.

PROP. XCVIII.

IF a ſtraight line be drawn within a circle given in magnitude, cutting off a ſegment containing a given angle: If the angle adjacent to the angle in the ſegment be biſected by a ſtraight line produced till it meet the circumference again and the baſe of the ſegment; the exceſs of the ſtraight lines which contain the given angle ſhall have a given ratio to the ſegment of the biſecting line which is within the circle; and the rectangle contained by the ſame exceſs and the ſegment of the biſecting line betwixt the baſe produced and the point where it again meets the circumference, ſhall be given.

Let the ſtraight line BC be drawn within the circle ABC given in magnitude cutting off a ſegment containing the given angle BAC, and let the angle CAF adjacent to BAC be biſected by the ſtraight line DAE meeting the circumference again in D, and BC the baſe of the ſegment produced in E; the exceſs of BA, AC has a given ratio to AD; and the rectangle which is contained by the ſame exceſs and the ſtraight line ED, is given.

Join BD, and through B, draw BG parallel to DE meeting AC produced in G: And becauſe BC cuts off from the circle ABC given in magnitude the ſegment BAC containing a given angle, BC is therefore given[a] in magnitude: By the ſame reaſon BD is given, becauſe the angle BAD is equal to the given angle EAF: therefore the ratio of BC to BD is given: And becauſe the angle CAE is equal to EAF, of which CAE is equal to the alternate angle AGB, and EAF to the interior and oppoſite angle ABG; therefore the angle AGB is equal to ABG, and the ſtraight line AB equal to AG; ſo that GC is the exceſs



[a] 91. dat.

of BA, AC: And becauſe the angle BGC is equal to GAE, that is, to EAF, or the angle BAD: And that the angle BCG is equal to the oppoſite interior angle BDA of the quadrilateral BCAD in the circle; therefore the triangle BGC is equiangular to BDA: Therefore as GC to CB, ſo is AD to DB; and, by permutation, as GC which is the exceſs of BA, AC to AD, ſo is BC to BD: And the ratio of CB to BD is given; therefore the ratio of the exceſs of BA, AC to AD is given.

And becauſe the angle GBC is equal to the alternate angle DEB, and the angle BCG equal to BDE; the triangle BCG is equiangular to BDE: Therefore as GC to CB, ſo is BD to DE; and conſequently the rectangle GC, DE is equal to the rectangle CB, BD which is given, becauſe its ſides CB, BD are given: Therefore the rectangle contained by the exceſs of BA, AC and the ſtraight line DE is given.

PROP. XCIX. 95.

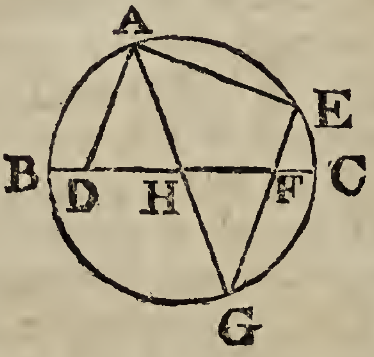
IF from a given point in the diameter of a circle given in poſition, or in the diameter produced, a ſtraight line be drawn to any point in the circumference, and from that point a ſtraight line be drawn at right angles to the firſt, and from the point in which this meets the circumference again, a ſtraight line be drawn parallel to the firſt; the point in which this parallel meets the diameter is given; and the rectangle contained by the two parallels is given.

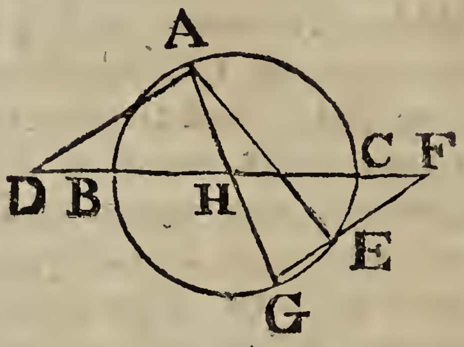
In BC the diameter of the circle ABC given in poſition, or in BC produced, let the given point D be taken, and from D let a ſtraight line DA be drawn to any point A in the circumference, and let AE be drawn at right angles to DA, and from the point E where it meets the circumference again let EF be drawn parallel to DA meeting BC in F; the point F is given, as alſo the rectangle AD, EF.

Produce EF to the circumference in G, and join AG: Becauſe GEA is a right angle, the ſtraight line AG is[a] the diameter of the circle ABC; and BC is alſo a diameter of it; therefore the point H where they meet is the centre of the circle, and conſequently H is given: And the point D is given, wherefore DH is given in magnitude: And becauſe AD is pa-

[a] Cor. 5. 4.

b 4. 6. rallel to FG, and GH equal to HA; DH is equal[b] to HF, and AD equal to GF: And DH is given, therefore HF is given in
magnitude; and it is alſo given in poſition, and the point H is
c 30. dat. given, therefore[c] the point F is given.





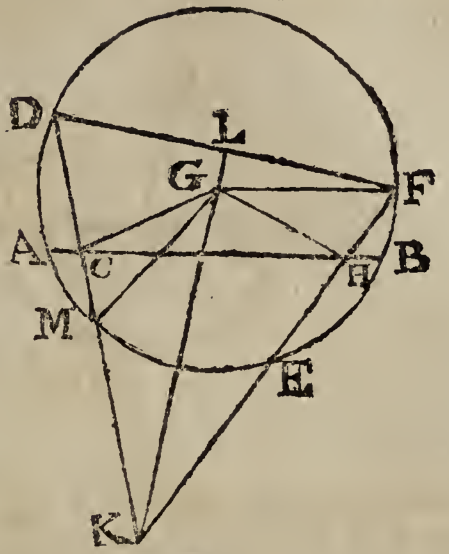
And becauſe the ſtraight line EFG is drawn from a given point F without or within the circle ABC given in poſition,
d 95. or 96 dat. therefore[d] the rectangle EF, FG is given: And GF is equal to AD, wherefore the rectangle AD, EF is given.

Q.

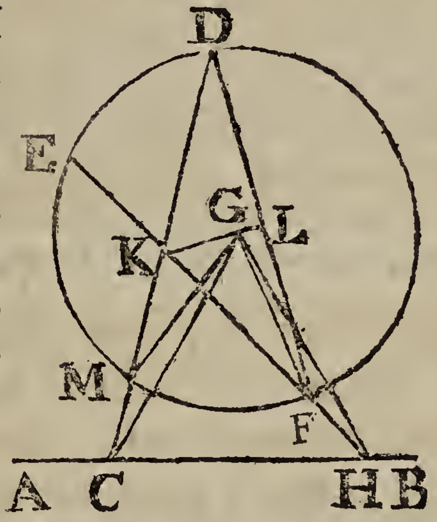
PROP. C.

IF from a given point in a ſtraight line given in poſition, a ſtraight line be drawn to any point in the circumference of a circle given in poſition; and from this point a ſtraight line be drawn making with the firſt an angle equal to the difference of a right angle, and the angle contained by the ſtraight line given in poſition, and the ſtraight line which joins the given point and the centre of the circle; and from the point in which the ſecond line meets the circumference again, a third ſtraight line be drawn making with the ſecond an angle equal to that which the firſt makes with the ſecond: The point in which this third line meets the ſtraight line given in poſition is given; as alſo the rectangle contained by the firſt ſtraight line and the ſegment of the third betwixt the circumference and the ſtraight line given in poſition, is given.

Let the ſtraight line CD be drawn from the given point C in the ſtraight line AB given in poſition, to the circumference of the circle DEF given in poſition, of which G is the centre; join CG, and from the point D let DF be drawn making the angle CDF equal to the difference of a right angle and the angle BCG, and from the point F let FE be drawn making the angle DFE equal to CDF, meeting AB in H: The point H is given; as alſo the rectangle CD, FH.



Let CD, FH meet one another in the point K, from which draw KL perpendicular to DF; and let DC meet the circumference again in M, and let FH meet the ſame in E, and join MG; GF, GH.



Becauſe the angles MDF, DFE are equal to one another, the circumferences MF, DE are equal[a]; and adding or taking away the common part ME, the circumference DM is equal to EF; therefore the ſtraight line DM is equal to the ſtraight line EF, and the angle GMD to the angle[b] GFE; and the angles GMC, GFH are equal to one another, becauſe they are either the ſame with the angles GMD, GFE, or adjacent to them: And becauſe the angles KDL, LKD are together equal[c] to a right angle, that is, by the hypotheſis, to the angles KDL, GCB; the angle GCB or GCH is equal to the angle (LKD, that is, to the angle) LKF or GKH: Therefore the points C, K, H, G are in the circumference of a circle; and the angle GCK is therefore equal to the angle GHF; and the angle GMC is equal to GFH, and the ſtraight line GM to GF; therefore[d] CG is equal to GH, and CM to HF: And becauſe CG is equal to GH, the angle GCH is equal to GHC; but the angle GCH is given: Therefore GHC is given, and conſequently the angle CGH is given; and CG is given in poſition, and the point G; therefore[e] GH is given in poſition; and CB is alſo given in poſition, whereof the point H is given.

And becauſe HF is equal to CM, the rectangle DC, FH is equal to DC, CM: But DC, CM is given[f], becauſe the point C is given, therefore the rectangle DC, FH is given.

FINIS.

[a] 26. 3.

[b] 8. 1.

[c] 32. 1.

[d] 26. 1.

[e] 32. dat.

[f] 95. or 96. dat.

NOTES

ON

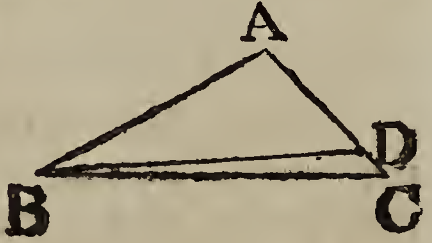
EUCLID's DATA.

DEFINITION II.

THIS is made more explicit than in the Greek text, to prevent a miſtake which the author of the ſecond demonſtration of the 24th propoſition in the Greek edition has fallen into, of thinking that a ratio is given to which another ratio is ſhown to be equal, though this other be not exhibited in given magnitudes. See the notes on that propoſition, which is the 13th in this edition. Beſides, by this definition, as it is now given, ſome propoſitions are demonſtrated, which in the Greek are not ſo well done by help of prop. 2.

DEF. IV.

In the Greek text, def. 4. is thus: "Points, lines, ſpaces, "and angles are ſaid to be given in poſition which have always "the ſame ſituation;" but this is imperfect and uſeleſs, becauſe there are innumerable caſes in which things may be given according to this definition, and yet their poſition cannot be found; for inſtance, let the triangle ABC be given in poſition, and let it be propoſed to draw a ſtraight line BD from the angle at B to the oppoſite ſide AC which ſhall cut off the angle DBC which ſhall be the ſeventh part of the angle ABC; ſuppoſe this is done, therefore the ſtraight line BD is invariable in its poſition, that is, has always the ſame ſituation; for any other ſtraight line drawn from the point B on either ſide of BD cuts off an angle greater or leſſer than the ſeventh part of the angle ABC; therefore, according to this definition, the ſtraight line BD is given in poſition as alſo[a] the point D in which it meets the ſtraight line AC which is given in poſition. But from the things here given, neither the ſtraight line BD nor the point D can be found by the help of Euclid's Elements only, by which every thing in his data is ſuppoſed may be



[a] 28. dat.

be found. This definition is therefore of no uſe. We have amended it by adding, "and which are either actually exhibited "or can be found;" for nothing is to be reckoned given, which cannot be found, or is not actually exhibited.

The definition of an angle given by poſition is taken out of the 4th, and given more diſtinctly by itſelf in the definition marked A.

DEF. XI. XII. XIII. XIV. XV.

The 11th and 12th are omitted, becauſe they cannot be given in Engliſh ſo as to have any tolerable ſenſe; and, therefore, wherever the terms defined occur, the words which expreſs their meaning are made uſe of in their place.

The 13th, 14th, 15th are omitted, as being of no uſe.

It is to be obſerved in general of the data in this book, that they are to be underſtood to be given geometrically, not always arithmetically, that is, they cannot always be exhibited in numbers; for inſtance, if the ſide of a ſquare be given, the ratio of it to its diameter is given[b] geometrically, but not in numbers; and the diameter is given[c]; but though the number of any equal parts in the ſide be given, for example 10, the number of them in the diameter cannot be given: And the like holds in many other caſes.

b 44. dat.
c 2. dat.

PROPOSITION I.

In this it is ſhown that A is to B, as C to D, from this, that A is to C, as B to D, and then by permutation; but it follows directly, without theſe two ſteps, from 7. 5.

PROP. II.

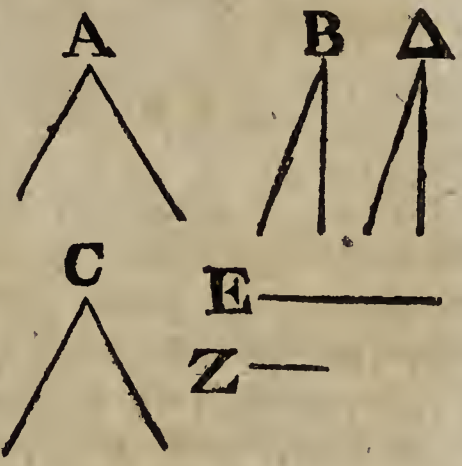
The limitation added at the end of this propoſition between the inverted commas is quite neceſſary, becauſe without it the propoſition cannot always be demonſtrated: For the author having ſaid *, "becauſe A is given, a magnitude equal to it "can be found[a]; let this be C; and becauſe the ratio of A to "B is given, a ratio which is the ſame to it can be found[b]," adds, "let it be found, and let it be the ratio of C to Δ." Now, from the ſecond definition nothing more follows than that ſome ratio, ſuppoſe the ratio of E to Z, can be found, which is the ſame with the ratio of A to B; and when the author ſuppoſes that the ratio of C to Δ, which is alſo

a 1. def.
b 2. def.

---

* See Dr Gregory's Edition of the Data.

alſo the ſame with the ratio of A to B, can be found, he neceſſarily ſuppoſes that to the three magnitudes E, Z, C, a fourth proportional Δ may be found; but this cannot always be done by the Elements of Euclid; from which it is plain Euclid muſt have underſtood the Propoſition under the limitation which is now added to his text. An example will make this clear; let A be a given angle, and B another angle to which A has a given ratio, for inſtance, the ratio of the given ſtraight line E to the given one Z; then, having found an angle C equal to A, how can the angle Δ be found to which C has the ſame ratio that E has to Z? Certainly no way, until it be ſhown how to find an angle to which a given angle has a given ratio, which cannot be done by Euclid's Elements, nor probably by any Geometry known in his time. Therefore, in all the propoſitions of this book which depend upon this ſecond, the above mentioned limitations muſt be underſtood, though it be not explicitly mentioned.



PROP. V.

The order of the Propoſitions in the Greek text between prop. 4. and prop. 25. is now changed into another which is more natural, by placing thoſe which are more ſimple before thoſe which are more complex; and by placing together thoſe which are of the ſame kind, ſome of which were mixed among others of a different kind. Thus, prop. 12. in the Greek is now made the 5th, and thoſe which were the 22d and 23d are made the 11th and 12th, as they are more ſimple than the propoſitions concerning magnitudes, the exceſs of one of which above a given magnitude has a given ratio to the other, after which theſe two were placed; and the 24th in the Greek text is, for the ſame reaſon, made the 13th.

PROP. VI. VII.

Theſe are univerſally true, though, in the Greek text, they are demonſtrated by prop. 2. which has a limitation; they are therefore now ſhown without it.

PROP. XII.

In the 23d prop. in the Greek text, which here is the 12th, the words, "μη τȣς αυτȣς δε," are wrong tranſlated by Claud. Hardy, in his edition of Euclid's Data, printed at Paris, ann. 1625, which was the firſt edition of the Greek text; and Dr Gregory follows him in tranſlating them by the words, "etſi, "non eaſdem," as if the Greek had been ει και μη τȣς αυτȣς, as in prop. 9. of the Greek text. Euclid's meaning is, that the ratios mentioned in the propoſition muſt not be the ſame; for, if they were, the propoſition would not be true. Whatever ratio the whole has to the whole, if the ratios of the parts of the firſt to the parts of the other be the ſame with this ratio, one part of the firſt may be double, triple, &c. of the other part of it, or have any other ratio to it, and conſequently cannot have a given ratio to it; wherefore, theſe words muſt be rendered by "non autem, eaſdem," but not the ſame ratios, as Zambertus has tranſlated them in his edition.

PROP. XIII.

Some very ignorant editor has given a ſecond demonſtration of this propoſition in the Greek text, which has been as ignorantly kept in by Claud. Hardy and Dr Gregory, and has been retained in the tranſlations of Zambertus and others; Carolus Renaldinus gives it only: The author of it has thought that a ratio was given, if another ratio could be ſhown to be the ſame to it, though this laſt ratio be not found: But this is altogether abſurd, becauſe from it would be deduced that the ratio of the ſides of any two ſquares is given, and the ratio of the diameters of any two circles, *&c.* And it is to be obſerved, that the moderns frequently take given ratios, and ratios that are always the ſame, for one and the ſame thing; and Sir Iſaac Newton has fallen into this miſtake in the 17th Lemma of his Principia, edit. 1713, and in other places; but this ſhould be carefully avoided, as it may lead into other errors.

PROP. XIV. XV.

Euclid in this book has ſeveral propoſitions concerning magnitudes, the exceſs of one of which above a given magnitude has a given ratio to the other; but he has given none concerning magnitudes whereof one together with a given magnitude has a given ratio to the other; though theſe laſt occur as frequently in the ſolution of problems as the firſt; the reaſon of which is, that the laſt may be all demonſtrated by help of the firſt; for, if a magnitude, together with a given magnitude, has a given ratio to another magnitude, the exceſs of this other above a given magnitude ſhall have a given ratio to the firſt, and on the contrary; as we have demonſtrated in prop. 14. And for a like reaſon prop. 15, has been added to the data. One example will make the thing clear; ſuppoſe it were to be demonſtrated, that if a magnitude A together with a given magnitude has a given ratio to another magnitude B, that the two magnitudes A and B, together with a given magnitude, have a given ratio to that other magnitude B; which is the ſame propoſition with reſpect to the laſt kind of magnitudes above mentioned, that the firſt part of prop. 16. in this edition is in reſpect of the firſt kind: This is ſhewn thus, from the hypotheſis, and by the firſt part of prop. 14. the exceſs of B above a given magnitude has unto A a given ratio; and, therefore, by the firſt part of prop. 17. the exceſs of B above a given magnitude has unto B and A together a given ratio; and by the ſecond part of prop. 14. A and B together with a given magnitude has unto B a given ratio; which is the thing that was to be demonſtrated. In like manner, the other propoſitions concerning the laſt kind of magnitudes may be ſhewn.

PROP. XVI. XVII.

In the third part of prop. 10. in the Greek text, which is the 16th in this edition, after the ratio of EC to CB has been ſhown to be given; from this, by inverſion and converſion the ratio of BC to BE is demonſtrated to be given; but without theſe two ſteps, the concluſion ſhould have been made only by citing the 6th propoſition. And in like manner, in the firſt part of prop. 11. in the Greek, which in this edition is the 17th from the ratio of DB to BC being given, the ratio of DC to DB is ſhewn to be given, by inverſion and compoſition inſtead of citing prop. 7. and the ſame fault occurs in the ſecond part of the ſame prop. 11.

PROP. XXI. XXII.

Theſe now are added, as being wanting to complete the ſubject treated of in the four preceding propoſitions.

PROP. XXIII.

This, which is prop. 20. in the Greek text, was ſeparated from prop. 14. 15. 16. in that text, after which it ſhould have been immediately placed, as being of the ſame kind; it is now put into its proper place; but prop. 21. in the Greek is left out, as being the ſame with prop. 14. in that text, which is here prop. 18.

PROP. XXIV.

This, which is prop. 13. in the Greek, is now put into its proper place, having been disjoined from the three following it in this edition, which are of the ſame kind.

PROP. XXVIII.

This, which in the Greek text is prop. 25. and ſeveral of the following propoſitions are there deduced from def. 4. which is not ſufficient, as has been mentioned in the note on that definition: They are therefore now ſhewn more explicitly.

PROP. XXXIV. XXXVI.

Each of theſe has a determination, which is now added, which occaſions a change in their demonſtrations.

PROP. XXXVII. XXXIX. XL. XLI.

The 35th and 36th propoſitions in the Greek text are joined into one, which makes the 39th in this edition, becauſe the ſame enunciation and demonſtration ſerves both: And for the ſame reaſon prop. 37. 38. in the Greek are joined into one, which here is the 40.

Prop. 37. is added to the data, as it frequently occurs in the ſolution of problems; and prop. 41. is added to complete the reſt.

PROP. XLII.

This is prop. 39. in the Greek text, where the whole conſtruction of prop. 22. of book I. of the Elements is put, without need, into the demonſtration, but is now only cited.

PROP. XLV.

This is prop. 42. in the Greek, where the three ſtraight lines made uſe of in the conſtruction are ſaid, but not ſhewn, to be ſuch that any two of them is greater than the third, which is now done.

PROP. XLVII.

This is prop. 44. in the Greek text; but the demonſtration of it is changed into another, wherein the ſeveral caſes of it are ſhewn, which, though neceſſary, is not done in the Greek.

PROP. XLVIII.

There are two caſes in this propoſition, ariſing from the two caſes of the third part of prop. 47. on which the 48th depends; and in the compoſition theſe two caſes are explicitly given.

PROP. LII.

The conſtruction and demonſtration of this, which is prop. 48. in the Greek, are made ſomething ſhorter than in that text.

PROP. LIII.

Prop. 63. in the Greek text is omitted, being only a caſe of prop. 49. in that text, which is prop. 53. in this edition.

PROP. LVIII.

This is not in the Greek text, but its demonſtration is contained in that of the firſt part of prop. 54. in that text; which propoſition is concerning figures that are given in ſpecies: This 58th is true of ſimilar figures, though they be not given in ſpecies, and, as it frequently occurs, it was neceſſary to add it.

PROP. LIX. LXI.

This is the 54th in the Greek; and the 77th in the Greek, being the very ſame with it, is left out, and a ſhorter demonſtration is given of prop. 61.

PROP. LXII.

This, which is moſt frequently uſeful, is not in the Greek, and is neceſſary to prop. 87. 88. in this edition, as alſo, though not mentioned, to prop. 86. 87. in the former editions. Prop. 66. in the Greek text is made a corollary to it.

PROP. LXIV.

This contains both prop. 74. and 73. in the Greek text; the firſt caſe of the 74th is a repetition of prop. 56. from which it is ſeparated in that text by many propoſitions; and as there is no order in theſe propoſitions, as they ſtand in the Greek, they are now put into the order which ſeemed moſt convenient and natural.

The

The demonſtration of the firſt part of prop. 73. in the Greek is groſsly vitiated. Dr Gregory ſays, that the ſentences he has incloſed betwixt two ſtars are ſuperfluous, and ought to be cancelled; but he has not obſerved, that what follows them is abſurd, being to prove that the ratio [See his figure] of A Γ to Γ K is given, which, by the hypotheſis at the beginning of the propoſition, is expreſsly given; ſo that the whole of this part was to be altered, which is done in this prop. 64.

PROP. LXVII. LXVIII.

Prop. 70. in the Greek text is divided into theſe two, for the ſake of diſtinctneſs; and the demonſtration of the 67th is rendered ſhorter than that of the firſt part of prop. 70. in the Greek, by means of prop. 23. of book 6. of the Elements.

PROP. LXX.

This is prop. 62. in the Greek text; prop. 78. in that text is only a particular caſe of it, and is therefore omitted.

Dr Gregory, in the demonſtration of prop. 62. cites the 49th prop. dat. to prove that the ratio of the figure AEB, to the parallelogram AH is given; whereas this was ſhewn a few lines before: And beſides, the 49th prop. is not applicable to theſe two figures; becauſe AH is not given in ſpecies, but is, by the ſtep for which the citation is brought, proved to be given in ſpecies.

PROP. LXXIII.

Prop. 83. in the Greek text is neither well enunciated nor demonſtrated. The 73d, which in this edition is put in place of it, is really the ſame, as will appear by conſidering [See Dr Gregory's edition] that A, B, Γ, E in the Greek text are four proportionals; and that the propoſition is to ſhew that Δ, which has a given ratio to E, is to Γ, as B is to a ſtraight line to which A has a given ratio; or, by inverſion, that Γ is to Δ, as a ſtraight line to which A has a given ratio, is to B; that is, if the proportionals be placed in this order, viz. Γ, E, A, B, that the firſt Γ is to Δ to which the ſecond E has a given ratio, as a ſtraight line to which the third A has a given ratio is to the fourth B; which is the enunciation of this 73d, and was thus changed that it might be made like to that of prop. 72. in this edition, which is the 82d in the Greek text: And the demonſtration of prop. 73. is the ſame with that of prop. 72. only making uſe of prop. 23. inſtead of prop. 22. of book 5. of the Elements.

PROP. LXXVII.

This is put in place of prop. 79. in the Greek text, which is not a datum, but a theorem premiſed as a lemma to prop. 80. in that text: And prop. 79. is made cor. 1. to prop. 77. in this edition. Cl. Hardy, in his edition of the data, takes notice, that in prop. 80. of the Greek text, the parallel KL in the figure of prop. 77. in this edition, muſt meet the circumference, but does not demonſtrate it, which is done here at the end of cor. 3. prop. 77. in the conſtruction for finding a triangle ſimilar to ABC.

PROP. LXXVIII.

The demonſtration of this, which is prop. 80. in the Greek, is rendered a good deal ſhorter by help of prop. 77.

PROP. LXXIX. LXXX. LXXXI.

Theſe are added to Euclid's data, as propoſitions which are often uſeful in the ſolution of problems.

PROP. LXXXII.

This, which is prop. 60. in the Greek text, is placed before the 83d and 84th, which in the Greek are the 58th and 59th, becauſe the demonſtration of theſe two in this edition are deduced from that of prop. 82. from which they naturally follow.

PROP. LXXXVIII. XC.

Dr Gregory, in his preface to Euclid's works, which he publiſhed at Oxford in 1703, after having told that he had ſupplied the defects of the Greek text of the data in innumerable places from ſeveral manuſcripts, and corrected Cl. Hardy's tranſlation by Mr Bernard's, adds, that the 86th theorem, "or "propoſition," ſeemed to be remarkably vitiated, but which could not be reſtored by help of the manuſcripts; then he gives three different tranſlations of it in Latin, according to which he thinks it may be read; the two firſt have no diſtinct meaning, and the third, which he ſays is the beſt, though it contains a true propoſition, which is the 90th in this edition, has no connection in the leaſt with the Greek text. And it is ſtrange that Dr Gregory did not obſerve, that, if prop. 86. was changed into this, the demonſtration of the 86th muſt be cancelled, and another put into its place: But the truth is, both the enunciation and the demonſtration of prop. 86. are quite entire and right, only prop. 87. which is more ſimple, ought to have been placed before it; and the deficiency which the Doctor juſtly obſerves to be in this part of Euclid's data, and which no doubt, is owing to the careleſſneſs and ignorance of the Greek editors, ſhould have been ſupplied, not by changing prop. 86. which is both entire and neceſſary, but by adding the two propoſitions, which are the 88th and 90th in this edition.

PROP. XCVIII. C.

Theſe were communicated to me by two excellent geometers, the firſt of them by the Right Honourable the Earl of Stanhope, and the other by Dr Matthew Stewart; to which I have added the demonſtrations.

Though the order of the propoſitions has been in many places changed from that in former editions, yet this will be of little diſadvantage, as the ancient geometers never cite the data, and the moderns very rarely.

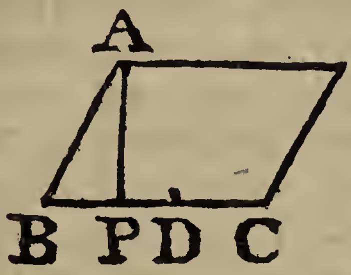
As that part of the compoſition of a problem which is its conſtruction may not be ſo readily deduced from the analyſis by beginners: For their ſake the following example is given, in which the derivation of the ſeveral parts of the conſtruction from the analyſis is particularly ſhown, that they may be aſſiſted to do the like in other problems.

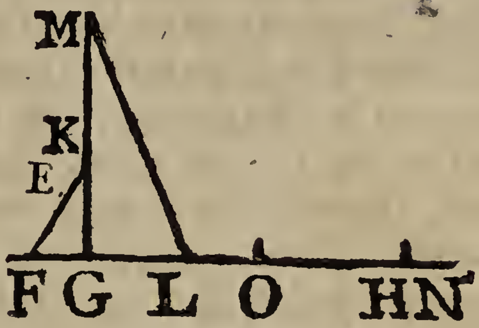
PROBLEM.

Having given the magnitude of a parallelogram, the angle of which ABC is given, and alſo the exceſs of the ſquare of its ſide BC above the ſquare of the ſide AB; to find its ſides and deſcribe it.

The analyſis of this is the ſame with the demonſtration of the 87th prop. of the data, and the conſtruction that is given of the problem at the end of that propoſition is thus derived from the analyſis.

Let EFG be equal to the given angle ABC, and becauſe in the analyſis it is ſaid that the ratio of the rectangle AB, BC to the parallelogram AC is given by the 62d prop. dat. therefore, from a point in FE, the perpendicular EG is drawn to FG, as the ratio of FE to EG is the ratio of the rectangle AB, BC to the parallelogram AC by what is ſhown at the end of prop. 62. Next, the magnitude of AC is exhibited by making the rectangle EG, GH equal to it; and the given exceſs of the ſquare of BC above the ſquare of BA, to which exceſs the rectangle CB, BD is equal, is exhibited by the rectangle HG, GL: Then, in the analyſis, the rectangle AB, BC is ſaid to be given, and this is equal to the rectangle FE, GH, becauſe the rectangle AB, BC is to the parallelogram AC, as (FE to EG, that is, as the rectangle) FE, GH to EG, GH; and the parallelogram AC is equal to the rectangle EG, GH, therefore the rectangle AB, BC, is equal to FE, GH: And conſequently the ratio of the rectangle CB, BD, that is, of the rectangle HG, GL, to AB, BC, that is, of the ſtraight line DB to BA, is the ſame with the ratio (of the rectangle GL, GH to FE, GH, that is) of the ſtraight line GL to FE, which ratio of DB to BA is the next thing ſaid to be given in the analyſis: From this it is plain that the ſquare of FE is to the ſquare of GL, as the ſquare of BA, which is equal to the rectangle BC, CD, is to the ſquare of BD: The ratio of which ſpaces is the next thing ſaid to be given: And from this it follows that four times the ſquare of FE is to the ſquare of GL, as four times the rectangle BC, CD is to the ſquare of BD; and, by compoſition, four times the ſquare of FE together with the ſquare of GL, is to the ſquare of GL, as four times the rectangle BC, CD together with the ſquare of BD, is to the ſquare of BD, that is (8. 6.) as the ſquare of the ſtraight lines BC, CD taken together is to the ſquare of BD, which ratio is the next thing ſaid to be given in the analyſis: And becauſe four times the ſquare of FE and the ſquare of GL are to be added together; therefore in the perpendicular EG there is taken KG equal to FE,





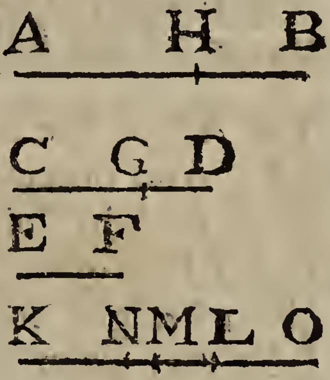
FE, and MG equal to the double of it, becauſe thereby the ſquares of MG, GL, that is, joining ML, the ſquare of ML, is equal to four times the ſquare of FE and to the ſquare of GL: And becauſe the ſquare of ML is to the ſquare of GL, as the ſquare of the ſtraight line made up of BC and CD is to the ſquare of BD, therefore (22. 6.) ML is to LG, as BC together with CD is to BD; and, by compoſition, ML and LG together, that is, producing GL to N, ſo that ML be equal to LN, the ſtraight line NG is to GL, as twice BC is to BD; and by taking GO equal to the half of NG, GO is to GL, as BC to BD, the ratio of which is ſaid to be given in the analyſis: And from this it follows, that the rectangle HG, GO is to HG, GL, as the ſquare of BC is to the rectangle CB, BD, which is equal to the rectangle HG, GL; and therefore the ſquare of BC is equal to the rectangle HG, GO; and BC is conſequently found by taking a mean proportional betwixt HG and GO, as is ſaid in the conſtruction: And becauſe it was ſhown that GO is to GL, as BC to BD, and that now the three firſt are found, the fourth BD is found by 12. 6. It was likewiſe ſhown that LG is to FE, or GK, as DB to BA, and the three firſt are now found, and thereby the fourth BA. Make the angle ABC equal to EFG, and complete the parallelogram of which the ſides are AB, BC, and the conſtruction is finiſhed; the reſt of the compoſition contains the demonſtration.

AS the propoſitions from the 13th to the 28th may be thought by beginners to be leſs uſeful than the reſt, becauſe they cannot ſo readily ſee how they are to be made uſe of in the ſolution of problems; on this account the two following problems are added, to ſhow that they are equally uſeful with the other propoſitions, and from which it may be eaſily judged that many other problems depend upon theſe propoſitions.

PROBLEM

TO find three ſtraight lines ſuch, that the ratio of the firſt to the ſecond is given; and if a given ſtraight line be taken from the ſecond, the ratio of the remainder to the third is given; alſo the rectangle contained by the firſt and third is given.

Let AB be the firſt ſtraight line, CD the ſecond, and EF the third: And becauſe the ratio of AB to CD is given, and that if a given ſtraight line be taken from CD, the ratio of the remainder to EF is given; therefore[a] the exceſs of the firſt AB above a given ſtraight line has a given ratio to the third EF: Let BH be that given ſtraight line; therefore AH, the exceſs of AB above it, has a given ratio to EF; and conſequently[b] the rectangle BA, AH, has a given ratio to the rectangle AB, EF, which laſt rectangle is given by the hypotheſis; therefore[c] the rectangle BA, AH is given, and BH the exceſs of its ſides is given; wherefore the ſides AB, AH are given[d]: And becauſe the ratios of AB to CD, and of AH to EF are given, CD and EF are[c] given.



a 24. dat.
b 1. 6.
c 2. dat.
d 85. dat.

The Compoſition.

Let the given ratio of KL to KM be that which AB is required to have to CD; and let DG be the given ſtraight line which is to be taken from CD, and let the given ratio of KM to KN be that which the remainder muſt have to EF; alſo let the given rectangle NK, KO be that to which the rectangle AB, EF is required to be equal: Find the given ſtraight line BH which is to be taken from AB, which is done, as plainly appears from prop. 24. dat. by making as KM to KL, ſo GD to HB. To the given ſtraight line BH apply[e] a rectangle equal to LK, KO exceeding by a ſquare, and let BA, AH be its ſides: Then is AB the firſt of the ſtraight lines required to be found, and by making as LK to KM, ſo AB to DC, DC will be the ſecond: And laſtly, make as KM to KN, ſo CG to EF, and EF is the third.

e 29. 6.

For as AB to CD, ſo is HB to GD, each of theſe ratios being the ſame with the ratio of LK to KM; therefore[f] AH is to CG, as (AB to CD, that is, as) LK to KM; and as CG to EF, ſo is KM to KN; wherefore, ex æquali, as AH to EF, ſo is LK, to KN: And as the rectangle BA, AH to the rectangle BA, EF, ſo is[g] the rectangle LK, KO to the rectangle KN, KO: And by the conſtruction, the rectangle BA, AH is equal to LK, KO: Therefore[h] the rectangle AB, EF is equal to the given rectangle NK, KO: And AB has to CD the given ratio of KL to KM; and from CD the given ſtraight line GD being taken, the remainder CG has to EF the given ratio of KM to KN. Q. E. D.

f 19. 5.
g 1. 6.
h 14. 5.

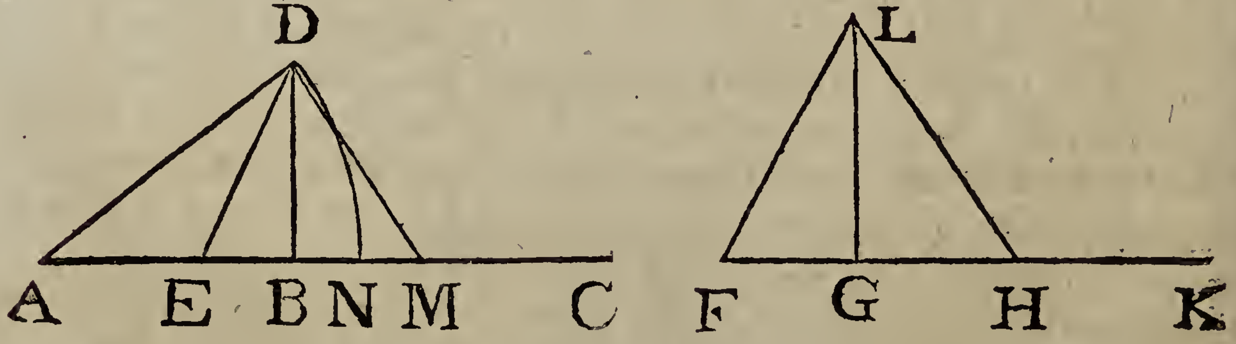
PROB. II.

TO find three ſtraight lines ſuch, that the ratio of the firſt to the ſecond is given; and if a given ſtraight line be taken from the ſecond, the ratio of the remainder to the third is given; alſo the ſum of the ſquares of the firſt and third is given.

Let AB be the firſt ſtraight line, BC the ſecond, and BD the
third: And becauſe the ratio of AB to BC is given, and that if
a given ſtraight line be taken from BC, the ratio of the remain-
a 24. dat. der to BD is given; therefore [a] the exceſs of the firſt AB above
a given ſtraight line, has a given ratio to the third BD: Let
AE be that given ſtraight line, therefore the remainder EB has
a given ratio to BD: Let BD be placed at right angles to EB,
b 44. dat. and join DE; then the triangle EBD is [b] given in ſpecies;
wherefore the angle BED is given: Let AE which is given in
magnitude, be given alſo in poſition, as alſo the point E, and
c 32. dat. the ſtraight line ED will be given [c] in poſition: Join AD, and
d 47. 1. becauſe the ſum of the ſquares of AB, BD, that is [d], the ſquare
of AD is given, therefore the ſtraight line AD is given in mag-
e 34. dat. nitude; and it is alſo given [e] in poſition, becauſe from the gi-
ven point A it is drawn to the ſtraight line ED given in poſition:
Therefore the point D, in which the two ſtraight lines AD, ED
f 28. dat. given in poſition, cut one another, is given [f]: And the ſtraight
g 33. dat. line DB, which is at right angles to AB, is given [g] in poſition,
and AB is given in poſition, therefore [f] the point B is given:
h 29. dat. And the points A, D are given, wherefore [h] the ſtraight lines
AB, BD are given: And the ratio of AB to BC is given, and
i 2. dat. therefore [i] BC is given.

The Compoſition.

Let the given ratio of FG to GH be that which AB is required to have to BC, and let HK be the given ſtraight line which is to be taken from BC, and let the ratio which the remainder is required to have to BD be the given ratio of HG to LG, and place GL at right angles to FH, and join LF, LH:



Next, as HG is to GF, ſo make HK to AE; produce AE to N, ſo that AN be the ſtraight line to the ſquare of which the ſum of the ſquares of AB, BD is required to be equal; and make the angle NED equal to the angle GFL; and from the centre A at the diſtance AN deſcribe a circle, and let its circumference meet ED in D, and draw DB perpendicular to AN, and DM making the angle BDM equal to the angle GLH. Laſtly, produce BM to C, ſo that MC be equal to KH; then is AB the firſt, BC the ſecond, and BD the third of the ſtraight lines that were to be found.

For the triangles EBD, FGL, as alſo DBM, LGH being equiangular, as EB to BD, ſo is FG to GL; and as DB to BM, ſo is LG to GH; therefore, ex æquali, as EB to BM, ſo is (FG to GH, and ſo is) AE to HK or MC; wherefore[k], AB is to BC, as AE to HK, that is, as FG to GH, that is, in the given ratio; and from the ſtraight line BC taking MC, which is equal to the given ſtraight line HK, the remainder BM has to BD the given ratio of HG to GL; and the ſum of the ſquares of AB, BD is equal[d] to the ſquare of AD or AN, which is the given ſpace. Q. E. D.

[k] 12. 5.

[d] 47. 1.

I believe it would be in vain to try to deduce the preceding conſtruction from an algebraical ſolution of the problem.

FINIS.

THE

ELEMENTS

OF

PLANE AND SPHERICAL

TRIGONOMETRY.

EDINBURGH:
Printed for C. Nourse, London; and J. Balfour, Edinburgh.

M,DCC,XCIV.

PLANE TRIGONOMETRY.

LEMMA I. Fig. 1.

LET ABC be a rectilineal angle, if about the point B as a centre, and with any diſtance BA, a circle be deſcribed, meeting BA, BC, the ſtraight lines including the angle ABC in A, C; the angle ABC will be to four right angles, as the arch AC to the whole circumference.

Produce AB till it meet the circle again in F, and through B draw DE perpendicular to AB, meeting the circle in D, E.

By 33. 6. Elem. the angle ABC is to a right angle ABD, as the arch AC to the arch AD; and quadrupling the conſequents, the angle ABC will be to four right angles, as the arch AC to four times the arch AD, or to the whole circumference.

LEMMA II. Fig. 2.

LET ABC be a plane rectilineal angle as before: About B as a centre with any two diſtances BD, BA, let two circles be deſcribed meeting BA, BC, in D, E, A, C; the arch AC will be to the whole circumference of which it is an arch, as the arch DE is to the whole circumference of which it is an arch.

By Lemma 1. the arch AC is to the whole circumference of which it is an arch, as the angle ABC is to four right angles; and by the ſame Lemma 1. the arch DE is to the whole circumference of which it is an arch, as the angle ABC is to four right angles; therefore the arch AC is to the whole circumference of which it is an arch, as the arch DE to the whole circumference of which it is an arch.

DEFINITIONS. Fig. 3.

I.

LET ABC be a plane rectilineal angle; if about B as a centre, with BA any diſtance, a circle ACF be deſcribed meeting BA, BC, in A, C; the arch AC is called the meaſure of the angle ABC.

II.

The circumference of a circle is ſuppoſed to be divided into

360 equal parts called degrees, and each degree into 60 equal parts called minutes, and each minute into 60 equal parts called ſeconds, &c. And as many degrees, minutes, ſeconds, &c. as are contained in any arch, of ſo many degrees, minutes, ſeconds, &c. is the angle, of which that arch is the meaſure, ſaid to be.

Cor. Whatever be the radius of the circle of which the meaſure of a given angle is an arch, that arch will contain the ſame number of degrees, minutes, ſeconds, &c. as is manifeſt from Lemma 2.

III.

Let AB be produced till it meet the circle again in F, the angle CBF, which, together with ABC, is equal to two right angles, is called the *Supplement* of the angle ABC.

IV.

A ſtraight line CD drawn through C, one of the extremities of the arch AC perpendicular upon the diameter paſſing through the other extremity A, is called the *Sine* of the arch AC, or of the angle ABC, of which it is the meaſure.

Cor. The *Sine* of a quadrant, or of a right angle, is equal to the radius.

V.

The ſegment DA of the diameter paſſing through A, one extremity of the arch AC between the ſine CD, and that extremity, is called the *Verſed Sine* of the arch AC, or angle ABC.

VI.

A ſtraight line AE touching the circle at A, one extremity of the arch AC, and meeting the diameter BC paſſing through the other extremity C in E, is called the *Tangent* of the arch AC, or of the angle ABC.

VII.

The ſtraight line BE between the centre and the extremity of the tangent AE, is called the *Secant* of the arch AC, or angle ABC.

Cor. to def. 4. 6. 7. the ſine, tangent, and ſecant of any angle ABC, are likewiſe the ſine, tangent, and ſecant of its ſupplement CBF.

It is manifeſt from def. 4. that CD is the ſine of the angle CBF. Let CB be produced till it meet the circle again in G; and it is manifeſt that AE is the tangent, and BE the ſecant, of the angle ABG or EBF, from def. 8. 7.

Cor. to def. 4. 5. 6. 7. The ſine, verſed ſine, tangent, and ſecant, of any arch which is the meaſure of any given angle ABC, is to the ſine, verſed ſine, tangent, and ſecant, of any other arch which is the meaſure of the ſame angle, as the radius of the firſt is to the radius of the ſecond.

Let AC, MN be meaſures of the angle ABC, according to def. 1. CD the ſine, DA the verſed ſine, AE the tangent, and BE the ſecant of the arch AC, according to def. 4. 5. 6. 7. and NO the ſine, OM the verſed ſine, MP the tangent, and BP the ſecant of the arch MN, according to the ſame definitions. Since CD, NO, AE, MP are parallel, CD is to NO as the radius CB to the radius NB, and AE to MP as AB to BM, and BC or BA to BD as BN or BM to BO; and, by converſion, DA to MO as AB to MB. Hence the corollary is manifeſt; therefore, if the radius be ſuppoſed to be divided into any given number of equal parts, the ſine, verſed ſine, tangent, and ſecant of any given angle, will each contain a given number of theſe parts; and, by trigonometrical tables, the length of the ſine, verſed ſine, tangent, and ſecant of any angle may be found in parts of which the radius contains a given number; and, vice verſa, a number expreſſing the length of the ſine, verſed ſine, tangent, and ſecant being given, the angle of which it is the ſine, verſed ſine, tangent, and ſecant may be found.

VIII.

Fig. 3.

The difference of an angle from a right angle is called the *complement* of that angle. Thus, if BH be drawn perpendicular to AB, the angle CBH will be the complement of the angle ABC, or of CBF.

IX.

Let HK be the tangent, CL or DB, which is equal to it, the ſine, and BK the ſecant of CBH, the complement of ABC, according to def. 4. 6. 7. HK is called the *co-tangent*, BD the *co-ſine*, and BK the *co-ſecant* of the angle ABC.

Cor. 1. The radius is a mean proportional between the tangent and co-tangent.

For, ſince HK, BA are parallel, the angles HKB, ABC will be equal, and the angles KHB, BAE are right; therefore

the

the triangles BAE, KHB are ſimilar, and therefore AE is to AB, as BH or BA to HK.

Cor. 2. The radius is a mean proportional between the co-ſine and ſecant of any angle ABC.

Since CD, AE are parallel, BD is to BC or BA, as BA to BE.

PROP. I. Fig. 5.

IN a right angled plain triangle, if the hypothenuſe be made radius, the ſides become the ſines of the angles oppoſite to them; and if either ſide be made radius, the remaining ſide is the tangent of the angle oppoſite to it, and the hypothenuſe the ſecant of the ſame angle.

Let ABC be a right angled triangle; if the hypothenuſe BC be made radius, either of the ſides AC will be the ſine of the angle ABC oppoſite to it; and if either ſide BA be made radius, the other ſide AC will be the tangent of the angle ABC oppoſite to it, and the hypothenuſe BC the ſecant of the ſame angle.

About B as a centre, with BC, BA for diſtances, let two circles CD, EA be deſcribed, meeting BA, BC in D, E: Since CAB is a right angle, BC being radius, AC is the ſine of the angle ABC, by def. 4. and BA being radius, AC is the tangent, and BC the ſecant of the angle ABC, by def. 6. 7.

Cor. 1. Of the hypothenuſe a ſide and an angle of a right angled triangle, any two being given, the third is alſo given.

Cor. 2. Of the two ſides and an angle of a right angled triangle, any two being given, the third is alſo given.

PROP. II. Fig. 6. 7.

THE ſides of a plain triangle are to one another, as the ſines of the angles oppoſite to them.

In right angled triangles, this prop. is manifeſt from prop. 1. for if the hypothenuſe be made radius, the ſides are the ſines of the angles oppoſite to them, and the radius is the ſine of a right angle (cor. to def. 4.) which is oppoſite to the hypothenuſe.

In any oblique angled triangle ABC, any two ſides AB, AC will be to one another as the ſines of the angles ACB, ABC which are oppoſite to them.

From C, B draw CE, BD perpendicular upon the oppoſite ſides AB, AC produced, if need be. Since CEB, CDB are right angles, BC being radius, CE is the ſine of the angle CBA, and BD the ſine of the angle ACB; but the two triangles CAE, DAB have each a right angle at D and E; and likewiſe the common angle CAB; therefore they are ſimilar, and conſequently, CA is to AB, as CE to DB; that is, the ſides are as the ſines of the angles oppoſite to them.

Cor. Hence of two ſides, and two angles oppoſite to them, in a plain triangle, any three being given, the fourth is alſo given.

PROP. III. Fig. 8.

In a plain triangle, the ſum of any two ſides is to their difference, as the tangent of half the ſum of the angles at the baſe, to the tangent of half their difference.

Let ABC be a plain triangle, the ſum of any two ſides AB, AC will be to their difference as the tangent of half the ſum of the angles at the baſe ABC, ACB to the tangent of half their difference.

About A as a centre, with AB the greater ſide for a diſtance, let a circle be deſcribed, meeting AC produced in E, F, and BC in D; join DA, EB, FB: and draw FG parallel to BC, meeting EB in G.

The angle EAB (32. 1.) is equal to the ſum of the angles at the baſe, and the angle EFB at the circumference is equal to the half of EAB at the centre (20. 3.); therefore EFB is half the ſum of the angles at the baſe; but the angle ACB (32. 1.) is equal to the angles CAD and ADC, or ABC together; therefore FAD is the difference of the angles at the baſe, and FBD at the circumference, or BFG, on account of the parallels FG, BD, is the half of that difference; but ſince the angle EBF in a ſemicircle is a right angle (1. of this) FB being radius, BE, BG, are the tangents of the angles EFB, BFG; but it is manifeſt that EC is the ſum of the ſides BA, AC, and CF their difference; and ſince BC, FG are parallel (2. 6.) EC is to CF, as EB to BG; that is, the ſum of the ſides is to their difference, as the tangent of half the ſum of the angles at the baſe to the tangent of half their difference.

PROP. IV. Fig. 18.

IN any plain triangle BAC, whoſe two ſides are BA, AC and baſe BC, the leſs of the two ſides which let be BA, is to the greater AC as the radius is to the tangent of an angle, and the radius is to the tangent of the exceſs of this angle above half a right angle as the tangent of half the ſum of the angles B and C at the baſe, is to the tangent of half their difference.

At the point A, draw the ſtraight line EAD perpendicular to BA; make AE, AF, each equal to AB, and AD to AC; join BE, BF, BD, and from D, draw DG perpendicular upon BF. And becauſe BA is at right angles to EF, and EA, AB, AF are equal, each of the angles EBA, ABF is half a right angle, and the whole EBF is a right angle; (alſo 4. 1. El.) EB is equal to BF. And ſince EBF, FGD are right angles, EB is parallel to GD, and the triangles EBF, FGD are ſimilar; therefore EB is to BF as DG to GF, and EB being equal to BF, FG muſt be equal to GD. And becauſe BAD is a right angle, BA the leſs ſide is to AD or AC the greater as the radius is to the tangent of the angle ABD; and becauſe BGD is a right angle, BG is to GD or GF as the radius is to the tangent of GBD, which is the exceſs of the angle ABD above ABF half a right angle. But becauſe EB is parallel to GD, BG is to GF as ED is to DF, that is, ſince ED is the ſum of the ſides BA, AC and FD their difference, (3. of this), as the tangent of half the ſum of the angles B, C, at the baſe to the tangent of half their difference. Therefore, in any plain triangle, &c. Q. E. D.

PROP. V. Fig. 9. and 10.

IN any triangle, twice the rectangle contained by any two ſides is to the difference of the ſum of the ſquares of theſe two ſides, and the ſquare of the baſe, as the radius is to the co-ſine of the angle included by the two ſides.

Let ABC be a plain triangle, twice the rectangle ABC contained by any two ſides BA, BC is to the difference of the ſum of the ſquares of BA, BC, and the ſquare of the baſe AC, as the radius to the co-ſine of the angle ABC.

From A, draw AD perpendicular upon the oppoſite ſide BC, then (by 12. and 13. 2. El.) the difference of the ſum of the ſquares of AB, BC, and the ſquare of the baſe AC, is equal to twice the rectangle CBD; but twice the rectangle CBA is to twice the rectangle CBD; that is, to the difference of the ſum of the ſquares of AB, BC, and the ſquare of AC, (1. 6.) as AB to BD; that is, by prop. 1. as radius to the ſine of BAD, which is the complement of the angle ABC, that is, as radius to the co-ſine of ABC.

PROP. VI. FIG. 11.

IN any triangle ABC, whoſe two ſides are AB, AC, and baſe BC, the rectangle contained by half the perimeter, and the exceſs of it above the baſe BC, is to the rectangle contained by the ſtraight lines by which the half of the perimeter exceeds the other two ſides AB, AC, as the ſquare of the radius is to the ſquare of the tangent of half the angle BAC oppoſite to the baſe.

Let the angles BAC, ABC be biſected by the ſtraight lines AG, BG; and producing the ſide AB, let the exterior angle CBH be biſected by the ſtraight line BK, meeting AG in K; and from the points G, K, let their be drawn perpendicular upon the ſides the ſtraight lines GD, GE, GF, KH, KL, KM. Since therefore (4. 4.) G is the centre of the circle inſcribed in the triangle ABC, GD, GF, GE will be equal, and AD will be equal to AE, BD to BF, and CE to CF. In like manner KH, KL, KM will be equal, and BH will be equal to BM, and AH to AL, becauſe the angles HBM, HAL are biſected by the ſtraight lines BK, KA: And becauſe in the triangles KCL, KCM, the ſides LK, KM are equal, KC is common and KLC, KMC are right angles, CL will be equal to CM: Since therefore BM is equal to BH, and CM to CL; BC will be equal to BH and CL together; and, adding AB and AC together, AB, AC, and BC will together be equal to AH and AL together: But AH, AL are equal: Wherefore each of them is equal to half the perimeter of the triangle ABC: But ſince AD, AE are equal, and BD, BF, and alſo CE, CF, AB together with FC, will be equal to half the perimeter of the triangle to which AH or AL was ſhewn to be equal; taking away therefore the common AB, the remainder FC will be equal to the remainder BH: In the ſame manner it is demonſtrated, that BF is equal to CL: And ſince the points B, D, G, F, are in a circle, the angle DGF will be equal to the exterior and oppoſite angle FBH, (22. 3.); wherefore their halves BGD, HBK will be equal to one another: The right angled triangles BGD, HBK will therefore be equiangular, and GD will be to BD, as BH to HK, and the rectangle contained by GD, HK will be equal to the rectangle DBH or BFC: But ſince AH is to HK, as AD to DG, the rectangle HAD (22. 6.) will be to the rectangle contained by HK, DG, or the rectangle BFC, (as the ſquare of AD is to the ſquare of DG, that is) as the ſquare of the radius to the ſquare of the tangent of the angle DAG, that is, the half of BAC: But HA is half the perimeter of the triangle ABC, and AD is the exceſs of the ſame above HD, that is, above the baſe BC; but BF or CL is the exceſs of HA or AL above the ſide AC, and FC, or HB is the exceſs of the ſame HA above the ſide AB; therefore the rectangle contained by half the perimeter, and the exceſs of the ſame above the baſe, viz. the rectangle HAD, is to the rectangle contained by the ſtraight lines by which the half of the perimeter exceeds the other two ſides, that is, the rectangle BFC, as the ſquare of the radius is to the ſquare of the tangent of half the angle BAC oppoſite to the baſe. Q. E. D.

PROP. VII. Fig. 12. 13.

IN a plain triangle, the baſe is to the ſum of the ſides as the difference of the ſides is to the ſum or difference of the ſegments of the baſe made by the perpendicular upon it from the vertex, according as the ſquare of the greater ſide is greater or leſs than the ſum of the ſquares of the leſſer ſide and the baſe.

Let ABC be a plain triangle; if from A the vertex be drawn a ſtraight line AD perpendicular upon the baſe BC, the baſe BC will be to the ſum of the ſides BA, AC, as the difference of the ſame ſides is to the ſum or difference of the ſegments CD, BD, according as the ſquare of AC the greater ſide is greater or leſs than the ſum of the ſquares of the leſſer ſide AB, and the baſe BC.

About A as a centre, with AC the greater ſide for a diſtance, let a circle be deſcribed meeting AB produced in E, F, and CB in G: It is manifeſt that FB is the ſum and BE, the difference of the ſides; and ſince AD is perpendicular to GC, GD, CD will be equal; conſequently GB will be equal to the ſum or difference of the ſegments CD, BD, according as the perpendicular AD meets the baſe, or the baſe produced; that is, (by Conv. 12. 13. 2.) according as the ſquare of AC is greater or leſs than the ſum of the ſquares of AB, BC: But (by 35. 3.) the rectangle CBG is equal to the rectangle EBF; that is, (16. 6.) BC is to BF, as BE is to BG; that is, the baſe is to the ſum of the ſides, as the difference of the ſides is to the ſum or difference of the ſegments of the baſe made by the perpendicular from the vertex, according as the ſquare of the greater ſide is greater or leſs than the ſum of the ſquares of the leſſer ſide and the baſe. Q. E. D.

PROP. VIII. PROB. FIG. 14.

THE ſum and difference of two magnitudes being given, to find them.

Half the given ſum added to half the given difference, will be the greater, and half the difference ſubtracted from half the ſum, will be the leſs.

For, let AB be the given ſum, AC the greater, and BC the leſs. Let AD be half the given ſum; and to AD, DB, which are equal, let DC be added, then AC will be equal to BD, and DC together; that is, to BC, and twice DC; conſequently twice DC is the difference, and DC half that difference; but AC the greater is equal to AD, DC; that is, to half the ſum added to half the difference, and BC the leſs is equal to the exceſs of BD, half the ſum above DC half the difference. Q. E. F.

SCHOLIUM.

Of the ſix parts of a plain triangle (the three ſides and three angles) any three being given, to find the other three is the buſineſs of plane trigonometry; and the ſeveral caſes of that problem may be reſolved by means of the preceding propoſition, as in the two following, with the tables annexed. In theſe, the ſolution is expreſſed by a fourth proportional to three given lines; but if the given parts be expreſſed by numbers from trigonometrical tables, it may be obtained arithmetically by the common Rule of Three.

SOLUTION.

NOTE. In the tables the following abbreviations are uſed: R, is put for the Radius; T, for Tangent; and S, for Sine. Degrees, minutes, ſeconds, &c. are written in this manner: 30° 25′ 13″, &c. which ſignifies 30 degrees, 25 minutes, 13 ſeconds, &c.

SOLUTION of the CASES of right angled TRIANGLES.

GENERAL PROPOSITION.

IN a right angled triangle, of the three ſides and three angles, any two being given beſides the right angle, the other three may be found, except when the two acute angles are given, in which caſe the ratios of the ſides are only given, being the ſame with the ratios of the ſines of the angles oppoſite to them.

It is manifeſt from 47. 1. that of the two ſides and hypothenuſe any two be given, the third may alſo be found. It is alſo manifeſt from 32. 1. that if one of the acute angles of a right angled triangle be given, the other is alſo given, for it is the complement of the former to a right angle.

If two angles of any triangle be given, the third is alſo given, being the ſupplement of the two given angles to two right angles.

Fig. 15. The other caſes may be reſolved by help of the preceding propoſitions, as in the following table:

	GIVEN.	SOUGHT.	
1	Two ſides, AB, AC.	The angles B, C.	AB : AC : : R : T, B, of which C is the complement.
2	AB, BC, a ſide and the hypothenuſe.	The angles B, C.	BC : BA : : R : S, C, of which B is the complement.
3	AB, B, a ſide and an angle.	The other ſide AC.	R : T, B : : BA : AC.
4	AB and B, a ſide and an angle.	The hypothenuſe BC.	S, C : R : : BA : BC.
5	BC and B, the hypothenuſe and an angle.	The ſide AC.	R : S, B : : BC : CA.

Theſe five caſes are reſolved by prop. 1.

SOLUTION of the CASES of OBLIQUE-ANGLED TRIANGLES.

GENERAL PROPOSITION.

IN an oblique-angled triangle, of the three ſides and three angles, any three being given, the other three may be found, except when the three angles are given; in which caſe the ratios of the ſides are only given, being the ſame with the ratios of the ſines of the angles oppoſite to them.

	GIVEN.	SOUGHT.	
1	A, B, and therefore C, and the ſide AB.	BC, AC.	S, C : S, A :: AB : BC, and alſo, S, C : S, B :: AB : AC. (2.)
2	AB, AC, and B, two ſides and an angle oppoſite to one of them.	The angles A and C.	AC : AB :: S, B, S, C. (2.) This caſe admits of two ſolutions; for C may be greater or leſs than a quadrant. (Cor. to def. 4.)
3	AB, AC, and A, two ſides, and the included angle.	The angles B and C.	AB+AC : AB—AC :: T, $\frac{C+B.}{2}$: T, $\frac{C-B}{2}$: (3.) the ſum and difference of the angles C, B, being given, each of them is given. (7.) *Otherwiſe.* FIG. 18. BA : AC :: R : T, ABC, and alſo R : T, ABC—45° : T, $\frac{B+C}{2}$: T, $\frac{B-C}{2}$: (4.) therefore B and C are given as before. (7.)

Fig. 16, 17.

	GIVEN.	SOUGHT.	
4	AB, BC, CA, the three ſides.	A, B, C, the three angles.	$2\ AC\times CB : ACq+CBq-ABq :: R : Co\ S,\ C.$ If $ABq+CBq$ be greater than ABq. FIG. 16. $2\ AC\times CB : ABq-ACq-CBq+ :: R : Co\ S,\ C.$ If ABq be greater than $ACq\times CBq$. FIG. (17. 4.) *Otherwiſe.* Let $AB+BC+AC=2P$. $P+\overline{P-AB} : \overline{P-AC}+\overline{P-BC} :: Rq : Tq,\ \frac{1}{2}\ C$, and hence C is known. (5.) *Otherwiſe.* Let AD be perpendicular to BC. 1. If ABq be leſs than $ACq+CBq$. FIG. 16. $BC : BA+AC :: BA-AC : BD-DC$, and BC the ſum of BD, DC is given; therefore each of them is given. (7.) 2. If ABq be greater than $ACq+CBq$. FIG. 17. $BC : BA+AC :: BA-AC : BD+DC$; and BC the difference of BD, DC is given, therefore each of them is given. (7.) And $CA : CD :: R : Co\ S,\ C.$ (1.) and C being found, A and B are found by caſe 2. or 3.

SPHERICAL

SPHERICAL TRIGONOMETRY.

DEFINITIONS.

I.

THE pole of a circle of the ſphere is a point in the ſuperficies of the ſphere, from which all ſtraight lines drawn to the circumference of the circle are equal.

II.

A great circle of the ſphere is any whoſe plane paſſes through the centre of the ſphere, and whoſe centre therefore is the ſame with that of the ſphere.

III.

A ſpherical triangle is a figure upon the ſuperficies of a ſphere comprehended by three arches of three great circles, each of which is leſs than a ſemicircle.

IV.

A ſpherical angle is that which on the ſuperficies of a ſphere is contained by two arches of great circles, and is the ſame with the inclination of the planes of theſe great circles.

PROP. I.

GREAT circles biſect one another.

As they have a common centre their common ſection will be a diameter of each which will biſect them.

PROP. II. FIG. 1.

THE arch of a great circle betwixt the pole and the circumference of another is a quadrant.

Let ABC be a great circle, and D its pole; if a great circle DC paſs through D, and meet ABC in C, the arch DC will be a quadrant.

Let the great circle CD meet ABC again in A, and let AC be the common ſection of the great circles, which will paſs through E the centre of the ſphere: Join DE, DA, DC: By def. 1. DA, DC are equal, and AE, EC are alſo equal, and DE is common; therefore (8. 1.) the angles DEA, DEC are equal; wherefore the arches DA, DC are equal, and conſequently each of them is a quadrant. Q. E. D.

PROP. III. Fig. 2.

IF a great circle be deſcribed meeting two great circles AB, AC paſſing through its pole A in B, C, the angles at the centre of the ſphere upon the circumference BC, is the ſame with the ſpherical angle BAC, and the arch BC is called the meaſure of the ſpherical angle BAC.

Let the planes of the great circles AB, AC interſect one another in the ſtraight line AD paſſing through D their common centre; join DB, DC.

Since A is the pole of BC, AB, AC will be quadrants, and the angles ADB, ADC right angles; therefore (6. def. 11.) the angle CDB is the inclination of the planes of the circles AB, AC; that is, (def. 4.) the ſpherical angle BAC. Q. E. D.

Cor. If through the point A, two quadrants AB, AC, be drawn, the point A will be the pole of the great circle BC, paſſing through their extremities B, C.

Join AC, and draw AE a ſtraight line to any other point E in BC; join DE: Since AC, AB are quadrants, the angles ADB, ADC are right angles, and AD will be perpendicular to the plane of BC: Therefore the angle ADE is a right angle, and AD, DC are equal to AD, DE, each to each; therefore AE, AC, are equal, and A is the pole of BC, by def. 1. Q. E. D.

PROP. IV. Fig. 3.

IN iſoſceles ſpherical triangles, the angles at the baſe are equal.

Let ABC be an iſoſceles triangle, and AC, CB the equal ſides; the angles BAC, ABC, at the baſe AC, are equal.

Let D be the centre of the ſphere, and join DA, DB, DC; in DA take any point E, from which draw, in the plane ADC, the ſtraight line EF at right angles to ED meeting CD in F, and draw, in the plane ADB, EG at right angles to the ſame ED; therefore the rectilineal angle FEG is (6. def. 11.) the inclination of the planes ADC, ADB, and therefore is the ſame with the ſpherical angle BAC: From F draw FH perpendicular to DB, and from H draw, in the plane ADB, the ſtraight line HG at right angles to HD meeting EG in G, and join GF. Becauſe DE is at right angles to EF and EG, it is perpendicular to the plane EFG, (4. 11.) and therefore the plane FEG is perpendicular to the plane ADB, in which DE is: (18. 11.) In the ſame manner the plane FHG is perpendicular to the plane ADB; and therefore GF the common ſection of the planes FEG, FHG is perpendicular to the plane ADB; (19. 11.) and becauſe the angle FHG is the inclination of the planes BDC, BDA, it is the ſame with the ſpherical angle ABC; and the ſides AC, CB of the ſpherical triangle being equal, the angles EDF, HDF, which ſtand upon them at the centre of the ſphere, are equal; and in the triangles EDF, HDF, the ſide DF is common, and the angles DEF, DHF are right angles; therefore EF, FH are equal; and in the triangles FEG, FHG the ſide GF is common, and the ſides EG, GH will be equal by the 47. 1. and therefore the angle FEG is equal to FHG; (8. 1.) that is, the ſpherical angle BAC is equal to the ſpherical angle ABC.

PROP. V. Fig. 3.

If, in a ſpherical triangle ABC, two of the angles BAC, ABC, be equal, the ſides BC, AC oppoſite to them are equal.

Read the conſtruction and demonſtration of the preceding propoſition, unto the words, "and the ſides of AC, CB," &c. and the reſt of the demonſtration will be as follows, viz.

And the ſpherical angles BAC, ABC, being equal, the rectilineal angles FEG, FHG, which are the ſame with them, are equal; and in the triangles FGE, FGH the angles at G are right angles, and the ſide FG oppoſite to two of the equal angles is common; therefore (26. 1.) EF is equal to FH: And in the right angled triangles DEF, DHF the ſide DF is common; wherefore (47. 1.) ED is equal to DH, and the angles EDF, HDF, are therefore equal, (4. 1.) and conſequently the ſides AC, BC of the ſpherical triangle are equal.

PROP. VI. FIG. 4.

ANY two ſides of a ſpherical triangle are greater than the third.

Let ABC be a ſpherical triangle, any two ſides AB, BC will be greater than the other ſide AC.

Let D be the centre of the ſphere: Join DA, DB, DC.

The ſolid angle at D, is contained by three plane angles, ADB, ADC, BDC; and by 20. 11. any two of them ADB, BDC are greater than the third ADC; that is, any two ſides AB, BC of the ſpherical triangle ABC, are greater than the third AC.

PROP. VII. FIG. 4.

THE three ſides of a ſpherical triangle are leſs than a circle.

Let ABC be a ſpherical triangle as before, the three ſides AB, BC, AC are leſs than a circle.

Let D be the centre of the ſphere: The ſolid angle at D is contained by three plane angles BDA, BDC, ADC, which together are leſs than four right angles, (21. 11.) therefore the ſides AB, BC, AC together, will be leſs than four quadrants; that is, leſs than a circle.

PROP. VIII. FIG. 5.

IN a ſpherical triangle the greater angle is oppoſite to the greater ſide; and converſely.

Let ABC be a ſpherical triangle, the greater angle A is oppoſed to the greater ſide BC.

Let the angle BAD be made equal to the angle B, and then BD, DA will be equal, (5. of this) and therefore AD, DC are equal to BC; but AD, DC are greater than AC, (6. of this), therefore BC is greater than AC, that is, the greater angle A is oppoſite to the greater ſide BC. The converſe is demonſtrated as prop. 19. 1. El. Q. E. D.

PROP. IX. FIG. 6.

IN any ſpherical triangle ABC, if the ſum of the ſides AB, BC, be greater, equal, or leſs than a ſemicircle, the internal angle at the baſe AC will be greater, equal, or leſs than the external and oppoſite BCD; and therefore the ſum of the angles A and ACB will be greater, equal, or leſs than two right angles.

Let AC, AB produced meet in D.

1. If AB, BC be equal to a ſemicircle, that is, to AD, BC, BD will be equal, that is, (4. of this) the angle D, or the angle A will be equal to the angle BCD.

2. If AB, BC together be greater than a ſemicircle, that is greater than ABD, BC will be greater than BD; and therefore (8. of this) the angle D, that is, the angle A, is greater than the angle BCD.

3. In the ſame manner is it ſhown, that if AB, BC together be leſs than a ſemicircle, the angle A is leſs than the angle BCD. And ſince the angles BCD, BCA are equal to two right angles, if the angle A be greater than BCD, A and ACB together will be greater than two right angles. If A be equal to BCD, A and ACB together will be equal to two right angles; and if A be leſs than BCD, A and ACB will be leſs than two right angles. Q. E. D.

PROP. X. FIG. 7.

IF the angular points A, B, C of the ſpherical triangle ABC be the poles of three great circles, theſe great circles by their interſections will form another triangle FDE, which is called ſupplemental to the former; that is, the ſides FD, DE, EF are the ſupplements of the meaſures of the oppoſite angles C, B, A, of the triangle ABC, and the meaſures of the angles F, D, E of the triangle FDE, will be the ſupplements of the ſides AC, BC, BA, in the triangle ABC.

Let AB produced meet DE, EF, in G, M, and AC meet FD, FE in K, L, and BC meet FD, DE in N, H.

Since A is the pole of FE, and the circle AC paſſes through A, EF will paſs through the pole of AC, (13. 15. 1. Th.) and ſince AC paſſes through C, the pole of FD, FD will paſs through the pole of AC; therefore the pole of AC is in the point F, in which the arches DF, EF interſect each other. In the ſame manner, D is the pole of BC, and E the pole of AB.

And ſince F, E are the poles of AL, AM, FL and EM are quadrants, and FL, EM together, that is, FE and ML together, are equal to a ſemicircle. But ſince A is the pole of ML, ML is the meaſure of the angle BAC, conſequently FE is the ſupplement of the meaſure of the angle BAC. In the ſame manner, ED, DF are the ſupplements of the meaſures of the angles ABC, BCA.

Since likewiſe CN, BH are quadrants, CN, BH together, that is, NH, BC together are equal to a ſemicircle; and ſince D is the pole of NH, NH is the meaſure of the angle FDE, therefore the meaſure of the angle FDE is the ſupplement of the ſide BC. In the ſame manner, it is ſhown that the meaſures of the angles DEF, EFD are the ſupplements of the ſides AB, AC, in the triangle ABC. Q. E. D.

PROP. XI. Fig. 7.

THE three angles of a ſpherical triangle are greater than two right angles, and leſs than ſix right angles.

The meaſures of the angles A, B, C, in the triangle ABC, together with the three ſides of the ſupplemental triangle DEF, are (10. of this) equal to three ſemicircles; but the three ſides of the triangle FDE, are (7. of this) leſs than two ſemicircles;

therefore the meaſures of the angles A, B, C are greater than a ſemicircle; and hence the angles A, B, C are greater than two right angles.

All the external and internal angles of any triangle are equal to ſix right angles; therefore all the internal angles are leſs than ſix right angles.

PROP. XII. Fig. 8.

IF from any point C, which is not the pole of the great circle ABD, there be drawn arches of great circles CA, CD, CE, CF, &c. the greateſt of theſe is CA, which paſſes through H the pole of ABD, and CB the remainder of ACB is the leaſt, and of any others CD, CE, CF, &c. CD, which is nearer to CA, is greater than CE, which is more remote.

Let the common ſection of the planes of the great circles ACB, ADB be AB; and from C, draw CG perpendicular to AB, which will alſo be perpendicular to the plane ADB; (4. def. 11.) join GD, GE, GF, CD, CE, CF, CA, CB.

Of all the ſtraight lines drawn from G to the circumference ADB, GA is the greateſt, and GB the leaſt; (7. 3.) and GD, which is nearer to GA, is greater than GE, which is more remote. The triangles CGA, CGD are right-angled at G, and they have the common ſide CG; therefore the ſquares of CG, GA together, that is, the ſquare of CA, is greater than the ſquares of CG, GD together, that is, the ſquare of CD: And CA is greater than CD, and therefore the arch CA is greater than CD. In the ſame manner, ſince GD is greater than GE, and GE than GF, &c. it is ſhown that CD is greater than CE, and CE than CF, &c. and conſequently, the arch CD greater than the arch CE, and the arch CE greater than the arch CF, &c. And ſince GA is the greateſt, and GB the leaſt of all the ſtraight lines drawn from G to the circumference ADB, it is manifeſt that CA is the greateſt, and CB the leaſt of all the ſtraight lines drawn from C to the circumference: And therefore the arch CA is the greateſt, and CB the leaſt of all the circles drawn through C, meeting ADB. Q. E. D.

PROP. XIII. Fig. 9.

In a right-angled ſpherical triangle the ſides are of the ſame affection with the oppoſite angles; that is, if the ſides be greater or leſs than quadrants, the oppoſite angles will be greater or leſs than right angles.

Let ABC be a ſpherical triangle right-angled at A, any ſide AB, will be of the ſame affection with the oppoſite angle ACB.

Caſe 1. Let AB be leſs than a quadrant, let AE be a quadrant, and let EC be a great circle paſſing through E, C. Since A is a right angle, and AE a quadrant, E is the pole of the great circle AC, and ECA a right angle; but ECA is greater than BCA, therefore BCA is leſs than a right angle. Q. E. D.

Fig. 10. Caſe 2. Let AB be greater than a quadrant, make AE a quadrant, and let a great circle paſs through C, E, ECA is a right angle as before, and BCA is greater than ECA, that is, greater than a right angle. Q. E. D.

PROP. XIV.

If the two ſides of a right-angled ſpherical triangle be of the ſame affection, the hypothenuſe will be leſs than a quadrant; and if they be of different affection, the hypothenuſe will be greater than a quadrant.

Let ABC be a right-angled ſpherical triangle, if the two ſides AB, AC be of the ſame or of different affection, the hypothenuſe BC will be leſs or greater than a quadrant.

Fig. 9 Caſe 1. Let AB, AC be each leſs than a quadrant. Let AE, AG be quadrants; G will be the pole of AB, and E the pole of AC, and EC a quadrant; but, by prop. 12. CE is greater than CB, ſince CB is farther off from CGD than CE. In the ſame manner, it is ſhown that CB, in the triangle CBD, where the two ſides CD, BD are each greater than a quadrant, is leſs than CE, that is, leſs than a quadrant. Q. E. D.

Caſe 2. Let AC be leſs, and AB greater than a quadrant; Fig. 10. then the hypothenuſe BC will be greater than a quadrant; for let AE be a quadrant, then E is the pole of AC, and EC will be a quadrant. But CB is greater than CE by prop. 12. ſince AC paſſes through the pole of ABD. Q. E. D.

PROP. XV.

IF the hypothenuſe of a right-angled triangle be greater or leſs than a quadrant, the ſides will be of different or the ſame affection.

This is the converſe of the preceding, and demonſtrated in the ſame manner.

PROP. XVI.

IN any ſpherical triangle ABC, if the perpendicular AD from A upon the baſe BC, fall within the triangle, the angles B and C at the baſe will be of the ſame affection; and if the perpendicular fall without the triangle, the angles B and C will be of different affection.

1. Let AD fall within the triangle; then (13. of this) ſince Fig. 11. ADB, ADC are right-angled ſpherical triangles, the angles B, C muſt each be of the ſame affection as AD.

2. Let AD fall without the triangle, then (13. of this) the Fig. 12. angle B is of the ſame affection as AD; and by the ſame the angle ACD is of the ſame affection as AD; therefore the angle ACB and AD are of different affection, and the angles B and ACB of different affection.

Cor. Hence if the angles B and C be of the ſame affection, the perpendicular will fall within the baſe; for, if it did not, (16. of this) B and C would be of different affection. And if the angles B and C be of oppoſite affection, the perpendicular will fall without the triangle; for, if it did not, (16. of this), the angles B and C would be of the ſame affection, contrary to the ſuppoſition.

PROP. XVII. Fig. 13.

IN right-angled ſpherical triangles, the ſine of either of the ſides about the right angle, is to the radius of the ſphere, as the tangent of the remaining ſide is to the tangent of the angle oppoſite to that ſide.

Let ABC be a triangle, having the right angle at A; and let AB be either of the ſides, the ſine of the ſide AB will be to the radius, as the tangent of the other ſide AC to the tangent of the angle ABC, oppoſite to AC. Let D be the centre of the ſphere; join AD, BD, CD, and let AE be drawn perpendicular to BD, which therefore will be the ſine of the arch AB, and from the point E, let there be drawn in the plane BDC the ſtraight line EF at right angles to BD, meeting DC in F, and let AF be joined. Since therefore the ſtraight line DE is at right angles to both EA and EF, it will alſo be at right angles to the plane AEF, (4. 11.) wherefore the plane ABD, which paſſes through DE is perpendicular to the plane AEF, (18. 11.) and the plane AEF perpendicular to ABD: The plane ACD or AFD is alſo perpendicular to the ſame ABD: Therefore the common ſection, viz. the ſtraight line AF, is at right angles to the plane ABD: (19. 11.) And FAE, FAD are right angles; (3. def. 11.) therefore AF is the tangent of the arch AC; and in the rectilineal triangle AEF having a right angle at A, AE will be to the radius as AF to the tangent of the angle AEF, (1 Pl. Tr.); but AE is the ſine of the arch AB, and AF the tangent of the arch AC, and the angle AEF is the inclination of the planes CBD, ABD, (6. def. 11.) or the ſpherical angle ABC: Therefore the ſine of the arch AB is to the radius as the tangent of the arch AC, to the tangent of the oppoſite angle ABC.

Cor. 1. If therefore of the two ſides, and an angle oppoſite to one of them, any two be given, the third will alſo be given.

Cor. 2. And ſince by this propoſition the ſine of the ſide AB is to the radius, as the tangent of the other ſide AC to the tangent of the angle ABC oppoſite to that ſide; and as the radius is to the co-tangent of the angle ABC, ſo is the tangent of the ſame angle ABC to the radius, (Cor. 2. def. Pl. Tr.) by equality, the ſine of the ſide AB is to the co-tangent of the angle ABC adjacent to it, as the tangent of the other ſide AC to the radius.

PROP. XVIII. Fig. 13.

IN right-angled ſpherical triangles the ſine of the hypothenuſe is to the radius, as the ſine of either ſide is to the ſine of the angle oppoſite to that ſide.

Let the triangle ABC be right-angled at A, and let AC be either of the ſides; the ſine of the hypothenuſe BC will be to the radius as the ſine of the arch AC is to the ſine of the angle ABC.

Let D be the centre of the ſphere, and let CG be drawn perpendicular to DB, which will therefore be the ſine of the hypothenuſe BC; and from the point G let there be drawn in the plane ABD the ſtraight line GH perpendicular to DB, and let CH be joined; CH will be at right angles to the plane ABD, as was ſhown in the preceding propoſition of the ſtraight line FA: Wherefore CHD, CHG are right angles, and CH is the ſine of the arch AC; and in the triangle CHG, having the right angle CHG, CG is to the radius as CH to the ſine of the angle CGH: (1. Pl. Tr.) But ſince CG, HG are at right angles to DGB, which is the common ſection of the planes CBD, ABD, the angle CGH will be equal to the inclination of theſe planes; (6. def. 11.) that is, to the ſpherical angle ABC. The ſine therefore, of the hypothenuſe CB is to the radius as the ſine of the ſide AC is to the ſine of the oppoſite angle ABC. Q. E. D.

Cor. Of theſe three, viz. the hypothenuſe, a ſide, and the angle oppoſite to that ſide, any two being given, the third is alſo given by prop. 2.

PROP. XIX. FIG. 14.

IN right-angled ſpherical triangles, the co-ſine of the hypothenuſe is to the radius as the co-tangent of either of the angles is to the tangent of the remaining angle.

Let ABC be a ſpherical triangle, having a right angle at A, the co-ſine of the hypothenuſe BC will be to the radius as the co-tangent of the angle ABC to the tangent of the angle ACB.

Deſcribe the circle DE, of which B is the pole, and let it meet AC in F and the circle BC in E; and ſince the circle BD paſſes through the pole B of the circle DF, DF will alſo paſs through the pole of BD. (13. 18. 1. Theod. Sph.) And ſince AC is perpendicular to BD, AC will alſo paſs through the pole of BD; wherefore the pole of the circle BD will be found in the point where the circles AC, DE meet, that is, in the point F: The arches FA, FD are therefore quadrants, and likewiſe the arches BD, BE: In the triangle CEF, right-angled at the point E, CE is the complement of the hypothenuſe BC of the triangle ABC, EF is the complement of the arch ED, which is the meaſure of the angle ABC, and FC the hypothenuſe of the triangle CEF, is the complement of AC, and the arch AD, which is the meaſure of the angle CFE, is the complement of AB.

But (17. of this) in the triangle CEF, the ſine of the ſide CE is to the radius, as the tangent of the other ſide is to the tangent of the angle ECF oppoſite to it, that is, in the triangle ABC, the co-ſine of the hypothenuſe BC is to the radius, as the co-tangent of the angle ABC is to the tangent of the angle ACB. Q. E. D.

Cor. 1. Of theſe three, viz. the hypothenuſe and the two angles, any two being given, the third will alſo be given.

Cor. 2. And ſince by this propoſition the co-ſine of the hypothenuſe BC is to the radius, as the co-tangent of the angle ABC to the tangent of the angle ACB. But as the radius is to the co-tangent of the angle ACB, ſo is the tangent of the ſame to the radius; (Cor. 2. def. Pl. Tr.) and, ex æquo, the co-ſine of the hypothenuſe BC is to the co-tangent of the angle ACB, as the co-tangent of the angle ABC to the radius.

PROP. XX. Fig. 14.

IN right-angled ſpherical triangles, the co-ſine of an angle is to the radius, as the tangent of the ſide adjacent to that angle is to the tangent of the hypothenuſe.

The ſame conſtruction remaining; in the triangle CEF, (17. of this) the ſine of the ſide EF is to the radius, as the tangent of the other ſide CE is to the tangent of the angle CFE oppoſite to it; that is, in the triangle ABC, the co-ſine of the angle ABC is to the radius as (the co-tangent of the hypothenuſe BC to the co-tangent of the ſide AB, adjacent to ABC or as) the tangent of the ſide AB to the tangent of the hypothenuſe, ſince the tangents of two arches are reciprocally proportional to their co-tangents. (Cor. 1. def. Pl. Tr.)

Cor. And ſince by this propoſition the co-ſine of the angle ABC is to the radius, as the tangent of the ſide AB is to the tangent of the hypothenuſe BC; and as the radius is to the co-tangent of BC, ſo is the tangent of BC to the radius; by equality, the co-ſine of the angle ABC will be to the co-tangent of the hypothenuſe BC, as the tangent of the ſide AB, adjacent to the angle ABC, to the radius.

PROP. XXI. Fig. 14.

IN right-angled ſpherical triangles, the co-ſine of either of the ſides is to the radius, as the co-ſine of the hypothenuſe is to the co-ſine of the other ſide.

The ſame conſtruction remaining; in the triangle CEF, the ſine of the hypothenuſe CF is to the radius, as the ſine of the ſide CE to the ſine of the oppoſite angle CFE; (18. of this) that is, in the triangle ABC the co-ſine of the ſide CA is to the radius as the co-ſine of the hypothenuſe BC to the co-ſine of the other ſide BA. Q. E. D.

PROP. XXII. Fig. 14.

IN right-angled ſpherical triangles, the co-ſine of either of the ſides is to the radius, as the co-ſine of the angle oppoſite to that ſide is to the ſine of the other angle.

The ſame conſtruction remaining; in the triangle CEF, the ſine of the hypothenuſe CF is to the radius as the ſine of the ſide EF is to the ſine of the angle ECF oppoſite to it; that is in the triangle ABC, the co-ſine of the ſide CA is to the radius, as the co-ſine of the angle ABC oppoſite to it, is to the ſine of the other angle. Q. E. D.

Of the CIRCULAR PARTS.

IN any right-angled ſpherical triangle ABC, the complement of the hypothenuſe, the complements of the angles and the two ſides are called *The circular parts of the triangle*, as if it were following each other in a circular order, from whatever part we begin: Thus, if we begin at the complement of the hypothenuſe, and proceed towards the ſide BA, the parts following in order will be the complement of the hypothenuſe, the complement of the angle B, the ſide BA the ſide AC, (for the right angle at A is not reckoned among the parts), and, laſtly, the complement of the angle C. And thus at whatever part we begin, if any three of theſe five be taken, they either will be all contiguous or adjacent, or one of them will not be contiguous to either of the other two: In the firſt caſe, the part which is between the other two is called the *Middle part*, and the other two are called *Adjacent extremes*. In the ſecond caſe, the part which is not contiguous to either of the other two is called the *Middle part*, and the other two *Oppoſite extremes*. For example, if the three parts be the complement of the hypothenuſe BC, the complement of the angle B, and the ſide BA; ſince theſe three are contiguous to each other, the complement of the angle B will be the middle part, and the complement of the hypothenuſe BC and the ſide BA will be adjacent extremes: But if the complement of the hypothenuſe BC, and the ſides BA, AC be taken; ſince the complement of the hypothenuſe is not adjacent to either of the ſides, viz. on account of the complements of the two angles B and C intervening between it and the ſides, the complement of the hypothenuſe BC will be the middle part, and the ſides, BA, AC oppoſite extremes. The moſt acute and ingenious Baron Napier, the inventor of Logarithims, contrived the two following rules concerning theſe parts, by means of which all the caſes of right-angled ſpherical triangles are reſolved with the greateſt eaſe. Fig. 15.

RULE I.

The rectangle contained by the radius and the ſine of the middle part, is equal to the rectangle contained by the tangents of the adjacent parts.

RULE II.

The rectangle contained by the radius, and the ſine of the middle part is equal to the rectangle contained by the coſines of the oppoſite parts.

Theſe rules are demonſtrated in the following manner:

Fig. 16. Firſt, Let either of the ſides, as BA, be the middle part, and therefore the complement of the angle B, and the ſide AC will be adjacent extremes. And by cor. 2. prop. 17. of this, S, BA is to the Co-T, B, as T, AC is to the radius, and therefore R×S, BA=Co-T, B×T, AC.

The ſame ſide BA being the middle part, the complement of the hypothenuſe, and the complement of the angle C, are oppoſite extremes; and by prop. 18. S. BC is to the radius, as S, BA to S, C; therefore R×S, BA=S, BC×S, C.

Secondly, Let the complement of one of the angles, as B, be the middle part, and the complement of the hypothenuſe, and the ſide BA will be adjacent extremes: And by cor. prop. 20. Co-S, B is to Co-T, BC, as T, BA is to the radius, and therefore R×Co-S, B=Co-T, BC×T, BA.

Again, Let the complement of the angle B be the middle part, and the complement of the angle C, and the ſide AC will be oppoſite extremes: And by prop. 22. Co-S, AC is to the radius, as Co-S, B is to S, C: And therefore R×Co-S, B=Co-S, AC×S, C.

Thirdly, Let the complement of the hypothenuſe be the middle part, and the complements of the angles B, C, will be adjacent extremes: But by cor. 2. prop. 19. Co-S, BC is to Co-T, B as Co-T, B to the radius: Therefore R×Co , BC=Co-T, C×Co-T, C.

Again, Let the complement of the hypothenuſe be the middle part, and the ſides AB, AC will be oppoſite extremes: But by prop. 21. Co-S, AC is to the radius, Co-S, BC to Co-S, BA; therefore R×Co-S, BC=Co-S, BA×Co-S, AC. Q. E. D.

SOLUTION of the Sixteen CASES of RIGHT-ANGLED SPHERICAL TRIANGLES.

GENERAL PROPOSITION.

IN a right-angled ſpherical triangle, of the three ſides and three angles, any two being given, beſides the right angle, the other three may be found.

In the following Table the ſolutions are derived from the preceding propoſitions. It is obvious that the ſame ſolutions may be derived from Baron Napier's two rules above demonſtrated, which, as they are eaſily remembered, are commonly uſed in practice.

Caſe.	Given.	Sought.	
1	AC, C	B	R : CoS, AC : : S, C : CoS, B : And B is of the ſame ſpecies with CA, by 22. and 13.
2	AC, B	C	CoS, AC : R : : CoS, B : S, C : By 22.
3	B, C	AC	S, C : CoS, B : : R CoS, AC : By 22. and AC is of the ſame ſpecies with B. 13.
4	BA, AC	BC	R : CoS, BA : : CoS, AC : CoS, BC. 21. and if both AB, AC be greater or leſs than a quadrant, BC will be leſs than a quadrant. But if they be of different affection, BC will be greater than a quadrant. 14.
5	BA, BC	AC	CoS, BA : R : : CoS, BC, CoS, AC. 21. and if BC be greater or leſs than a quadrant, BA, AC will be of different or the ſame affection : By 15.
6	BA, AC	B	S, BA : R : : T, CA : T, B. 17. and B is of the ſame affection with AC. 13.

Caſe.	Given.	Sought.	
7	BA, B	AC	R : S, BA : : T, B : T, AC 17. And AC is of the ſame affection with B. 13.
8	AC, B	BA	T, B : R : : T, CA : S, BA. 17.
9	BC, C	AC	R : CoS, C : : T, BC : T, CA. 20. If BC be leſs or greater than a quadrant, C and B will be of the ſame or different affection. 15. 13.
10	AC, C	BC	CoS, C : R : : T, AC : T, BC. 20. And BC is leſs or greater than a quadrant, according as C and AC or C and B are of the ſame or different affection. 14. 1.
11	BC, CA	C	T, BC : R : : T, CA : CoS. C. 20. If BC be leſs or greater than a quadrant. CA and AB, and therefore CA and C, are of the ſame or different affection. 15.
12	BC, B	AC	R : S, BC : : S, B : S, AC. 18. And AC is of the ſame affection with B.
13	AC, B	BC	S, B : S, AC : : R : S, BC : 18.
14	BC, AC	B	S, BC : R : : S, AC : S, B : 18. And B is of the ſame affection with AC.
15	B, C	BC	T. C : R : : CoT, B : CoS, BC. 19. And according as the angles B and C are of different or the ſame affection, BC will be greater or leſs than a quadrant. 14.
16	BC, C	B	R : CoS, BC : : T, C : CoT, B. 19. If BC be leſs or greater than a quadrant, C and B will be of the ſame or different affection. 15.

The ſecond, eight, and thirteenth caſes, which are commonly called ambiguous, admit of two ſolutions: For in theſe it is not determined whether the ſide or meaſure of the angle ſought be greater or leſs than a quadrant.

PROP. XXIII. FIG. 16.

IN ſpherical triangles whether right-angled or oblique-angled, the ſines of the ſides are proportional to the ſines of the angles oppoſite to them.

Firſt, Let ABC be a right-angled triangle, having a right angle at A; therefore by prop. 18. the ſine of the hypothenuſe BC is to the radius (or the ſine of the right angle at A) as the ſine of the ſide AC to the ſine of the angle B. And in like manner, the ſine of BC is to the ſine of the angle A, as the ſine of AB to the ſine of the angle C; wherefore (11. 5.) the ſine of the ſide AC is to the ſine of the angle B, as the ſine of AB to the ſine of the angle C.

Secondly, Let BCD be an oblique-angled triangle, the ſine of either of the ſides BC, will be to the ſine of either of the other two CD, as the ſine of the angle D oppoſite to BC is to the ſine of the angle B oppoſite to the ſide CD. Through the point C, let there be drawn an arch of a great circle CA perpendicular upon BD; and in the right-angled triangle ABC (18. of this) the ſine of BC is to the radius, as the ſine of AC to the ſine of the angle B; and in the triangle ADC (by 18. of this): And, by inverſion, the radius is to the ſine of DC as the ſine of the angle D to the ſine of AC: Therefore ex æquo perturbate, the ſine of BC is to the ſine of DC, as the ſine of the angle D to the ſine of the angle B. Q. E. D. Fig. 17. 18.

PROP. XXIV. FIG. 17. 18.

IN oblique-angled ſpherical triangles having drawn a perpendicular arch from any of the angles upon the oppoſite ſide, the co-ſines of the angles at the baſe are proportional to the ſines of the verticle angles.

Let BCD be a triangle, and the arch CA perpendicular to the baſe BD; the co-ſine of the angle B will be to the co-ſine of the angle D, as the ſine of the angle BCA to the ſine of the angle DCA.

For by 22. the co-ſine of the angle B is to the ſine of the angle BCA as (the co-ſine of the ſide AC is to the radius; that is, by prop. 22. as) the co-ſine of the angle D to the ſine of the angle DCA; and, by permutation, the co-ſine of the angle B is to the co-ſine of the angle D, as the ſine of the angle BCA to the ſine of the angle DCA. Q. E. D.

PROP. XXV. Fig. 17. 18.

THE ſame things remaining, the co-ſines of the ſides BC, CD, are proportional to the co-ſines of the baſes BA, AD.

For by 21. the co-ſine of BC is to the co-ſine of BA, as (the co-ſine of AC to the radius; that is, by 21. as) the co-ſine of CD is to the co-ſine of AD: Wherefore, by permutation, the co-ſines of the ſides BC, CD are proportional to the co-ſines of the baſes BA, AD. Q. E. D.

PROP. XXVI. Fig. 17. 18.

THE ſame conſtruction remaining, the ſines of the baſes BA, AD are reciprocally proportional to the tangents of the angles B and D at the baſe.

For by 17. the ſine of BA is to the radius, as the tangent of AC to the tangent of the angle B; and by 17. and inverſion the radius is to the ſine of AD, as the tangent of D to the tangent of AC: Therefore ex æquo perturbate, the ſine of BA is to the ſine of AD, as the tangent of D to the tangent of B.

PROB.

PROP. XXVII. Fig. 17. 18.

THE co ſines of the vertical angles are reciprocally proportional to the tangents of the ſides.

For by prop. 20. the coſine of the angle BCA, is to the radius as the tangent CA is to the tangent of BC; and by the ſame prop. 20. and by inverſion, the radius is to the co-ſine of the angle DCA, as the tangent of DC to the tangent of CA: Therefore, ex æquo perturbate, the co-ſine of the angle BCA is to the co-ſine of the angle DCA, as the tangent of DC is to the tangent of BC. Q. E. D.

LEMMA. Fig. 19. 20.

IN right-angled plain triangles, the hypothenuſe is to the radius, as the exceſs of the hypothenuſe above either of the ſides to the verſed ſine of the acute angle adjacent to that ſide, or as the ſum of the hypothenuſe, and either of the ſides to the verſed ſine of the exterior angle of the triangle.

Let the triangle ABC have a right angle at B; AC will be to the radius as the exceſs of AC above AB, to the verſed ſine of the angle A adjacent to AB; or as the ſum of AC, AB to the verſed ſine of the exterior angle CAK.

With any radius DE, let a circle be deſcribed, and from D the centre let DF be drawn to the circumference, making the angle EDF equal to the angle BAC, and from the point F, let FG be drawn perpendicular to DE: Let AH, AK be made equal to AC, and DL to DE; DG therefore is the co-ſine of the angle EDF or BAC, and GE its verſed ſine: And becauſe of the equiangular triangles ACB, DFG, AC or AH is to DF or DE, as AB to DG: Therefore (19. 5.) AC is to the radius DE as BH to GE, the verſed ſine of the angle EDF or BAC: And ſince AH is to DE, as AB to DG, (12. 5.) AH or AC will be to the radius DE as KB to LG, the verſed ſine of the angle LDF or KAC. Q. E. D.

PROP. XXVIII. Fig. 21. 22.

In any ſpherical triangle, the rectangle contained by the ſines of two ſides, is to the ſquare of the radius, as the exceſs of the verſed ſines of the third ſide or baſe, and the arch, which is the exceſs of the ſides, is to the verſed ſine of the angle oppoſite to the baſe.

Let ABC be a ſpherical triangle, the rectangle contained by the ſines of AB, BC will be to the ſquare of the radius, as the exceſs of the verſed ſines of the baſe AC, and of the arch, which is the exceſs of AB, BC to the verſed ſine of the angle ABC oppoſite to the baſe.

Let D be the centre of the ſphere, and let AD, BD, CD be joined, and let the ſines AE, CF, CG of the arches AB, BC, AC be drawn; let the ſide BC be greater than BA, and let BH be made equal to BC; AH will therefore be the exceſs of the ſides BC, BA; let HK be drawn perpendicular to AD, and ſince AG is the verſed ſine of the baſe AC, and AK the verſed ſine of the arch AH, KG is the exceſs of the verſed ſines of the baſe AC, and of the arch AH, which is the exceſs of the ſides BC, BA: Let GL likewiſe be drawn parallel to KH, and let it meet FH in L, let CL, DH be joined, and let AD, FH meet each other in M.

Since therefore in the triangles CDF, HDF, DC, DH are equal, DF is common, and the angle FDC equal to the angle FDH, becauſe of the equal arches BC, BH, the baſe HF will be equal to the baſe FC, and the angle HFD equal to the right angle CFD: The ſtraight line DF therefore (4. 11.) is at right angles to the plain CFH: Wherefore the plain CFH is at right angles to the plain BDH, which paſſes through DF, (18. 11.) In like manner, ſince DG is at right angles to both GC and GL, DG will be perpendicular to the plane CGL; therefore the plane CGL is at right angles to the plane BDH, which paſſes through DG: And it was ſhown, that the plane CFH or CFL was perpendicular to the ſame plane BDH; therefore the common ſection of the planes CFL, CGL, viz. the ſtraight line CL, is perpendicular to the plane BDA, (19. 11.) and therefore CLF is a right angle: In the triangle CFL having the right angles CLF, by the lemma CF is to the radius as LH, the exceſs, viz. of CF or FH above FL, is to the verſed ſine of the angle CFL; but the angle CFL is the inclination of the planes BCD, BAD, ſince FC, FL are drawn in them at right angles to the common ſection BF: The ſpherical angle ABC is therefore the ſame with the angle CFL; and therefore CF is to the radius as LH to the verſed ſine of the ſpherical angle ABC; and ſince the triangle AED is equiangular (to the triangle MFD, and therefore) to the triangle MGL, AE will be to the radius of the ſphere AD, (as MG to ML; that is, becauſe of the parallels as) GK to LH: The ratio therefore which is compounded of the ratios of AE to the radius, and of CF to the ſame radius; that is, (23. 6.) the ratio of the rectangle contained by AE, CF to the ſquare of the radius, is the ſame with the ratio compounded of the ratio of GK to LH, and the ratio of LH to the verſed ſine of the angle ABC; that is, the ſame with the ratio of GK to the verſed ſine of the angle ABC; therefore, the rectangle contained by AE, CF, the ſines of the ſides AB, BC, is to the ſquare of the radius as GK, the exceſs of the verſed ſines AG, AK, of the baſe AC, and the arch AH, which is the exceſs of the ſides to the verſed ſine of the angle ABC oppoſite to the baſe AC. Q. E. D.

PROP. XXIX. FIG. 23.

THE rectangle contained by half of the radius, and the exceſs of the verſed ſines of two arches, is equal to the rectangle contained by the ſines of half the ſum, and half the difference of the ſame arches.

Let AB, AC, be any two arches, and let AD be made equal to AC the leſs; the arch DB therefore is the ſum, and the arch CB the difference of AC, AB: Through E the centre of the circle, let there be drawn a diameter DEF, and AE joined, and CD likewiſe perpendicular to it in G; and let BH be perpendicular to AE, and AH will be the verſed ſine of the arch AB, and AG the verſed ſine of AC, and HG the exceſs of theſe verſed ſines: Let BD, BC, BF be joined, and FC alſo meeting BH in K.

Since therefore BH, CG are parallel, the alternate angles BKC, KCG will be equal; but KCG is in a ſemicircle, and therefore a right angle; therefore BKC is a right angle; and in the triangles DFB, CBK, the angles FDB, BCK, in the ſame ſegment are equal, and FBD, BKC are right angles; the triangles DFB, CBK are therefore equiangular; wherefore DF is to DB, as BC to CK, or HG; and therefore the rectangle contained by the diameter DF, and HG is equal to that contained by DB, BC; wherefore the rectangle contained by a fourth part of the diameter, and HG, is equal to that contained by the halves of DB, BC: But half the chord DB is the ſine of half the arch DAB, that is, half the ſum of the arches AB, AC; and half the chord of BC is the ſine of half the arch BC, which is the difference of AB, AC. Whence the propoſition is manifeſt.

PROP. XXX. Fig. 19. 24.

THE rectangle contained by half of the radius, and the verſed ſine of any arch, is equal to the ſquare of the ſine of half the ſame arch.

Let AB be an arch of a circle, C its centre, and AC, CB, BA being joined: Let AB be biſected in D, and let CD be joined, which will be perpendicular to BA, and biſect it in E, (4. 1.) BE or AE therefore is the ſine of the arch DB or AD, the half of AB: Let BF be perpendicular to AC, and AF will be the verſed ſine of the arch BA; but, becauſe of the ſimilar triangles CAE, BAF, CA is to AE as AB, that is, twice AE to AF; and by halving the antecedents, half of the radius CA is to AE the ſine of the arch AD, as the ſame AE to AF the verſed ſine of the arch AB. Wherefore by 16. 6. the propoſition is manifeſt.

PROP. XXXI. Fig. 25.

IN a ſpherical triangle, the rectangle contained by the ſines of the two ſides, is to the ſquare of the radius, as the rectangle contained by the ſine of the arch which is half the ſum of the baſe and the exceſs of the ſides, and the ſine of the arch, which is half the difference of the ſame to the ſquare of the ſine of half the angle oppoſite to the baſe.

Let ABC be a ſpherical triangle, of which the two ſides are AB, BC, and baſe AC, and let the leſs ſide BA be produced, ſo that BD ſhall be equal to BC: AD therefore is the exceſs of BC, BA; and it is to be ſhown, that the rectangle contained by the ſines of BC, BA is to the ſquare of the radius, as the rectangle contained by the ſine of half the ſum of AC, AD, and the ſine of half the difference of the ſame AC, AD to the ſquare of the ſine of half the angle ABC, oppoſite to the baſe AC.

Since by prop. 28. the rectangle contained by the ſines of the ſides BC, BA is to the ſquare of the radius, as the exceſs of the verſed ſines of the baſe AC and AD, to the verſed ſine of the angle B; that is, (1. 6.) as the rectangle contained by half the radius, and that exceſs, to the rectangle contained by half the radius, and the verſed ſine of B; therefore (29. 30.) of this), the rectangle contained by the ſines of the ſides BC, BA is to the ſquare of the radius, as the rectangle contained by the ſine of the arch, which is half the ſum of AC, AD, and the ſine of the arch which is half the difference of the ſame AC, AD is to the ſquare of the ſine of half the angle ABC. Q. E. D.

SOLUTION of the twelve CASES of OBLIQUE-ANGLED SPHERICAL TRIANGLES.

GENERAL PROPOSITION.

IN an oblique-angled ſpherical triangle, of the three ſides and three angles, any three being given, the other three may be found.

Fig. 26. 27.

	Given.	Sought.	
1	B, D, and BC, two angles and a ſide oppoſite one of them.	C.	CoS, BC : R : : CoT, B : T, BCA. 19. Likewiſe by 24. CoS, B : S, BCA : : CoS, D : S, DCA; wherefore BCD is the ſum or difference of the angles DCA, BCA according as the perpendicular CA falls within or without the triangle BCD; that is, (16. of this,) according as the angles B, D are of the ſame or different affection.
2	B, C, and BC two angles and the ſide between them.	D.	CoS, BC : R : : CoT, B : T, BCA, 19. and alſo by 24. S, BCA : S, DCA : : CoS, B : CoS, D; and according as the angle BCA is leſs or greater than BCD, the perpendicular CA falls within or without the triangle BCD; and therefore (16. of this,) the angles B, D will be of the ſame or different affection.
3	BC, CD, and B.	BD.	R : CoS, B : : T, BC : T, BA. 20. and CoS, BC : CoS, BA : : CoS, DC : CoS, DA. 25. and BD is the ſum or difference of BA, DA.
4	BC, DB, and B.	CD.	R : CoS, B : : T, BC : T, BA. 20. and CoS, BA : CoS, BC : : CoS, DA : CoS, DC. 25. and according as DA, AC are of the ſame or different affection, DC will be leſs or greater than a quadrant. 14.

	Given.	Sought.	
5	B, D and BC.	DB.	R : CoS, B :: T, BC : T, BA. 20. and T, D : T, B :: S, BA : S, DA. 26. and BD is the ſum or difference of BA DA.
6	BC, BD and B.	D.	R : CoS, B :: T, BC : T, BA. 20. and S, DA : S, BA :: T, B : T, D; and according as BD is greater or leſs than BA, the angles B, D are of the ſame or different affection. 16.
7	BC, DC and B.	C.	CoS, BC : R :: CoT, B : T, BCA. 19. and T, DC : T, BC :: CoS, BCA : CoS, DCA, 27. the ſum or difference of the angles BCA, DCA is equal to the angle BCD.
8	B, C and BC.	DC.	CoS, BC : R :: CoT, B : T, BCA, 19. alſo by 27. CoS, DCA : CoS, BCA :: T, BC : T, DC. 27. if DCA and B be of the ſame affection; that is, (13.) if AD and CA be ſimilar, DC will be leſs than a quadrant. 14. and if AD, CA be not of the ſame affection, DC is greater than a quadrant. 14.
9	BC, DC and B.	D.	S, CD : S, B :: S, BC : S, D.
10	B, D and BC.	DC.	S, D : S, BC :: S, B : S, DC.
11	BC, BA, AC, Fig. 25.	B.	S, AB×S, BC : Rq :: S, $\frac{AC+AD}{2}$ ×S, $\frac{AC—AD}{2}$: Sq $\frac{ABC}{2}$. See Fig. 24. AD being the difference of the ſides BC, BA.

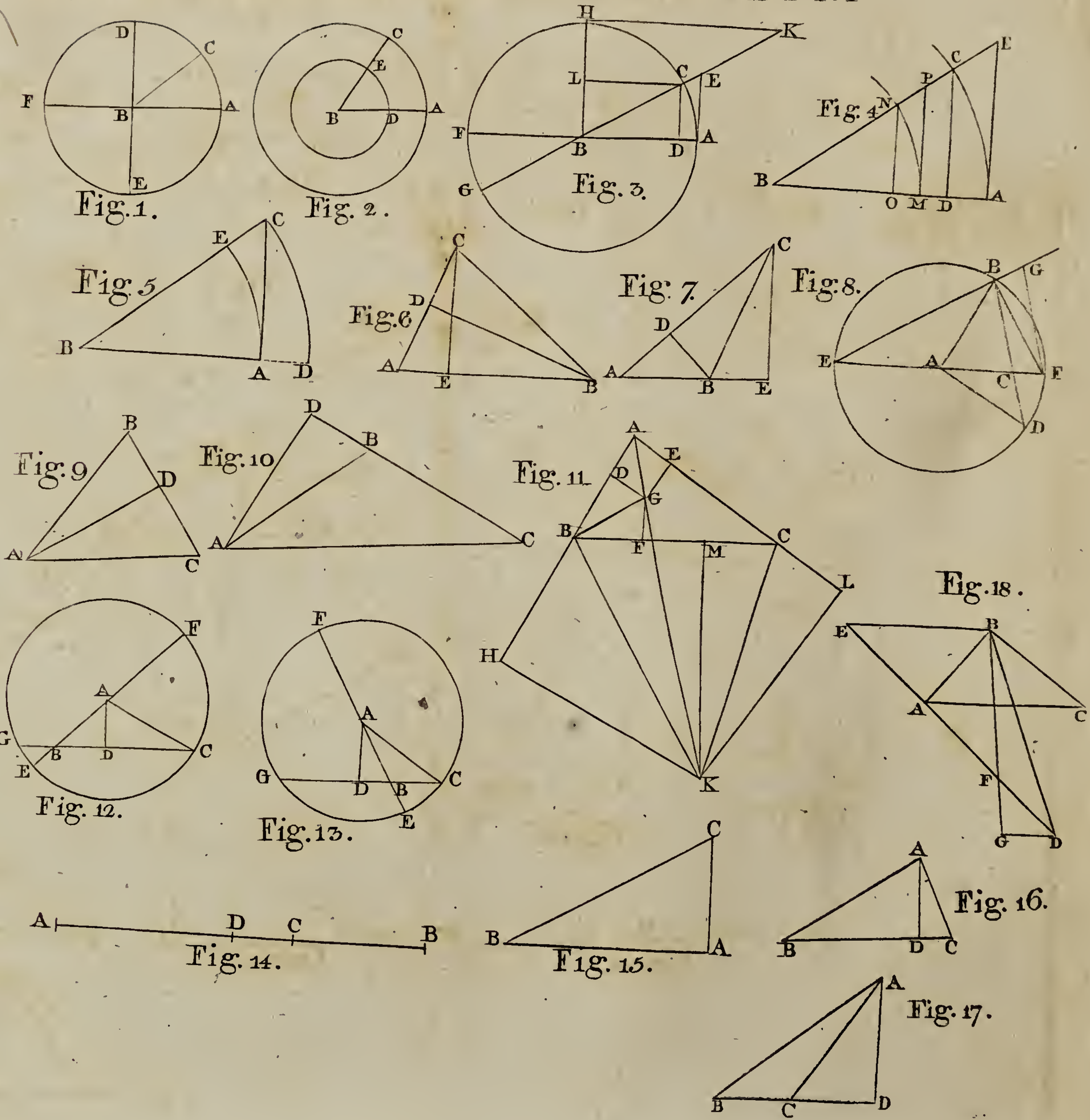
	Given.	Sought.	
12	A, B, C. Fig. 7.	The ſides.	See Fig. 7. In the triangles DEF, DE, EF, FD are reſpectively the ſupplements of the meaſures of the given angles B, A, C in the triangle BAC; the ſides of the triangle DEF are therefore given, and by the preceding caſe the angles D, E, F may be found, and the ſides BC, BA, AC, are the ſupplements of the meaſures of theſe angles.

The 3d, 5th, 7th, 9th, 10th, caſes which are commonly called ambiguous, admit of two ſolutions, either of which will anſwer the conditions required; for, in theſe caſes, the meaſure of the angle or ſide ſought, may be either greater or leſs than a quadrant, and the two ſolutions will be ſupplements to each other. (Cor. to def. 4. 6. Pl. Tr.)

If from any of the angles of an oblique-angled ſpherical triangle, a perpendicular arch be drawn upon the oppoſite ſide, moſt of the caſes of oblique-angled triangles may be reſolved by means of Napier's rules.

FINIS.

PLANE TRIGONOMETRY



SPHERICAL TRIGONOMETRY Plate 1st.

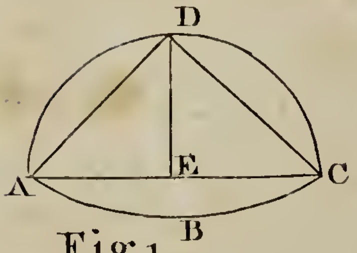


Fig. 1.

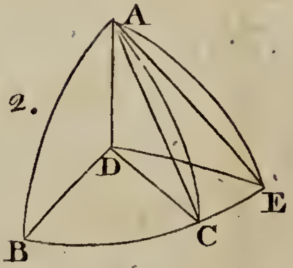


Fig. 2.

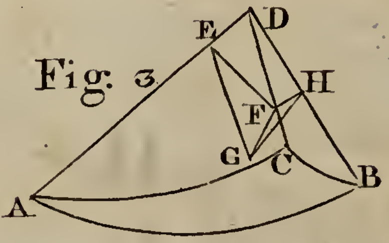


Fig. 3

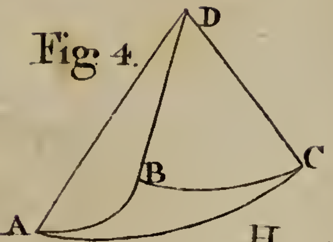


Fig. 4.

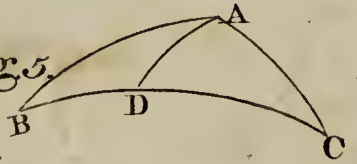


Fig. 5.

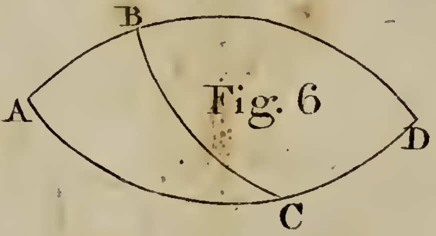


Fig. 6

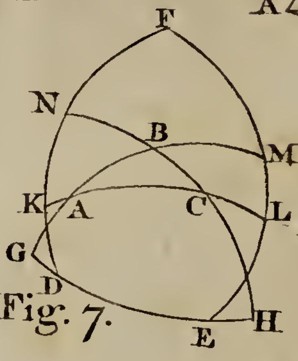


Fig. 7.

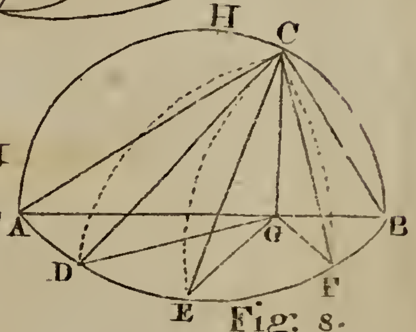


Fig. 8.

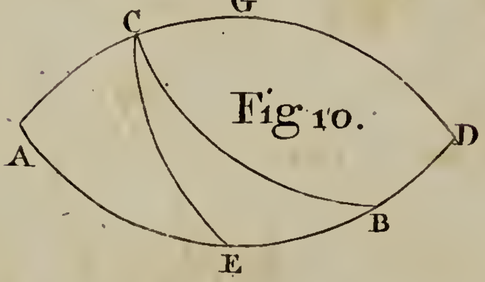


Fig. 10.

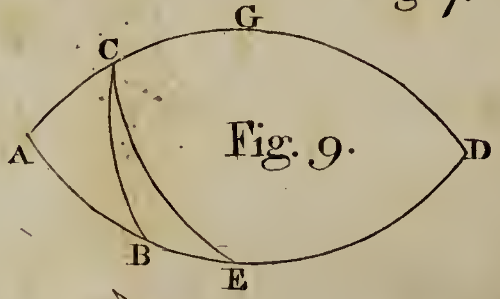


Fig. 9.

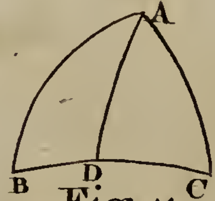


Fig. 11.

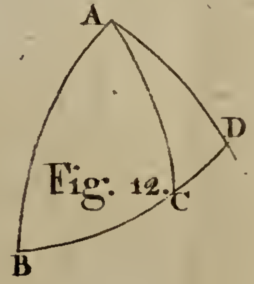


Fig. 12.

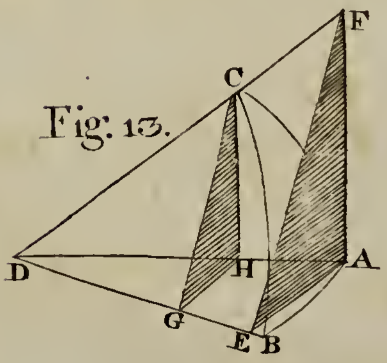


Fig. 13.

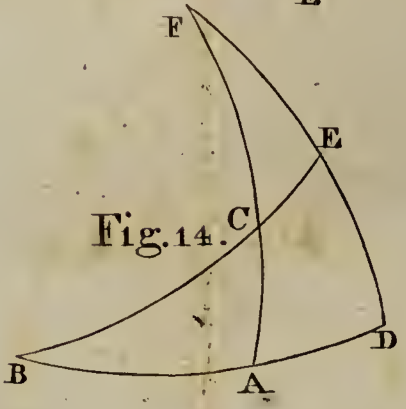


Fig. 14.

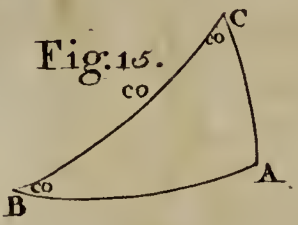


Fig. 15.

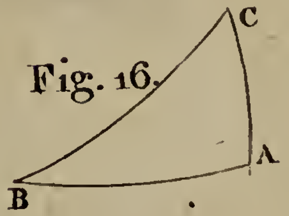


Fig. 16.

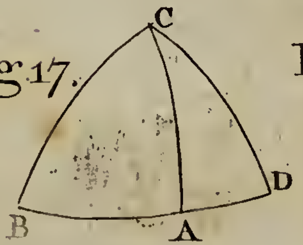


Fig. 17.

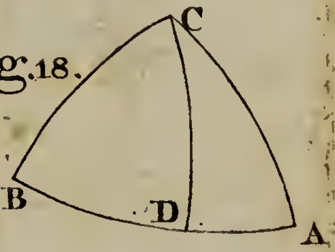


Fig. 18.

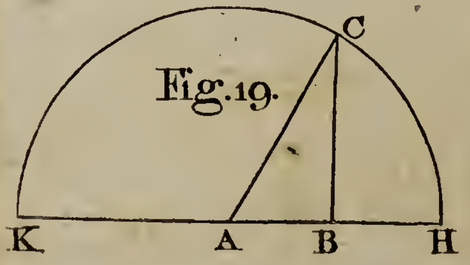


Fig. 19.

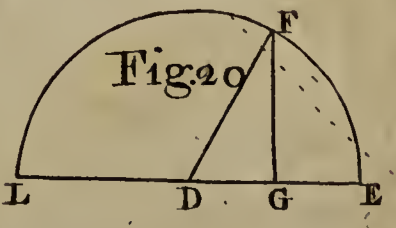


Fig. 20

SPHERICAL TRIGONOMETRY Plate II.

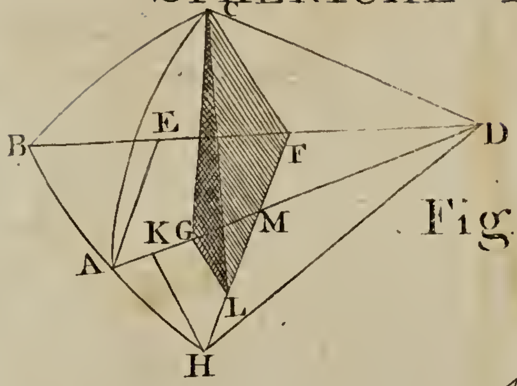


Fig. 21.

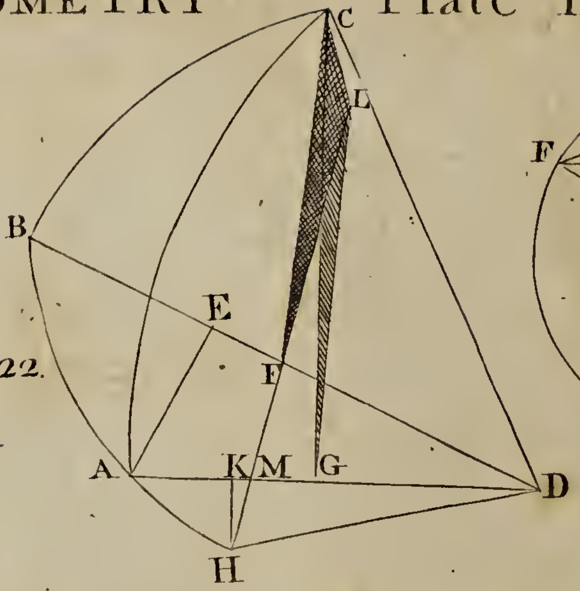


Fig. 22.

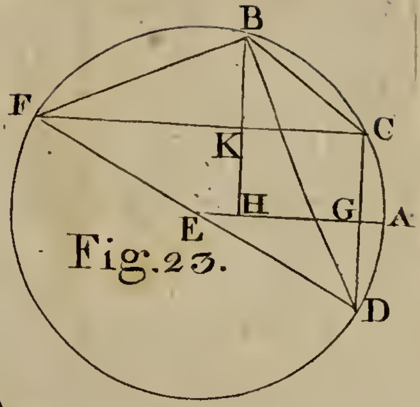


Fig. 23.

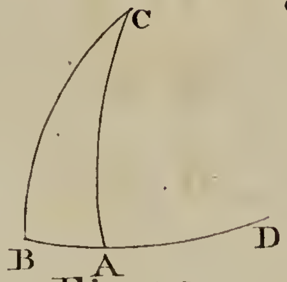


Fig. 25.

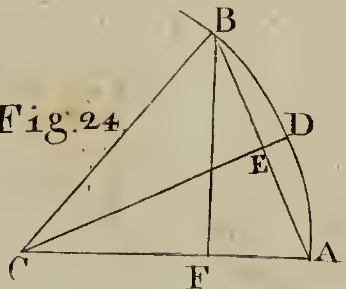


Fig. 24.

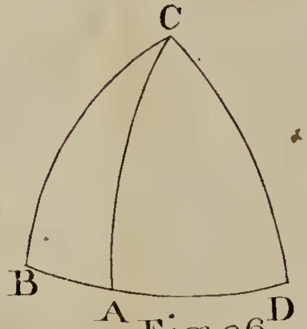


Fig. 26

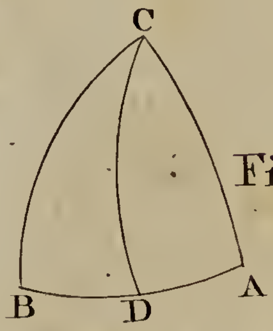


Fig. 27.